Low dimensional tori

Novosibirsk

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General remarks.

We deal with the Hamiltonian system

$$\dot{\mathbf{q}} = \partial_p H(\mathbf{p}, \mathbf{q}), \quad \dot{\mathbf{p}} = -\partial_q H(\mathbf{p}, \mathbf{q})$$

where the Hamiltonian

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0(\boldsymbol{p}) + \varepsilon H_1(\boldsymbol{p}, \boldsymbol{q}). \tag{1}$$

Functions H_0 and H_1 are analytic on $D \times \mathbb{T}^n$, where D is a neighborhood of zero in $\mathbb{R}^n$ and $\mathbb{T}^n$ is n-dimensional torus. H_1 is an analytic 2π-periodic function in $\boldsymbol{q}$.

Integrable case. Invariant torus.

Let

$$\varepsilon = 0 \text{ and hence } H = H_0(\mathbf{p}).$$

Then $\dot{\mathbf{p}} \equiv 0$ and for every $\mathbf{p}_0 \in D$, the system has a solution

$$\mathbf{p} = \mathbf{p}_0, \quad \mathbf{q} = \boldsymbol{\omega} t, \quad t \in \mathbb{R},$$

where the frequency vector

$$\boldsymbol{\omega} = \nabla_p H_0(\mathbf{p}_0).$$

In particular, $\{\mathbf{p}_0\} \times \mathbb{T}^n$ is the invariant tori of our hamiltonian system

The case $\varepsilon > 0$

The system

$$\dot{\mathbf{q}} = \partial_p H_0 + \varepsilon \partial_p H_1, \quad \dot{\mathbf{p}} = -\partial_q H_0 - \varepsilon \partial_q H_1 \tag{2}$$

has n-dimensional invariant tori if there is a canonical transform

$$\mathbf{q} = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}, \boldsymbol{\eta}), \quad \mathbf{p} = \mathbf{v}(\boldsymbol{\xi}, \boldsymbol{\eta}), \boldsymbol{\xi} \in \mathbb{T}^n, \boldsymbol{\eta} \in \mathbb{R}^n$$

which takes (2) to

$$\dot{\boldsymbol{\xi}} = \omega + O(\boldsymbol{\eta}), \quad \dot{\boldsymbol{\eta}} = O(\boldsymbol{\eta}) \tag{3}$$

$$\boldsymbol{\xi} = \boldsymbol{\omega} t, \quad \boldsymbol{\eta} = 0.$$

$$\mathbf{q} = \boldsymbol{\omega} t + \mathbf{u}(\boldsymbol{\omega} t, 0), \quad \mathbf{p} = \mathbf{v}(\boldsymbol{\omega} t, 0).$$

Theorem (Kolmogorov 1954, Arnold 1963)*Let H_i are analytic in $D \times \mathbb{T}^n$. Let* $\omega = \nabla_p H_0(0)$ *satisfy the Diophantine condition*

$$\omega \cdot \boldsymbol{m} > c|\mathbf{m}|^{-\tau} \quad \text{for all} \quad \boldsymbol{m} \in \mathbb{Z}^n \setminus \{0\} \tag{4}$$

and

$$det \left\{ \frac{\partial^2 H_0}{\partial p_i p_j} \right\} \neq 0.$$

Then for all sufficiently small ε *system* (2) *has n-dimensional invariant tori.*

Low-dimensional tori.

Our task is to prove the Kolmogorov theorem for the resonant case. We consider the case on invariant tori of dimension $n-1$ and assume that

$$\nabla_p H_0(0) = (\boldsymbol{\omega}, 0), \quad \boldsymbol{\omega} = (\omega_1, ..., \omega_{n-1})$$

where

$$\boldsymbol{\omega} \cdot \boldsymbol{m} > c|\mathbf{m}|^{-\tau} \text{ for all } \boldsymbol{m} \in \mathbb{Z}^{n-1} \setminus \{0\} \tag{5}$$

The problem is *to formulate conditions, which being imposed only on H_0, provide the existence at least one $(n-1)$-dimensional invariant tori of the perturbed system for all analytic H_1 and all small ε*

Notation

Let

$$\boldsymbol{p} = (\boldsymbol{y}, z_2) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad \boldsymbol{q} = (\boldsymbol{x}, z_1) \in \mathbb{T}^{n-1} \times \mathbb{T}.$$

Thus we get

$$H = H_0(\mathbf{y}, z_2) + \varepsilon H_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

$$\nabla_{\boldsymbol{y}} H_0(0) = \boldsymbol{\omega} \in \mathbb{R}^{n-1}, \quad H'_{0,z_2}(0) = 0. \tag{6}$$

Notation

$$\dot{\boldsymbol{x}} = \nabla_{\boldsymbol{y}} H, \quad \dot{\boldsymbol{y}} = -\nabla_{\boldsymbol{x}} H, \quad \dot{\boldsymbol{z}} = \mathbf{J} \nabla_{\boldsymbol{z}} H, \tag{7}$$

where $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\boldsymbol{z} = (z_1, z_2)$.

Definition .

System (7) has a reducible $(n-1)$-dimensional invariant torus with frequency vector ω, if there exists a canonical transformation

$$\vartheta : (\xi, \eta, \zeta) \to (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \quad \xi \in \mathbb{T}^{n-1}, \eta \in \mathbb{R}^{n-1}, \zeta \in \mathbb{R}^2$$

$$\boldsymbol{x} = \xi + \boldsymbol{u}(\xi), \ \ \boldsymbol{y} = \boldsymbol{v}(\xi) + O(|\eta|, |\zeta|), \ \ \boldsymbol{z} = \boldsymbol{w}(\xi) + O(|\zeta|, |\eta|),$$

such that

$$H \circ \vartheta = \omega \cdot \eta + \frac{1}{2}\Omega\zeta \cdot \zeta + O(|\eta|^2, |\zeta|^3, |\eta||\zeta|^2).$$

Definition .

An invariant torus is weak-hyperbolic, if eigenvalues of the symmetric matrix Ω has different signs.

$$\boldsymbol{\Omega} = \mathrm{diag}(-k, 1).$$

Every hyperbolic invariant torus is reducible (Bolotin, Trechev, 2000)

Remarks.

If $\varepsilon = 0$, then $\boldsymbol{u} = \boldsymbol{v} = 0$, $\boldsymbol{w} = (\alpha, 0)$, and a normal form is a degenerate one

$$H_0(\boldsymbol{\eta}, \zeta_2) = \boldsymbol{\omega} \cdot \boldsymbol{\eta} + \frac{1}{2}\zeta_2^2 + \boldsymbol{T}_0 \cdot \boldsymbol{\eta}\zeta_2 + \frac{1}{2}\mathbf{S}_0\boldsymbol{\eta} \cdot \boldsymbol{\eta} + o(|\boldsymbol{\eta}|^2, |\boldsymbol{\eta}||\zeta_2|, |\zeta_2|^2), \tag{8}$$

where $\mathbf{S}_0 = H''_{0,\boldsymbol{yy}}(0)$, $\boldsymbol{T}_0 = H''_{0,\boldsymbol{y}z_2}(0)$. For $\varepsilon > 0$ the system has the solution

$$\boldsymbol{\xi} = \omega t, \zeta = 0, \boldsymbol{\eta} = 0.$$

The transversal dynamics is defined by

$$\dot{\zeta} = \nabla^{\perp}(\boldsymbol{\Omega}\zeta \cdot \zeta) + O(\zeta^2), \quad \boldsymbol{\Omega} = \text{diag}\,\{0, 1\}.$$

There are many results concerning a persistence of lower dimensional tori. The first result for elliptic tori was formulated by Melnikov . Elliptic invariant tori were treated by Moser , Kuksin, Eliasson, Poschel. A hyperbolic case was investigated by Graff , Zehnder, You, Li . The persistence of low dimensional tori in reversible systems was studied by Sevryuk. These results can't be applied in a degenerate case, i.e. neither hyperbolic nor elliptic case. The persistence of hyperbolic $(n-1)$-dimensional tori was proved by Treschev in the resonance case under non-degeneracy restriction on the perturbation H_1.

Remarks

For the subject we refer to the monograph by ·

- V I. Arnold, V. V. Kozlov, and A. I. Neishtadt.
- Broer, H.W., Huitema, G.B., and Sevryuk, M.B.
- S.B. Kuksin
- L. Nirenberg,
- J. Poschel.

Remarks

Almost all results are related to the Hamiltonian

$$H = \omega \cdot \mathbf{y} + \frac{1}{2}\Omega\mathbf{z} \cdot \mathbf{z} + O(\mathbf{y}^2) + \varepsilon H_1(\mathbf{y}, \mathbf{z}, \mathbf{x})$$

or

$$H = \omega \cdot \mathbf{y} + \varepsilon\frac{1}{2}\Omega\mathbf{z} \cdot \mathbf{z} + O(\mathbf{y}^2) + \varepsilon^2 H_1(\mathbf{y}, \mathbf{z}, \mathbf{x})$$

These conditions are more restrictive than the conditions of the Kolmogorov theorem.

Remarks

In the papers by Cheng and You (1996, 1999, 2005) Corsi L., Feola R., Gentile G. (2013) for the low dimensional tori were studied without restrictions on the perturbations.

Conditions.

We assume that

$$H = H_0(\mathbf{y}, z_2) + \varepsilon H_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

$$\nabla_{\mathbf{y}} H_0(0,0) = \boldsymbol{\omega},$$

$$\partial_{z_2} H_0(0,0) = 0, \quad \partial^2_{z_2} H_0(0,0) = 1, \tag{9}$$

Conditions.

Recall $H_0 = H_0(\mathbf{y}, z_2)$. Set

$$\mathbf{S}_0 = H''_{0,\boldsymbol{yy}}(0,0)$$

$$\boldsymbol{t}_0 = H''_{0,\boldsymbol{y}z_2}(0,0)$$

and

$$\mathbf{K}_0 = \mathbf{S}_0 - \boldsymbol{t}_0 \otimes \boldsymbol{t}_0. \tag{10}$$

assume that

$$\det \mathbf{K}_0 \neq 0. \tag{11}$$

Remark

$$D^2 H_0(0,0)[\boldsymbol{p}, \boldsymbol{p}] = \mathbf{K}_0 \mathbf{y} \cdot \mathbf{y} + Y^2, \quad Y = z_2 - (\mathbf{t}_0 \cdot \mathbf{y})$$

Conditions.

Denote by $\Re_\varrho$ the strip

$$\{\boldsymbol{\xi} : \operatorname{Re}\xi \in \mathbb{R}^{n-1}, |\operatorname{Im}\xi_j| \leq \varrho, \ \ 1 \leq j \leq n-1\} \subset \mathbb{C}^{n-1}.$$

We denote by B^n_ϱ the complex neighborhood

$$B^n_\varrho = \{\boldsymbol{p} \in \mathbb{C}^n : |\mathrm{Re}\boldsymbol{p}| \leq \varrho \in D, \ |\operatorname{Im} p_j| \leq \varrho, \ \ 1 \leq j \leq n\} \subset \mathbb{C}^n. \tag{12}$$

.

Conditions.

We assume that

H.1 There is $\varrho > 0$ with the properties. The function $H_0(\boldsymbol{y}, z_2)$ is analytic in the complex ball $(\mathbf{y}, z_2) \in B^n_{3\varrho}$; the perturbation $H_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is analytic on the cartesian product of the complex strip $(\mathbf{x}, z_1) \in \Re_{3\varrho}$ and the ball $(\mathbf{y}, z_2) \in B^n_{3\varrho}$. In particular, H_1 is 2π-periodic in $\boldsymbol{x}$ and z_1. Moreover, there is $c > 0$ such that

$$\sup_{(\boldsymbol{y}, z_2) \in B^n_{3\rho}} |H_0(\boldsymbol{y}, z_2)| + \sup_{(\boldsymbol{y}, z_2) \in B^n_{3\rho},\ (\boldsymbol{x}, z_1) \in \Re_{3\rho}} |H_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})| \leq c, \tag{13}$$

H.2 The frequency vector $\overline{\boldsymbol{\omega}} = (\boldsymbol{\omega}, 0)$ satisfies the diophantine condition

$$\big| (\boldsymbol{\omega}^\top \cdot \boldsymbol{s})^{-1} \big| \leq c_0 |\boldsymbol{s}|^{-n} \text{ for all } \boldsymbol{s} \in \mathbb{Z}^{n-1} \setminus \{0\}. \tag{14}$$

Theorem

Let Conditions (**H**.**1**) - (**H**.**2**) *be satisfied. Furthermore, assume that*

$$\mathbf{K}_0\boldsymbol{\eta} \cdot \boldsymbol{\eta} < 0 \qquad \textit{for all } \ \boldsymbol{\eta} \in \mathbb{R}^{n-1} \setminus \{0\}. \tag{15}$$

Then there is $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*, Hamiltonian system* (7) *has an invariant* $(n-1)$*-dimensional weakly hyperbolic torus.*

Given: H
Unknowns: The canonical transform ϑ, α, and k.

Modified Hamiltonian.

J. Moser: Convergent series expansions for quasi periodic motions, Mathematische Annalen, 169, 136-176 (1967).

$$H_{mod}(\mathbf{x}, \mathbf{y}, \mathbf{z}) := H(\mathbf{x}, \mathbf{y}, \mathbf{z}) + mz_1 + \frac{1}{2}Mz_1^2. \qquad (16)$$

where $m, M \in \mathbb{R}^1$ are new parameters.
H, ε, $\alpha \in \mathbb{T}^1$, $k \in [0, 1]$ - are given
ϑ, m, and M are unknowns.

Modified problem

For given $\alpha \in \mathbb{T}^1$ and $k \in [0, 1]$ to find m, M and a canonical transform

$$\boldsymbol{\vartheta} : (\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta) \to (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \quad \boldsymbol{\xi} \in \mathbb{T}^{n-1}, \boldsymbol{\eta} \in \mathbb{R}^{n-1}, \zeta \in \mathbb{R}^2$$

$$\boldsymbol{x} = \xi + \boldsymbol{u}(\xi), \;\; \boldsymbol{y} = \boldsymbol{v}(\xi) + O(|\eta|, |\zeta|), \;\; \boldsymbol{z} = \boldsymbol{w}(\xi) + O(|\zeta|, |\eta|),$$

which puts H in the normal form

$$H \circ \boldsymbol{\vartheta} = \boldsymbol{\omega} \cdot \boldsymbol{\eta} + \frac{1}{2}\boldsymbol{\Omega}\zeta \cdot \zeta + O(|\boldsymbol{\eta}|^2, |\zeta|^3, |\boldsymbol{\eta}||\zeta|^2).$$

$$\Omega = \text{ diag } (-k, 1), \quad (2\pi)^{1-n} \int_{\mathbb{T}^{n-1}} w_1 \, d\boldsymbol{\xi} = \alpha.$$

Modified problem. $\varepsilon = 0$

$$\mathbf{x} = \xi, \mathbf{y} = \eta, \mathbf{z} = \alpha \mathbf{e}_1 + \zeta,$$

$$m = \alpha k, M = -k.$$

Canonical transform

Let $(\boldsymbol{x}, z_1) \in \mathbb{T}^{n-1} \times \mathbb{R}^1$, $(\boldsymbol{y}, z_2) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ be the angle-action variables. Let us consider an analytical transform

$$\begin{gathered} \vartheta : (\xi, \eta, \zeta) \to (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \\ \boldsymbol{x} = \xi + \boldsymbol{u}(\xi), \\ \boldsymbol{y} = \boldsymbol{v}(\xi) + \mathbf{V}(\xi)\eta + \boldsymbol{\Lambda}(\xi)\zeta + \frac{1}{2}\mathbf{R}(\xi)\zeta\zeta, \\ \boldsymbol{z} = \boldsymbol{w}(\xi) + \mathbf{W}(\xi)\zeta. \end{gathered} \tag{17}$$

Here $\mathbf{R}$ is a vector valued quadratic form:

$$\mathbf{R}(\xi)\zeta\zeta = (\mathbf{R}_i\zeta \cdot \zeta)^\top, \quad \mathbf{R}_i(\xi) = \mathbf{R}_i(\xi)^\top.$$

Functions $\boldsymbol{u}$, $\boldsymbol{v}$, and matrices $\mathbf{V}$, $\mathbf{W}$,$\mathbf{R}_i$, $\boldsymbol{\Lambda}$ are analytic functions of $\xi \in \mathbb{T}^{n-1}$.

Canonical transform

ϑ is canonical if and only if

$$\mathbf{V} = (\mathbf{I} + \mathbf{u}')^{-\top}, \tag{18a}$$

$$\det \mathbf{W} = 1, \tag{18b}$$

$$\mathbf{\Lambda} = -\mathbf{V}\,(\mathbf{w}')^{\top}\,\mathbf{J}\,\mathbf{W}, \tag{18c}$$

$$\mathbf{R}_i = -V_{ik}\,\frac{\partial}{\partial \xi_k}\Big(\mathbf{W}^{\top}\Big)\,\mathbf{J}\,\mathbf{W}, \tag{18d}$$

$$d(\xi_k + u_k) \wedge dv_k + dw_1 \wedge dw_2 = 0. \tag{18e}$$

Moreover, there exist $\beta \in \mathbb{R}^{n-1}$ and 2π -periodic scalar function $\varphi_0(\boldsymbol{\xi})$ such that

$$\mathbf{v} = \beta + \mathbf{V}\,\big(\,\nabla\varphi_0 - w_2\nabla w_1\,\big). \tag{18f}$$

Canonical transform

Canonical transforms (17)-(18) forms the subgroup $\mathfrak{G}$ of the group of canonical diffeomorphisms of $\mathbb{T}^{n-1}\times\mathbb{R}^2\times\mathbb{R}^{n-1}$. This group can be regarded as a nonlinear manifold in the linear space of all mappings $\boldsymbol{\vartheta}$ in the form (17).
The vector

$$\boldsymbol{\varphi} = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21})$$

can be regarded as *t*he local coordinates on this manifold.

Banach spaces

Let $\varrho > 0$ bi given by Condition (**H.1**). For every $\sigma \in [0, 1]$ and any integer $d \geq 0$ denote by $\mathcal{A}_{\sigma,d}$ the Banach space of all functions

$$\mathbf{u} : \Re_{\sigma\varrho} \to \mathbb{C}, \quad \Re_{\sigma\varrho} = \left\{ \boldsymbol{\xi} : \ \mathrm{Re}\, \boldsymbol{\xi} \in \mathbb{T}^{n-1}, \ |\mathrm{Im}\, \boldsymbol{\xi}| \leq \sigma\varrho \right\},$$

with the finite norm

$$\|u\|_{\sigma,d} = \max_{0 \leq |k| \leq d} \sup_{\boldsymbol{\xi} \in \Re_{\sigma\varrho}} |\partial^k u(\boldsymbol{\xi})|. \tag{19}$$

Banach spaces

For every $s \in \mathbb{R}$, we denote by H_s the Hilbert space which consists of all distributions u on the tori $\mathbb{T}^{n-1}$ such that

$$|u|_s = \Big(\sum_{m \in \mathbb{Z}^{n-1}} (1 + |m|^2)^s |\hat{u}(m)|^2 \Big)^{1/2}, \tag{20}$$

where $\hat{u}$ is the Fourier transform of u.

Modified problem. Result

$$
\begin{aligned}
\|\partial_\alpha^r \mathbf{u}\|_{\sigma/2,d} + \|\partial_\alpha^r \mathbf{v}\|_{\sigma/2,d} &\leq c(r)|\varepsilon|, \\
\|\partial_\alpha^r (\mathbf{w} - \alpha \mathbf{e}_1)\|_{\sigma/2,d} &\leq c(r)|\varepsilon|, \\
|\partial_\alpha^r (m + \alpha M)| + |\partial_\alpha^r (M + k)| &\leq c(r)|\varepsilon|,
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
\|\partial_k \mathbf{u}\|_{\sigma/2,d} + \|\partial_k \mathbf{u}\|_{\sigma/2,d} &\leq c|\varepsilon|, \\
\|\partial_k (\mathbf{w} - \alpha \mathbf{e}_1)\|_{\sigma/2,d} &\leq c|\varepsilon|, \\
|\partial_k (m + \alpha M)| + |\partial_k (M + k)| &\leq c|\varepsilon|,
\end{aligned}
\tag{22}
$$

$$m(\alpha, k) = 0 \quad M(\alpha, k) = 0$$

Variational principle

Consider the action functional

$$\Psi(\alpha, k) = \int\limits_{\mathbb{T}^{n-1}} \big((\boldsymbol{\omega}+\partial \boldsymbol{u})\cdot \boldsymbol{v} - \frac{1}{2}\mathbf{J}\partial \boldsymbol{w}\cdot \boldsymbol{w} - H(\boldsymbol{\xi}+\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\big)\, d\boldsymbol{\xi}, \quad (23)$$

where

$$\boldsymbol{w} = \boldsymbol{u}(\alpha, k; \boldsymbol{\xi}), \quad \boldsymbol{v} = \boldsymbol{v}(\alpha, k; \boldsymbol{\xi}), \quad \boldsymbol{w} = \boldsymbol{w}(\alpha, k; \boldsymbol{\xi}).$$

$\Psi(\alpha, k)$ is periodic in α, and smooth in α, k. In other words Ψ is a smooth function on the cylinder $\mathbb{T}^1 \times [0, 1]$.

Jacobi vector fields

Let $\tau = \alpha, k$.

$$\boldsymbol{\chi}^{(\tau)} = \mathbf{V}^{\top}\,\partial_\tau \mathbf{u} \tag{24a}$$

$$\boldsymbol{\lambda}^{(\tau)} = \mathbf{W}^{-1}\partial_\tau \mathbf{w} - \chi_i^{(\tau)}\,\mathbf{W}^{-1}\frac{\partial}{\partial \xi_i}\mathbf{w} \tag{24b}$$

$$\boldsymbol{\mu}^{(\tau)} = \mathbf{V}^{-1}\Big(\partial_\tau \mathbf{v} + \chi_i^{(\tau)}\,\frac{\partial}{\partial \xi_i}\mathbf{v} - \boldsymbol{\Lambda}\boldsymbol{\lambda}^{(\tau)}\Big). \tag{24c}$$

The vector fields $\boldsymbol{\chi}^{(\tau)}$, $\boldsymbol{\lambda}^{(\tau)}$, and $\boldsymbol{\mu}^{(\tau)}$ can be regarded as the Jacobi vector fields for the invariant tori problem. We also set

$$p^{(\tau)} = \partial_\tau m + \alpha \partial_\tau M. \tag{25}$$

$$
\begin{aligned}
(\boldsymbol{\mu}^{(\alpha)}, \boldsymbol{\lambda}^{(\alpha)}, \boldsymbol{\chi}^{(\alpha)}, p^{(\alpha)}) &= (\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\chi}^{(1)}, p^{(1)}) \\
&\quad + \partial_\alpha M(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}, \boldsymbol{\chi}^{(2)}, p^{(2)}) \qquad (26) \\
(\boldsymbol{\mu}^{(k)}, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\chi}^{(k)}, p^{(k)}) &= \partial_k M(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}, \boldsymbol{\chi}^{(2)}, p^{(2)}),
\end{aligned}
$$

Notation

$$\overline{w} = (2\pi)^{1-n} \int_{\mathbb{T}^{n-1}} w\, d\xi, \quad w^* = w - \overline{w}$$

$$\boldsymbol{\partial} = \sum \omega_i \frac{\partial}{\partial \xi_i}$$

Equations

$$\partial\, \boldsymbol{\mu}^{(1)} = -p^{(1)}\nabla w_1, \tag{27a}$$

$$\mathbf{J}\partial\, \boldsymbol{\lambda}^{(1)} + \boldsymbol{\Omega}\, \boldsymbol{\lambda}^{(1)} + \mathbf{T}\, \boldsymbol{\mu}^{(1)} + p^{(1)}\mathbf{W}^\top \boldsymbol{e}_1 = 0, \tag{27b}$$

$$-\, \partial\, \boldsymbol{\chi}^{(1)} + \mathbf{S}\boldsymbol{\mu}^{(1)} + \mathbf{T}^\top \boldsymbol{\lambda}^{(1)} = 0 \tag{27c}$$

$$\overline{\{W\boldsymbol{\lambda}^{(1)} \cdot \mathbf{e}_1\}} + \overline{\{\boldsymbol{\chi}^{(1)} \cdot \nabla w_1\}} = 1, \tag{27d}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}^{(1)}\}} = 0, \tag{27e}$$

$$\partial\,\boldsymbol{\mu}^{(2)} = -p^{(2)}\nabla w_1 - \frac{1}{2}\,\nabla(w_1^*)^2, \tag{28a}$$

$$\mathbf{J}\partial\,\boldsymbol{\lambda}^{(2)} + \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(2)} + \mathbf{T}\,\boldsymbol{\mu}^{(2)} + p^{(2)}\mathbf{W}^\top \boldsymbol{e}_1 = -\,w_1^*\mathbf{W}^\top \boldsymbol{e}_1, \tag{28b}$$

$$-\,\partial\,\boldsymbol{\chi}^{(2)} + \mathbf{S}\boldsymbol{\mu}^{(2)} + \mathbf{T}^\top\boldsymbol{\lambda}^{(2)} = 0 \tag{28c}$$

$$\overline{\{W\boldsymbol{\lambda}^{(2)}\cdot\mathbf{e}_1\}} + \overline{\{\boldsymbol{\chi}^{(2)}\cdot\nabla w_1\}} = 0, \tag{28d}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}^{(2)}\}} = 0 \tag{28e}$$

Here the matrices $\mathbf{S}$ and $\mathbf{T}$ are defined by

$$\begin{aligned} \mathbf{S} &= \mathbf{V}^\top \frac{\partial^2 H}{\partial \mathbf{y}^2}(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V}, \\ \mathbf{T} = \mathbf{W}^\top \frac{\partial^2 H}{\partial \mathbf{z} \partial \mathbf{y}}(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V} &+ \mathbf{\Lambda}^\top \frac{\partial^2 H}{\partial \mathbf{y}^2}(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V}. \end{aligned} \tag{29}$$

$$\|\mathbf{S} - \mathbf{S}_0\|_{\sigma/2, d-1} + \|\mathbf{T} - \mathbf{T}_0\|_{\sigma/2, d-1} \le c|\varepsilon|, \tag{30}$$

where the constant matrices $\mathbf{S}_0$, $\mathbf{T}_0$ are given by

$$\mathbf{S}_0 = \frac{\partial^2 H_0}{\partial \mathbf{y}^2}(0,0), \quad \mathbf{T}_0 = \frac{\partial^2 H_0}{\partial \mathbf{z} \partial \mathbf{y}}(0,0) \tag{31}$$

Estimates of Jacobi vector fields

Theorem

There exists $\varepsilon_0 > 0$ *with the following property. For every* $|\varepsilon| \leq \varepsilon_0$, *problem* (28) *has a unique analytic solution* $(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}, \boldsymbol{\chi}^{(2)}) \in \mathcal{A}_{\sigma/4,0}$, $p^{(2)} \in \mathbb{C}$. *This solution admits the estimate*

$$\|\boldsymbol{\mu}^{(2)}\|_{\sigma/4,0} + \|\boldsymbol{\lambda}^{(2)}\|_{\sigma/4,0} + \|\boldsymbol{\chi}^{(2)}\|_{\sigma/4,0} + |p^{(2)}| \leq c\|w_1^*\|_{\sigma/2,0} \leq c|\varepsilon|, \tag{32}$$

$$|\boldsymbol{\mu}^{(2)}|_0 \leq c\varepsilon_0 \, |w_1^*|_{-1}, \quad |\lambda_2^{(2)}|_0 + |k| \, |\lambda_1^{(2)}|_0 \geq c^{-1} \, |w_1^*|_{-1}. \tag{33}$$

Here the strictly positive constant c is independent of α, *k, and* ε.

Estimates of Jacobi vector fields

$$\mathbf{T}^{\top} = [\mathbf{t}_1, \mathbf{t}_2]$$

$$\|\mathbf{t}_1\|_{\sigma/2,d-1} + \|\mathbf{t}_2 - \mathbf{t}_0\|_{\sigma/2,d-1} \le c|\varepsilon|. \tag{34}$$

$$\overline{\mathbf{t}_1} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \mathbf{t}_1 \, d\xi.$$

Introduce the constant vector

$$\mu_0 = \mathbf{K}_0^{-1} \, \overline{\mathbf{t}_1}. \tag{35}$$

Estimates of Jacobi vector fields

Theorem

There exists $\varepsilon_0 > 0$ *with the following property. For every* $|\varepsilon| \leq \varepsilon_0$*, problem* (27) *has a unique analytic solution* $(\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\chi}^{(1)}) \in \mathcal{A}_{\sigma/4,0}$*,* $p^{(1)} \in \mathbb{C}$*. This solution admits the estimate*

$$\|\boldsymbol{\mu}^{(1)}\|_{\sigma/4,0} + \|\boldsymbol{\lambda}^{(1)}\|_{\sigma/4,0} + \|\boldsymbol{\chi}^{(1)}\|_{\sigma/4,0} + |p^{(1)}| \leq c. \tag{36}$$

If $k = 0$*, then this solution has the decomposition*

$$\boldsymbol{\mu}^{(1)} = -\boldsymbol{\mu}_0 + \boldsymbol{\mu}_\varepsilon, \quad \boldsymbol{\lambda}^{(1)} = \mathcal{C}\mathbf{e}_1 - (\boldsymbol{\mu}_0 \cdot \mathbf{t}_0)\mathbf{e}_2 + \boldsymbol{\lambda}_\varepsilon. \tag{37}$$

Here

$$|\mathcal{C} - 1| \leq c|\varepsilon|, \quad \|\boldsymbol{\mu}_\varepsilon\|_{\sigma/4,0} + \|\boldsymbol{\lambda}_\varepsilon\|_{\sigma/4,0} \leq c|\varepsilon||\boldsymbol{\mu}_0|. \tag{38}$$

Quadratic form

$$L_{ij} = \int\limits_{\mathbb{T}^{n-1}} \left(\mathbf{S}\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j - (\mathbf{J}\partial\boldsymbol{\lambda}_i + \boldsymbol{\Omega}\boldsymbol{\lambda}_i) \cdot \boldsymbol{\lambda}_j \right) d\xi. \tag{39}$$

Quadratic form. Estimates

Theorem

There exists $\varepsilon_0 > 0$ with the following property. For every $|\varepsilon| \le \varepsilon_0$, the elements of the quadratic form L satisfy the inequalities

$$\begin{gathered} |L_{12}| \le c|\varepsilon|, \quad |L_{11}| \le c, \\ c^{-1}|w_1^*|^2_{-1} \le |L_{22}| \le c\|w_1^*\|^2_{\sigma/2,0} \le c|\varepsilon|^2. \end{gathered} \tag{40}$$

If $k = 0$, then

$$|L_{11} - (2\pi)^{n-1}\mathbf{K}_0\boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0| \le c|\varepsilon||\boldsymbol{\mu}_0|^2, \quad |L_{12}| \le c|\varepsilon|\,|\boldsymbol{\mu}_0|. \tag{41}$$

Here the strictly positive constant c is independent of α, k, and ε, the vector $\boldsymbol{\mu}_0$ is given by (35).

Main formulae

Theorem

$$\partial_\alpha \Psi(\alpha, k) = (2\pi)^{n-1}(m + \alpha M) + \varsigma_1 M, \tag{42}$$

$$\partial_k \Psi(\alpha, k) = M\,\partial_k M\, L_{22}, \tag{43}$$

$$\partial_\alpha^2 \Psi(\alpha, k) = (2\pi)^{n-1} M + \varsigma_2 M + L_{11} + 2L_{12}\partial_\alpha M + L_{22}(\partial_\alpha M)^2. \tag{44}$$

Here

$$|\varsigma_i| + |\partial_\alpha \varsigma_i| \leq c|\varepsilon|^2. \tag{45}$$

Main formulae

Let $k = 0$. Then the third derivative of Ψ admits the estimate

$$|\partial_\alpha^3 \Psi(\alpha, 0) - (2\pi)^{n-1}\partial_\alpha M| \le c|\varepsilon|(|M| + |\boldsymbol{\mu}_0| + |\partial_\alpha M|). \qquad (46)$$

Critical points

$$\partial_k \Psi(\alpha, k) \geq 0 \ \text{ for } \ k \geq 1/2 \ \text{ and } \ |\varepsilon| \leq \varepsilon_0.$$

Critical points

Let

$$\Psi(\alpha_0, k_0) = \min_{\mathbb{T}^1 \times [0,1]} \Psi(\alpha, k)$$

we have $0 \le k_0 < 1$. If $k_0 > 0$, then $M(\alpha_0, k_0) = m(\alpha_0, k_0) = 0$. If $k_0 = 0$ then

$$M(\alpha_0, 0) \le 0,$$

$$0 \le M(1 + O(\varepsilon)) + L_{11} + 2L_{12}M_\alpha + L_{22}M_\alpha^2.$$

Main observation:

$$\mathbf{K}_0 \le 0 \Rightarrow L_{11} \le 0.$$

This yields

$$|M| \le |\varepsilon||M_\alpha|^2$$

Method.

$$\partial_\alpha^2 \Psi(\alpha_0, 0) \le c|\varepsilon||\partial_\alpha M|^2$$

$$|\partial_\alpha^3 \Psi(\alpha_0, 0) - 2(2\pi)^{n-1}\partial_\alpha M(\alpha_0, 0)| \le c|\varepsilon||\partial_\alpha M(\alpha_0, 0)|,$$