KAM for gravity capillary water waves

Riccardo Montalto
University of Zürich

June 5, 2015

Water waves:

Euler's equations for an irrotational, incompressible fluid, ∞ depth, with gravity and capillarity.

Unknowns:

- Free surface $\mathcal{S}(t) := \{(x, y) \in \mathbb{T} \times \mathbb{R} : y = \eta(t, x)\}$
- Velocity potential $\Phi(t, x, y)$ on the domain $\Omega(t) := \{(x, y) \in \mathbb{T} \times \mathbb{R} : y < \eta(t, x)\}$

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) & \text{on } \mathcal{S}(t) \\ \Delta\Phi = 0 & \text{in } \Omega(t) \\ \nabla\Phi \to 0 & \text{as } y \to -\infty \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{on } \mathcal{S}(t) \end{cases}$$

$$u = \nabla\Phi = \text{velocity field}, \ \ \mathrm{rot}\, u = 0 \text{ (irrotational)},$$

$$\mathrm{div}\, u = \Delta\Phi = 0 \text{ (uncompressible)}$$

$$g = \text{gravity}, \ \kappa = \text{surface tension coefficient}$$

Zakharov-Craig-Sulem formulation

$$\begin{cases} \partial_t \eta = G(\eta)\psi = \nabla_\psi H(\eta,\psi) \\ \partial_t \psi = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{\left(G(\eta)\psi + \eta_x\psi_x\right)^2}{2(1+\eta_x^2)} + \dfrac{\kappa \eta_{xx}}{(1+\eta_x^2)^{3/2}} = -\nabla_\eta H(\eta,\psi) \end{cases}$$

Hamiltonian: *kinetic energy* + *potential energy* + *area surface int.*

$$H(\eta,\psi) := \tfrac{1}{2}(\psi, G(\eta)\psi)_{L^2(\mathbb{T}_x)} + \int_{\mathbb{T}} g\frac{\eta^2}{2} + \kappa\sqrt{1+\eta_x^2}\, dx$$

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n \Phi|_{y=\eta(x)} = \{\Phi_y - \eta_x \Phi_x\}_{|y=\eta(x)}$$

Reversible solutions

$$H(\eta,\psi) = H(\eta,-\psi)\,,$$

$$\{(\eta,\psi) : \eta = \mathrm{even}(t)\mathrm{even}(x)\,, \quad \psi = \mathrm{odd}(t)\mathrm{even}(x)\}$$

Theorem (KAM for capillary-gravity water waves. Berti, M.)

For every choice of the tangential sites $S \subset \mathbb{N} \setminus \{0\}$, *there exists* $\bar{s} > \frac{|S|+1}{2}$, $\varepsilon_0 \in (0,1)$ *such that: for all* $\xi_j \in (0, \varepsilon_0)$, $j \in S$, $\exists$ **a Cantor like set** $\mathcal{G} \subset [\kappa_1, \kappa_2]$ **with asymptotically full measure as** $\xi \to 0$**, i.e.** $\lim_{\xi \to 0} |\mathcal{G}| = (\kappa_2 - \kappa_1)$, *such that, for any surface tension coefficient* $\kappa \in \mathcal{G}$, *the* CAPILLARY-GRAVITY WATER WAVES EQUATIONS *have a reversible, quasi-periodic standing wave solution* $(\eta, \psi) \in H^{\bar{s}}$, *even in* x, *of the form*

$$\eta(t,x) = \sum_{j \in S} \sqrt{\xi}_j \cos(\omega_j t) \cos(jx) + o(\sqrt{\xi})$$

$$\psi(t,x) = -\sum_{j \in S} \sqrt{\xi}_j j^{-1} \omega_j \sin(\omega_j t) \cos(jx) + o(\sqrt{\xi})$$

with frequency vector $\omega = (\omega_j)_{j \in S} \in \mathbb{R}^S$ *satisfying*

$$\omega_j \to \sqrt{j(1 + \kappa j^2)} \quad \text{as} \quad \xi \to 0\,, \quad \forall j \in S$$

The solutions are linearly **stable**.

Ideas of the proof

1. **Degenerate KAM Theory for PDEs:** (Bambusi, Berti, Magistrelli) Use the surface tension parameter κ to verify the non-resonance conditions
2. **Nash-Moser implicit function iterative scheme**
3. **Spectral analysis of the linearized equation** at any approximate solution. This requires
   - Theory of Pseudo differential operators and Fourier integral operators (the linearized equation is a Pseudo PDE)
   - KAM reducibility scheme

Analysis of the linearized operator

WW equation

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta + \frac{\psi_x^2}{2} - \frac{\left(G(\eta)\psi + \eta_x \psi_x\right)^2}{2(1+\eta_x^2)} - \frac{\kappa \eta_{xx}}{(1+\eta_x^2)^{3/2}} = 0 \end{cases}$$

Linearization on $u = \varepsilon(\eta, \psi)$ in the direction $h = (\widehat{\eta}, \widehat{\psi})$

$$\mathcal{L}h := \begin{pmatrix} \omega \cdot \partial_\varphi + \partial_x V + G(\eta)B & -G(\eta) \\ 1 + BV_x + BG(\eta)B - \kappa\partial_x c \partial_x & \omega \cdot \partial_\varphi + V\partial_x - BG(\eta) \end{pmatrix} \begin{pmatrix} \widehat{\eta} \\ \widehat{\psi} \end{pmatrix}$$

$$\|c - 1\|_{H^s}, \|V\|_{H^s}, \|B\|_{H^s} = O(\varepsilon)$$

Conjugacy of $\mathcal{L}$ to constant coefficients

The conjugacy procedure is splitted in two parts:

1. **∂_x-reduction in decreasing symbols**

$$\mathcal{L}_1 := \Phi^{-1}\mathcal{L}\Phi = \omega \cdot \partial_\varphi + \mathcal{D} + \mathcal{R}_0$$

- $\mathcal{D} := \mathrm{diag}_j \mu_j$, $\mu_j := \lambda_3\sqrt{j(1+\kappa j^2)} + \lambda_1\sqrt{j}$, $\lambda_3 - 1, \lambda_1 = O(\varepsilon)$,
- $\mathcal{R}_0(\varphi) : H^s_x \to H^s_x$, $\quad \partial^\beta_\varphi \mathcal{R}_0(\varphi) : H^s_x \to H^s_x$, for some $\beta = \beta(\nu) > 0$

Use changes of variables, Egorov Theorem (FIO), pseudo-differential operators

2. **ε-reduction, reducibility scheme**

$$\mathcal{L}_\nu := \Phi_\nu^{-1}\mathcal{L}_1\Phi_\nu = \omega \cdot \partial_\varphi + \mathcal{D} + r^{(\nu)} + \mathcal{R}_\nu$$

- $\mathcal{R}_\nu = \mathcal{R}_\nu(\varphi) = O(\mathcal{R}_0^{2^\nu}) = O(\varepsilon^{2^\nu})$
- $r^{(\nu)} = \mathrm{diag}_{j\in\mathbb{Z}}(r_j^{(\nu)})$, $\sup_j |r_j^{(\nu)}| = O(\varepsilon)$,

KAM-type scheme, now transformations of $H^s_x \to H^s_x$

Preliminary steps (Alazard-Baldi 2014, periodic case)

We introduce the good unknown of Alhinac

$$\mathcal{Z} : (\widehat{\eta}, \widehat{\psi}) \mapsto (\widehat{\eta}, \widehat{\psi} - B\widehat{\eta})$$

and the linearized operator simplifies as

$$\mathcal{L}_0 = \mathcal{Z}^{-1}\mathcal{L}\mathcal{Z} = \begin{pmatrix} \omega \cdot \partial_\varphi + \partial_x V & -G(\eta) \\ a - \kappa\partial_x c \partial_x & \omega \cdot \partial_\varphi + V\partial_x \end{pmatrix}$$

Since $G(\eta) = |D| + \mathcal{R}_G(\eta)$, $\mathcal{R}_G(\eta) \in OPS^{-\infty}$, we get

$$\mathcal{L}_0 = \begin{pmatrix} \omega \cdot \partial_\varphi + \partial_x V & -|D| \\ a - \kappa\partial_x c \partial_x & \omega \cdot \partial_\varphi + V\partial_x \end{pmatrix} + \text{smoothing terms}$$

Changes of variables

To reduce to constant coefficients the highest order $c(\varphi, x)\partial_{xx}$ we perform

change of variables

$$h(\varphi, x) \mapsto h(\varphi, x + \beta(\varphi, x))$$

Quasi-periodic reparametrization of time

$$h(\varphi, x) \mapsto h(\varphi + \omega\alpha(\varphi), x)$$

and then

$$\mathcal{L}_1 = \begin{pmatrix} \omega \cdot \partial_\varphi + a_1\partial_x & -\lambda_3|D| \\ \lambda_3(1 - \kappa\partial_{xx}) + a_2\partial_x & \omega \cdot \partial_\varphi + a_1\partial_x \end{pmatrix} + O(\varepsilon|D|^0) \,,$$

$\lambda_3 \in \mathbb{R}\,,\ \lambda_3 = 1 + O(\varepsilon)$, $\|a_1\|_{H^s}, \|a_2\|_{H^s} = O(\varepsilon)$.

Symmetrization of the highest order

In $\mathcal{L}_1$ symmetrize

$$\mathcal{L}_1 = \begin{pmatrix} \dots & -\lambda_3|D| + \dots \\ \lambda_3(1-\kappa\partial_{xx}) + \dots & \dots \end{pmatrix}$$

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda := \langle D\rangle^{-\frac{1}{2}}(1-\kappa\partial_{xx})^{\frac{1}{2}}$$

then

$$\mathcal{L}_2 = \begin{pmatrix} \omega\cdot\partial_\varphi + a_1\partial_x & -\lambda_3 T(D) + a_3\mathcal{H}|D|^{\frac{1}{2}} \\ \lambda_3 T(D) + a_4\mathcal{H}|D|^{\frac{1}{2}} & \omega\cdot\partial_\varphi + a_1\partial_x \end{pmatrix} + O(\varepsilon|D|^0)$$

$T(D) := \sqrt{|D|(1-\kappa\partial_{xx})}$, $\|a_3\|_{H^s}, \|a_4\|_{H^s} = O(\varepsilon)$, $\mathcal{H}$ (Hilbert transform)

New steps: Decoupling of h, $\bar{h}$

We write $\mathcal{L}_1$ as an operator acting on the variables $z = \widehat{\eta} + \mathrm{i}\widehat{\psi}$ and then

$$\mathcal{L}_1[h, \bar{h}] = \omega \cdot \partial_\varphi h + \mathrm{i}\lambda_3 T(D)h + a_1 \partial_x h + \mathrm{i} a_0 \mathcal{H}|D|^{\frac{1}{2}} h + \mathcal{R}_0 h + \mathcal{Q}_0 \bar{h}\,,$$

where $a_1(\varphi, x), a_0(\varphi, x) \in H^s$, $\mathcal{R}_0 \in OPS^0$, $\mathcal{Q}_0 \in OPS^{\frac{1}{2}}$.

Goal

Conjugate to

$$\mathcal{L}_N[h, \bar{h}] = \omega \cdot \partial_\varphi h + \mathrm{i}\lambda_3 T(D)h + a_1 \partial_x h + a_0 \mathcal{H}|D|^{\frac{1}{2}} h + \mathcal{R}_N h + \mathcal{Q}_N \bar{h}\,,$$

$\mathcal{R}_N \in OPS^0$, $\mathcal{Q}_N \in OPS^{-N}$ ($\mathcal{Q}_N \simeq |D|^{-N}$)

Decoupling

By induction: At the n-th step we have

$$\mathcal{L}_n[h,\bar{h}] = \omega\cdot\partial_\varphi h + \mathrm{i}\lambda_3 T(D)h + a_1\partial_x h + \mathrm{i}a_0\mathcal{H}|D|^{\frac{1}{2}}h + \mathcal{R}_n h + \mathcal{Q}_n\bar{h}\,,$$

where $\mathcal{R}_n \in OPS^0$ and $\mathcal{Q}_n \in OPS^{-n}$. We look for

$$\Phi_n h = h + \Psi_n\bar{h}\,,\quad \Psi_n = \mathrm{Op}\Big(\psi_n(x,\xi)\Big) \in OPS^{-n-\frac{3}{2}}$$

In the conjugation we get

$$\mathcal{L}_{n+1}[h,\bar{h}] = \omega\cdot\partial_\varphi h + \mathrm{i}\lambda_3 T(D)h + a_1\partial_x h + \mathrm{i}a_0\mathcal{H}|D|^{\frac{1}{2}}h + \mathcal{R}_{n+1}h + \mathcal{Q}_{n+1}\bar{h}$$

and

$$\mathcal{R}_{n+1} \in OPS^0\,,\quad \mathcal{Q}_{n+1} = \mathrm{i}\lambda_3(T(D)\Psi_n + \Psi_n T(D)) + \mathcal{Q}_n + O(|D|^{-n-1})$$

Then, using that
$T(D)\Psi_n + \Psi_n T(D) = 2\mathrm{Op}\Big(T(\xi)\psi_n(x,\xi)\Big) + O(|D|^{-n-1})$, to solve

$$\mathrm{i}\lambda_3\Big(T(D)\Psi_n + \Psi_n T(D)\Big) + \mathcal{Q}_n = O(|D|^{-n-1})\,.$$

it is enough to choose

$$\psi_n(x,\xi) = \frac{q_n(x,\xi)}{\mathrm{i}\lambda_3 T(\xi)}$$

and then

$$\mathcal{L}_{n+1}[h,\bar h] = \omega\cdot\partial_\varphi h + \mathrm{i}T(D)h + a_1\partial_x h + \mathrm{i}a_0\mathcal{H}|D|^{\frac{1}{2}}h + \mathcal{R}_{n+1}h + \mathcal{Q}_{n+1}\bar h\,,$$

$$\mathcal{Q}_{n+1} \simeq \varepsilon|D|^{-n-1}\,.$$

New steps: Egorov method

- GOAL: ELIMINATION OF $O(\partial_x)$. Now we deal with

$$\mathcal{L}_N[h, \bar{h}] = \omega \cdot \partial_\varphi h + P_0(\varphi, x, D)h + \mathcal{Q}_N \bar{h}\,,$$

where

$$P_0(\varphi, x, D) := \mathrm{i}\lambda_3 T(D) + a_1(\varphi, x)\partial_x + \dots\,, \quad \mathcal{Q}_N = O(|D|^{-N})\,.$$

We conjugate the operator $\mathcal{L}$ by means of the map $\Phi(\varphi, t)$, which is the flow of the Pseudo PDE

$$\partial_t u = \mathrm{i}a(\varphi, x)|D|^{\frac{1}{2}} u\,.$$

Egorov method: diagonal terms

The operator $P(\varphi,t) = \Phi(\varphi,t)\circ P_0 \circ \Phi(\varphi,t)^{-1}$ obeys to

Heisenberg equation

$$\partial_t P(\varphi,t) = [\mathrm{i}a|D|^{\frac{1}{2}}, P(\varphi,t)]$$

The equation can be solved in *decreasing symbols* finding

$$P = \mathrm{Op}(p) \in OPS^{\frac{3}{2}}\,, \quad p \sim \sigma_0 + \sigma_1 + \dots\,, \quad \sigma_k \in S^{\frac{3}{2}-\frac{k}{2}}\,.$$

It turns out that

$$p = \mathrm{i}\lambda_3 T(\xi) + \mathrm{i}\Big(a_1(x) - \frac{3}{4}\lambda_3(\partial_x a)(x)\Big)\xi + O(|\xi|^{\frac{1}{2}})\,.$$

To remove the order $O(\partial_x)$ it is enough to look for $a(x)$ so that

$$a_1(x) - \frac{3}{4}\lambda_3(\partial_x a)(x) = 0\,.$$

Egorov method: off-diagonal terms

The off diagonal term $\mathcal{Q}(\varphi,t) = \Phi(\varphi,t)\circ\mathcal{Q}_N\circ\overline{\Phi}(\varphi,t)^{-1}$ obeys to

$$\partial_t\mathcal{Q}(\varphi,t) = \mathrm{i}\Big(a|D|^{\frac{1}{2}}\circ\mathcal{Q}(\varphi,t)+\mathcal{Q}(\varphi,t)\circ a|D|^{\frac{1}{2}}\Big)$$

which has a solution

$$\mathcal{Q}(\varphi,t) = \mathrm{Op}\big(e^{2\mathrm{i}a(x)|\xi|^{\frac{1}{2}}}v(t,\varphi,x,\xi)\big)\,,\quad v\in S^{-N}\,.$$

We have $\mathcal{Q}(\varphi,t)\in S^{-N}_{\frac{1}{2},\frac{1}{2}}$, then

$$\partial_\varphi^\beta\mathcal{Q}\simeq\mathrm{Op}\big(\partial_\varphi^\beta(e^{2\mathrm{i}a(\varphi,x)|\xi|^{\frac{1}{2}}})v\big)\simeq|\xi|^{\frac{\beta}{2}-N}\,,$$

thus $\partial_\varphi^\beta\mathcal{Q}(\varphi,t): H^s_x\to H^s_x$, if $N\sim\beta/2$.

Further steps:

Now we deal with

$$\mathcal{L} = \omega \cdot \partial_\varphi + \mathrm{i}\lambda_3 T(D) + O(|D|)^{\frac{1}{2}}$$

- **The term** $O(|D|^{\frac{1}{2}})$ We conjugate by means of a pseudo-differential operator $c(x, D) \in OPS^0$ of order 0, then

$$\mathcal{L} = \omega \cdot \partial_\varphi + \mathrm{i}\lambda_3 T(D) + \mathrm{i}\lambda_1 |D|^{\frac{1}{2}} + \mathcal{R}_0 \,, \qquad \mathcal{R}_0 \sim \varepsilon |D|^0 \,.$$

- **Superquadratic KAM reducibility scheme**. II order Melnikov conditions to complete the diagonalization by perturbative arguments.

Reducibility

ITERATIVE SCHEME:

$$\mathcal{L}_\nu = \omega \cdot \partial_\varphi + \mathcal{D}_\nu + \mathcal{R}_\nu \,, \quad \mathcal{D}_\nu := \mathrm{diag}_j \mu_j^\nu$$

$$\mu_j^\nu = \lambda_3 \sqrt{|j|(1+\kappa j^2)} + \lambda_1 |j|^{\frac{1}{2}} + r_j^\nu \,, \quad \sup_j |r_j^\nu| = O(\varepsilon)$$

and $\mathcal{R}_\nu$ satisfies tame estimates

$$\|\mathcal{R}_\nu h\|_s \leq M_\nu(s)\|h\|_{s_0} + M_\nu(s_0)\|h\|_s \,, \quad \forall s \geq s_0 \,,$$

$$\|(\partial_\varphi^\beta \mathcal{R}) h\|_s \leq K_\nu(s,\beta)\|h\|_{s_0} + K_\nu(s_0,\beta)\|h\|_s \,, \quad \forall s \geq s_0 \,,$$

Imposing the second order Melnikov conditions

$$|\omega\cdot\ell+\mu_j^\nu-\mu_{j'}^\nu|\geq\frac{\gamma|j^{\frac{3}{2}}-j'^{\frac{3}{2}}|}{|\ell|^\tau}\,,\quad\forall(\ell,j,j')\neq(0,j,j)\,,\quad|\ell|\leq N_\nu$$

$$|\omega\cdot\ell+\mu_j^\nu+\mu_{j'}^\nu|\geq\frac{\gamma|j^{\frac{3}{2}}+j'^{\frac{3}{2}}|}{|\ell|^\tau}\,,\quad\forall(\ell,j,j')\quad|\ell|\leq N_\nu$$

$(N_\nu=N_0^{\chi^\nu},\chi\in(1,2))$, we find $\Phi_\nu=\mathrm{Id}+\Psi_\nu$ such that

$$\mathcal{L}_{\nu+1}=\omega\cdot\partial_\varphi+\mathcal{D}_{\nu+1}+\mathcal{R}_{\nu+1}$$

and $\mathcal{R}_{\nu+1}$, $\partial_\varphi^\beta\mathcal{R}_{\nu+1}$ satisfy tame estimates with tame constants

$$M_{\nu+1}(s)\lesssim N_\nu^{-\beta}K_\nu(s,\beta)+N_\nu^C\gamma^{-1}M_\nu(s)M_\nu(s_0)\,,$$

$$K_{\nu+1}(s,\beta)\lesssim N_\nu^C K_\nu(s,\beta)$$

CONVERGENCE provided $M_0(s_0)\gamma^{-1}\ll 1$

Thanks for the attention!