On Two-Sided Estimates for the Nonlinear Fourier Transforms of KdV and NLS

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Hamiltonian systems and their applications
St. Petersburg, Russia

June 7, 2015

KdV on the circle

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is a Hamiltonian PDE with Phase Space

$$H_0^m = \left\{ u \in H^m(\mathbb{T}, \mathbb{R}) : [u] = \int_{\mathbb{T}} u \,\mathrm{d}x = 0 \right\},$$

Poisson Structure

$$\{F, G\} = \int_{\mathbb{T}} (\partial_u F \partial_x \partial_u G) \,\mathrm{d}x,$$

and Hamiltonian

$$\mathcal{H}(u) = \frac{1}{2} \int_{\mathbb{T}} (u_x^2 + 2u^3) \,\mathrm{d}x.$$

In view of this

$$u_t = \{u, \mathcal{H}\} = \partial_x \partial_u \mathcal{H}.$$

Model space (endowed with standard Poisson structure)

$$\begin{aligned} h^m &= \ell^2_{m+1/2}(\mathbb{N},\mathbb{R}) \times \ell^2_{m+1/2}(\mathbb{N},\mathbb{R}) \\ &= \left\{ (x_n, y_n)_{n\geqslant 1} \, : \, \sum_{n\geqslant 1} (2n\pi)^{2m+1}(x_n^2 + y_n^2) < \infty) \right\}. \end{aligned}$$

The Birkhoff mapping (Kappeler et. al 1995 - 2003)

$$\Omega \colon H^0_0 \to h^0, \qquad u \mapsto (x_n, y_n)_{n\geqslant 1}$$

- is a global, bi-real-analytic, canonical diffeo,
- $\mathcal{H} \circ \Omega^{-1}\big|_{h^1}$ is a real analytic function of the actions $I_n = (x_n^2 + y_n^2)/2$ alone,
- the KdV evolution becomes trivial

$$\dot{x}_n = -\omega_n y_n, \qquad \dot{y}_n = \omega_n x_n, \qquad \omega_n \coloneqq \partial_{I_n} \mathcal{H}.$$

Ω is a nonlinear Fourier transform

- $d_0\Omega$ is a weighted Fourier transform,
- the restriction

$$\Omega\colon H_0^m \to h^m, \qquad m \geqslant 1,$$

  is again a bi-real-analytic diffeo.
- **Kuksin & Perelman '10, Kappeler, Schaad & Topalov '13**

$$\Omega - d_0\Omega\colon H_0^m \to h^{m+1} \quad (1 \text{ smoothing}).$$

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$$\Omega - \mathrm{d}_0\Omega\colon H_0^m \to h^{m+1} \quad \text{(1 smoothing)}.$$

- $\|\Omega(u)\|_0 = \|u\|_0$ Parsevall's identity,
- **Korotyaev '00-'06** for any $m \geqslant 1$ there exist absolute constants $c_m, d_m > 0$,

$$\|\Omega(u)\|_{h^m} \leqslant c_m(1 + \|u\|_m)^{(4m+1)/3}\|u\|_m$$
$$\|u\|_m \leqslant d_m(1 + \|\Omega(u)\|_{h^m})^{N_m}\|\Omega(u)\|_{h^m}$$

with

$$N_m \sim m^2(1 + m + \cdots + m!).$$

Main Results

Theorem 1 (M 2015) For any $m \geqslant 1$ there exist $c_m, d_m > 0$ s.t.

(i) $\|\Omega(u)\|_{h^m} \leqslant c_m\Big[\|u\|_m + (1 + \|u\|_{m-1})^m \|u\|_{m-1}\Big]$,

(ii) $\|u\|_m \leqslant d_m\Big[\|\Omega(u)\|_{h^m} + (1 + \|\Omega(u)\|_{h^{m-1}})^m \|\Omega(u)\|_{h^{m-1}}\Big]$.

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Corollary If $\|u_0\|_m = \lambda$ and $\|u_0\|_{m-1} = \varepsilon$ then for any $t \in \mathbb{R}$,

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Theorem 2 (M 2015) For any $m \geqslant 1$ there exist $c_m, d_m > 0$ s.t.

(i) $\|I(u)\|_{\ell^1_{2m+1}} \leqslant c_m^2\Big[\|u\|_m^2 + (1+\|u\|_{m-1})^{2m}\|u\|_{m-1}^2\Big]$.

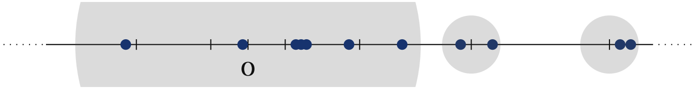
(ii) $\|u\|_m^2 \leqslant d_m^2\Big[\|I(u)\|_{\ell^1_{2m+1}} + (1+\|I(u)\|_{\ell^1_{2m-1}})^{m}\|I(u)\|_{\ell^1_{2m-2}}\Big]$.

KdV Lax Operator

$$L(u) = -\partial_x^2 + u,$$

Periodic Spectrum   pure point

$$\lambda_n^{\pm} = n^2\pi^2 + \ell_n^2, \qquad \lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \lambda_2^- \leqslant \lambda_2^+ < \cdots$$



Method of Proof (1)

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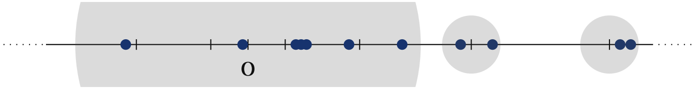

Localizable in L^2 $|\lambda_n^{\pm} - n^2\pi^2| \leqslant \pi/4 \quad \text{if} \quad |n| \geqslant 4\|u\|_0.$

Zero Set of $\Delta^2 - 4$ where Δ is the discriminant of L.

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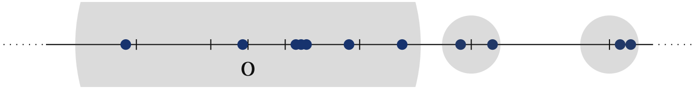

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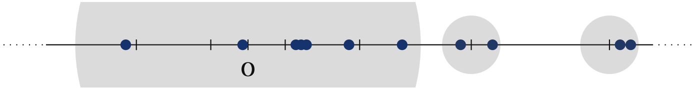

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$$I_{n,m} = \frac{2}{\pi}\int_{\lambda_n^-}^{\lambda_n^+} \lambda^m \cosh^{-1}\frac{(-1)^n\Delta(\lambda)}{2}\,\mathrm{d}\lambda, \qquad I_n = I_{n,0}.$$

Method of Proof (2)

Asymptotic Behavior if $|n| \geqslant 4\|u\|_0$

$$I_{n,m} \sim (\lambda_n^{\pm})^m I_n \sim (n\pi)^{2m} I_n,$$

Trace Formula

$$\sum_{n\geqslant 1} (2n\pi) I_{n,m} = \frac{\mathcal{H}_m}{4^m} - \frac{2}{4^m} \sum_{0\leqslant k\leqslant m-2} \mathcal{H}_{m-2-k}\mathcal{H}_k$$

KdV Hamiltonian Hierarchy

$$\mathcal{H}_0 = \frac{1}{2}\int_{\mathbb{T}} u^2 \, \mathrm{d}x, \qquad \mathcal{H}_1 = \frac{1}{2}\int_{\mathbb{T}} ((\partial_x u)^2 + 2u^3) \, \mathrm{d}x, \qquad \dots$$

$$\mathcal{H}_m = \frac{1}{2}\int_{\mathbb{T}} (\partial_x^m u)^2 + p_m(u, \dots, \partial_x^{m-1} u) \, \mathrm{d}x.$$

Conclusion

$$\sum_{n\geqslant 1} (2n\pi)^{2m+1} I_n \sim \sum_{n\geqslant 1} (2n\pi) I_{n,m} \sim \|u\|_m^2 + P_m(\|u\|_{m-1}).$$

Method of Proof (Summary)

- Proves Theorem 2 ($\Rightarrow$ Theorem 1) for any $m \geqslant 1$.
- Does not use conformal mapping theory.
- Essential ingredient is the uniform localization of the periodic spectrum.

The defocusing NLS on the circle

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$$H_r^m = \{(u, \overline{u}) : u \in H^m(\mathbb{T}, \mathbb{C})\},$$

Poisson Structure

$$\{F, G\} = -\mathrm{i} \int_{\mathbb{T}} (\partial_u F \partial_{\overline{u}} G - \partial_{\overline{u}} F \partial_u G) \, \mathrm{d}x,$$

and Hamiltonian

$$\mathcal{H}(u, \overline{u}) = \int_{\mathbb{T}} (u_x \overline{u_x} + u^2 \overline{u}^2) \, \mathrm{d}x.$$

In view of this

$$\mathrm{i}u_t = \mathrm{i}\{u, \mathcal{H}\} = \partial_{\overline{u}} \mathcal{H}.$$

Model space

$$h_r^m = \ell_m^2(\mathbb{Z}, \mathbb{R}) \times \ell_m^2(\mathbb{Z}, \mathbb{R}),$$

with elements $(x_n, y_n)_{n\in\mathbb{Z}}$ and standard Poisson structure.

The Birkhoff mapping (Kappeler et. al 1995 - 2014)

$$\Omega \colon H_r^0 \to h_r^0, \qquad u \mapsto (x_n, y_n)_{n\in\mathbb{Z}}$$

- is a global, bi-real-analytic, canonical diffeo.
- $\mathcal{H} \circ \Omega^{-1}\big|_{h_r^1}$ is a real analytic function of the actions $I_n = (x_n^2 + y_n^2)/2$ alone,
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$$\dot{x}_n = -\omega_n y_n, \qquad \dot{y}_n = \omega_n x_n, \qquad \omega_n \coloneqq \partial_{I_n}\mathcal{H}.$$

Ω is a Nonlinear Fourier Transform

- $\mathrm{d}_0\Omega$ is the Fourier transform,
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$$\Omega\colon H_r^m \to h_r^m, \qquad m \geqslant 1,$$

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- **Kappeler, Schaad & Topalov '15**

$$\Omega - \mathrm{d}_0\Omega\colon H_r^m \to h_r^{m+1}, \qquad \text{(1-smoothing).}$$

- $\|\Omega(u)\|_0 = \|u\|_0$ Parsevall's identity,
- **Korotyaev '05** there exist absolute constants $c, d > 0$,

$$\|\Omega(u)\|_{h_r^1} \leqslant c(\|u\|_1 + \|u\|_0^2), \quad \|u\|_1 \leqslant d(\|\Omega(u)\|_{h_r^1} + \|\Omega(u)\|_{h_r^0}^2).$$

Question Can we obtain explicit estimates of $\|\Omega(u)\|_{h_r^m}$ in terms of $\|u\|_m$ also for $m \geqslant 2$?

Answer

Theorem 1 (M, IMRN 2014) $\forall\ m \geqslant 1 \quad \exists\ c_m, d_m > 0$

$$\text{(i) } \|\Omega(u)\|_{h_r^m} \leqslant c_m \Big[\|u\|_m + (1 + \|u\|_{m-1})^{2m} \|u\|_{m-1} \Big],$$

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Note: For $m = 1$ this is due to Korotyaev '05.

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Difficulties

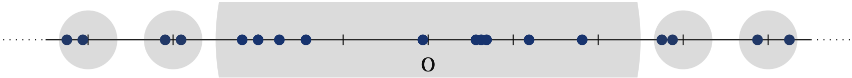
NLS Lax Operator

$$L(u) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & u \\ \overline{u} & \end{pmatrix}$$

Periodic Spectrum

pure point $(\lambda_n^\pm)_{n\in\mathbb{Z}}$,

$$\lambda_n^\pm = n\pi + \ell_n^2, \qquad \cdots \leqslant \lambda_{n-1}^+ \leqslant \lambda_n^- \leqslant \lambda_n^+ \leqslant \lambda_{n+1}^- \leqslant \cdots$$



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Periodic Spectrum pure point $(\lambda_n^\pm)_{n\in\mathbb{Z}}$,

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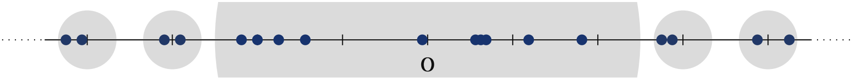

Localizable only locally uniformly in u on H_r^0

$$\operatorname{spec}(\mathrm{e}^{-\mathrm{i}2\pi m x}u) = \operatorname{spec}(u) + m\pi$$

Localization

Quadratic Localization in H_r^1 For $|n| \geqslant 8\|u\|_1^2$,

$$|\lambda_n^\pm - n\pi| \leqslant \frac{2(1+\|u\|_1)\|u\|_1}{n} \leqslant \frac{\pi}{5},$$

while the remaining eigenvalues for $|n| < 8\|u\|_1^2$ satisfy

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Quadratic Localization in H^1_r For $|n| \geqslant 8\|u\|_1^2$,

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Zero Set of $\Delta^2 - 4$ where Δ is the discriminant of L.

Actions

$$I_{n,m} = \frac{2}{\pi}\int_{\lambda_n^-}^{\lambda_n^+} \lambda^{2m} \cosh^{-1} \frac{(-1)^n \Delta(\lambda)}{2}\, \mathrm{d}\lambda, \qquad I_n = I_{n,0}.$$

Completion of Proof

Asymptotic Behavior for $|n| \geqslant 8\|u\|_1^2$

$$I_{n,m} \sim (\lambda_n^{\pm})^{2m} I_n \sim (n\pi)^{2m} I_n,$$

Satisfy the Trace Formula

$$\sum_{n\in\mathbb{Z}} I_{n,m} = \frac{(-1)^{m+1}}{4^m}\mathcal{H}_m,$$

NLS Hamiltonian Hierarchy

$$\mathcal{H}_0 = \int_{\mathbb{T}} |u|^2 \, \mathrm{d}x, \qquad \mathcal{H}_1 = \int_{\mathbb{T}} (|u_x|^2 + |u|^4) \, \mathrm{d}x, \qquad \dots$$

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Conclusion

$$\sum_{n\in\mathbb{Z}} (n\pi)^{2m} I_n \sim \sum_{n\in\mathbb{Z}} I_{n,m} \sim \|u\|_m^2 + P_m(\|u\|_{m-1}).$$