Freezing of energy of a soliton in an external potential

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joint work with D. Bambusi

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Outline

The equation

NLS with small external potential V

$$\mathrm{i}\partial_t \psi = -\Delta\psi - \beta'(|\psi|^2)\psi + \epsilon V(x)\psi \ , \quad x \in \mathbb{R}^3 \ ,$$

- V Schwartz class
- β focusing nonlinearity, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$

$$\left|\beta^{(k)}(u)\right| \leq C_k \langle u \rangle^{1+p-k} \ , \quad \beta'(0) = 0 \quad p < 2/3 \ ,$$

Under this assumptions: global unique solution for initial data in H^1

Case $\epsilon = 0$

Look for moving solitary waves:

$$\psi(x,t) = e^{\mathrm{i}\gamma(t)} e^{\mathrm{i}p(t)\cdot(x-q(t))/m(t)} \eta_{m(t)}(x - q(t))$$

where η_m is the ground state of NLS with mass m, and $p, q \in \mathbb{R}^3$, $\gamma \in \mathbb{R}$ time-dependent parameter which fulfill

$$\begin{cases} \dot{p} = 0 \\ \dot{q} = \frac{p}{m} \\ \dot{m} = 0 \\ \dot{\gamma} = \mathcal{E}(m) + \frac{|p|^2}{4m} \end{cases} \tag{1}$$

- first two equations are hamiltonian equations of free mechanical particle
- Solitary wave is traveling through space with a constant velocity!

Case $\epsilon \neq 0$

Solitary wave *is not* a solution anymore.
But if the initial data is close to solitary wave

$$\|\psi(0) - e^{\mathrm{i}\gamma} e^{\mathrm{i}p\cdot(x-q)/m} \eta_m(\cdot - q)\|_{H^1} < \epsilon$$

then $\psi(x,t)$ stays close to solitary wave with time-dependent parameter $p(t), q(t), \gamma(t), m(t)$ which fulfill

$$\begin{cases} \dot{p} = -\epsilon \nabla V^{eff}(q) + \mathcal{O}(\epsilon^2) \\ \dot{q} = \frac{p}{m} + \mathcal{O}(\epsilon^2) \\ \dot{m} = \mathcal{O}(\epsilon^2) \\ \dot{\gamma} = \mathcal{E}(m) + \frac{|p|^2}{4m} - \epsilon V^{eff}(q) + \mathcal{O}(\epsilon^2) \end{cases} \quad (2)$$

with $V^{eff}(q) = \int_{\mathbb{R}^3} V(x+q)\, \eta_m^2 \, dx$

- first two equations are hamiltonian equations of mechanical particle interacting with external field

$$H^{\epsilon}_{mech}(p,q) = \frac{|p|^2}{2m} + \epsilon V^{eff}(q)$$

- Solitary waves moves "like a mechanical particle in an external potential"
- for which interval of time is the argument rigorous?

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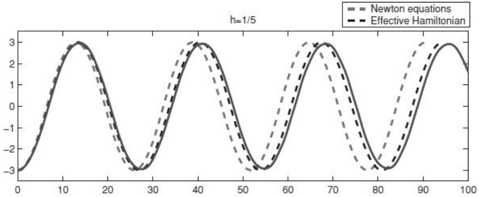
Previous results

- Fröhlich-Gustafson-Jonsson-Sigal '04, '06, Holmer-Zworski '08:

$$dist\Big((p(t), q(t)), (p_{mech}(t), q_{mech}(t))\Big) \ll 1 \quad \text{for times } |t| \leq T\epsilon^{-3/2}$$

- **Question:** can we do better? **Answer:** NO!



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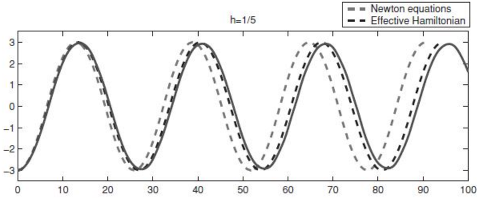

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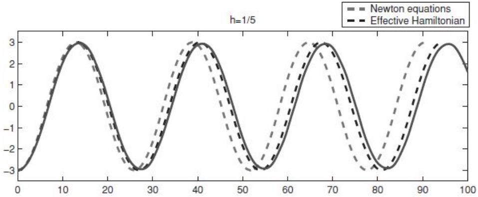

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New point of view

- Not surprising! in classical mechanics motions starting nearby get far away after quite short time scales.
- to control the dynamics for longer times, **control** only on some relevant quantities: **actions** or **energy of subsystem**
- in our case: system is composed by two subsystems evolving on different time-scales:

$$H_{NLS} = H^{\epsilon}_{mech}(p,q) + H_{field}(\phi) + \text{ high order coupling}$$

New Question: is it possible to control

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Outline

Assumptions

- existence of a smooth family of ground states $m \mapsto \eta_m$, $\forall m \in \mathcal{I}$
- **Linearization:** $\psi = \eta_m + \chi$ and linearize for $\epsilon = 0$:

$$\dot{\chi} = L_0 \chi$$

where, with $\chi = (Re\chi, Im\chi)$

$$L_0 := \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \qquad \begin{array}{l} L_+ = -\Delta + \mathcal{E} - \beta'(\eta_m^2) \\ L_- = -\Delta + \mathcal{E} - \beta'(\eta_m^2) - 2\beta''(\eta_m^2)\eta_m^2 \end{array}$$

We assume

1. $\sigma_d(L_0) = \{0\}$, $\sigma_c(L_0) = \bigcup_\pm \pm \mathrm{i}[\mathcal{E}, \pm\infty)$
2. $Ker(L_+) = span(\eta_m)$, $Ker(L_-) = span(\partial_{x_j}\eta_m)_{j=1,\dots,3}$, 0 has multiplicity 8
3. $\pm \mathrm{i}\mathcal{E}$ are not resonances, i.e. $L_0\chi = \mathrm{i}\mathcal{E}\chi$ has no solution with $\langle x \rangle^{-\delta} \chi \in L^2, \forall \delta > 1/2$

Theorem (Bambusi, M.)

Fix arbitrary $r \in \mathbb{N}$. Then $\exists \epsilon_r$ s.t. for $0 \le \epsilon < \epsilon_r$, it holds the following: let $\psi_0 \in H^1$ s.t. there exist $(\bar{m}, \bar{\alpha}, \bar{\mathbf{p}}, \bar{\mathbf{q}})$ with

$$\begin{aligned} \|\psi_0 - e^{\mathrm{i}\bar{\alpha}} \eta_{\bar{m}}(\bar{\mathbf{p}}, \bar{\mathbf{q}})\|_{H^1} &\le K_1 \epsilon^{1/2} \\ H^{\epsilon}_{mech}(\bar{\mathbf{p}}, \bar{\mathbf{q}}) &< K_2 \epsilon \ , \end{aligned} \tag{3}$$

then, for $|t| \le T_0 \epsilon^{-r}$, the solution $\psi(t)$ exists in H^1 and admits the decomposition

$$\psi(t) := e^{\mathrm{i}\alpha(t)} \eta_m(\mathbf{p}(t), \mathbf{q}(t)) + \phi(t) \ , \tag{4}$$

with a constant m and smooth functions $\mathbf{p}(t), \mathbf{q}(t), \alpha(t)$ s.t.

$$|H^{\epsilon}_{mech}(\mathbf{p}(t), \mathbf{q}(t)) - H^{\epsilon}_{mech}(\mathbf{p}(0), \mathbf{q}(0))| \le C_1 \epsilon^{3/2} \ , \quad |t| \le \frac{T_0}{\epsilon^r} \ . \tag{5}$$

Furthermore, for the same times one has

$$\|\phi(t)\|_{H^1} \le C_1 \epsilon^{1/2} \ .$$

Outline

Main steps

1) Canonical variables (Darboux theorem)
2) Birkhoff normal form (continuous spectrum)
3) dispersive estimates (Strichartz)

Framework

- Phase space:
  - $H^{s,k}$, scale of Hilbert spaces, $\|\psi\|_{H^{s,k}} = \| \langle x \rangle^s (\Delta - 1)^{k/2} \psi \|_{L^2}$
  - $\langle \psi_1 ; \psi_2 \rangle = 2Re \int \psi_1 \bar{\psi}_2$
  - $\omega(\psi_1, \psi_2) := \langle E\psi_1 ; \psi_2 \rangle$, with $E = \mathrm{i}$, $J = E^{-1}$ Poisson tensor
- Symmetries:
  - $A_j := \mathrm{i}\partial_{x_j}$, $j = 1, 2, 3$, $\quad A_4 = \mathbb{1}$,
  - $e^{q_j J A_j} \psi \equiv \psi(\cdot - q_j \mathbf{e}_j)$
- Soliton manifold: $\mathcal{T} := \bigcup_{q,p} e^{q^j J A_j} \eta_p$, where $\eta_p := e^{\mathrm{i} \frac{p_k x^k}{p_4}} \eta_{p_4}$

  restriction $\omega|_{\mathcal{T}} = dp \wedge dq$
- Natural decomposition:

$$L^2 \equiv T_{\eta_p} L^2 \simeq T_{\eta_p} \mathcal{T} \oplus T^{\angle}_{\eta_p} \mathcal{T}$$

  with $T^{\angle}_{\eta_p} \mathcal{T} := \{ U \ : \ \omega(U; X) = 0 \ , \ \forall X \in T_{\eta_p} \mathcal{T} \}$

  Let $\Pi_p : L^2 \to T^{\angle}_{\eta_p} \mathcal{T}$ the projector on the symplectic orthogonal

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Step 1: Adapted coordinates and Darboux theorem

- Coordinate system: Fix $p_0 := (0, 0, 0, m)$ and $\mathcal{V}^{s,k} := \Pi_{p_0} H^{s,k}$. Define

$$\mathcal{F} : \mathcal{J} \times \mathbb{R}^4 \times \mathcal{V}^{1,0} \to H^{1,0}, \quad (p, q, \phi) \mapsto e^{q^j JA_j} (\eta_p + \Pi_p \phi)$$

- Problems:
  - p, q, ϕ are **not** canonical
  - $\mathcal{F}$ **not smooth**, only continuous. Indeed $\mathbb{R}^4 \times H^1 \ni (q, \phi) \mapsto e^{q^j JA_j} \phi \in H^1$ only continuous

Darboux theorem

There exists a map of the form

$$\mathcal{D}(p', q', \phi') = \left(p' - N + P,\ q' + Q,\ \Pi_{p_0} e^{\alpha_j JA_j} (\phi' + S)\right) ,$$

with the following properties

1. $S : \mathcal{J} \times \mathbb{R}^4 \times \mathcal{V}^{-\infty} \to \mathcal{V}^{\infty}$ is smooth.
2. $P, Q, \alpha_j : \mathcal{J} \times \mathbb{R}^4 \times \mathcal{V}^{-\infty} \to \mathbb{R}^4$ are smooth.
3. $N_j := \frac{1}{2} \langle A_j \phi', \phi' \rangle$
4. p', q', ϕ' are canonical.

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Step 2: The Hamiltonian in Darboux coordinates and normal form

Scaling of variables: $\mu := \epsilon^{1/4}$, $p \mapsto \mu^2 p$, $\phi \mapsto \mu\phi$

- Hamiltonian in Darboux coordinates: Let $\mathcal{D}$ be the Darboux map. Then

$$H \circ \mathcal{F} \circ \mathcal{D} = \mu^2 h + H_L + H_R$$

with

$$h = \frac{p^2}{2m} + V^{eff}(q), \quad H_L = \frac{1}{2} \langle EL_0\phi, \phi \rangle \,, \quad H_R = \mu^3 \left(H_{R0} + H_{R1} + H_{R2} + H_{R3} \right)$$

and

$$H_{R1} = \langle S(\mu, p, q, N), \phi \rangle \,, \quad S \text{ smoothing}$$

- Birkhoff normal form: eliminate terms *linear* in ϕ up to order r in μ.
- Lie transform: time 1 flow Φ_{χ_r} of $\chi_r = \mu^r \langle \chi^{(r)}(N, p, q), \phi \rangle$
- Homological equation: $L_0 \chi^{(r)} = \Psi$, L_0 with continuous spectrum
- Problem: hamiltonian vector field of χ_r is not smooth, but its flow is well defined and it has the same structure as before! The structure of the hamiltonian is preserved

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Step 3: Analysis of the normal form

We have $H \circ \mathcal{T}^{r+2}$ in normal form. Equation for ϕ:

$$\dot{\phi} = L_0\phi + \mu^3 w^j(t) JA_j\phi + \mu^3 W\phi + J\nabla_\phi H_{R3} + \mu^{r+2} S \qquad (6)$$

linear equation has *variable coefficients* and *unbounded operators*

Aim: Nonlinear stability for ϕ!

- Strichartz stable under some unbounded small perturbations:

$$\|\phi\|_{L^2_t[0,T]W^{1,6}_x} + \|\phi\|_{L^\infty_t[0,T]H^1_x} \leq K\mu \ , \quad \forall |t| \leq T\mu^{-r}$$

- Using this, one proves that

$$|H_L(\phi(t)) - H_L(\phi(0))| \leq \mu^4 \qquad \forall |t| \leq T\mu^{-r}$$

- using conservation of energy

$$|h(p(t), q(t)) - h(p(0), q(0))| \leq \mu^4 \qquad \forall |t| \leq T\mu^{-r}$$

Thanks for your attention!