Ensemble Controllability by Lie Algebraic Methods

Andrey Sarychev

Department of Mathematics and Informatics U.Dini,
University of Florence, Italy

[1] Based on joint work with A.Agrachev and Yu.Baryshnikov

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Ensembles of Control Systems

Consider an ensemble (a parameterized family) of control systems

$$\frac{dx_\theta}{d\theta} = f_\theta(x_\theta, u). \tag{1}$$

- The state x_θ of each system belongs to a finite-dim. manifold M.
- A family $\theta \mapsto f_\theta$ of vector fields on M is parameterized by $\theta \in \Theta$ - a compact subset of $\mathbb{R}^d$.

For Θ being a singleton one ends up with a single control system, for which controllability property or its lack is an important feature.

The same property appears to be important for ensembles of control systems and is called ensemble controllability

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Ensemble Controllability: brief review

The interest in ensemble controllability appeared in 2000's in relation to the study of control of quantum ensembles, by using single controlling field. One wishes to steer approximately a system with 'dispersion in parameters' to a desired target.

Examples:

- NMR experiments, in which the spins of an ensemble may have dispersion in frequencies (Larmor dispersion).
- dispersion in the strength of the applied rf-field.
- dispersion in orientations of the spins etc .

These studies have been pioneered by *N. Khaneja and J.-S. Li* (2005 and later on).

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Ensemble Controllability: brief review ctd.

In the above cited examples the *dynamics is evolving on a Lie group*.
Example

$$\dot{M}(t,\varepsilon,\omega) = (\omega\Omega_z + \varepsilon u\Omega_y + \varepsilon v\Omega_x)\, M(t,\varepsilon,\omega),$$

$$M \in SO(3),\ \Omega_{x,y,z} \in so(3),\ \varepsilon \in [1-\delta, 1+\delta],\ \omega \in [-b, b],$$

$$\Omega_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \Omega_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \Omega_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore some Lie algebraic tools, such as Campbell-Hausdorff formula and Lie series appear naturally.

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Main idea of N. Khaneja and J.-S. Li: "generating higher order Lie brackets by use of the control vector fields which carry higher order powers of the dispersion parameters to investigating ensemble controllability".

Beauchard, Coron & Rouchon considered (2010) Bloch equations with dispersed Larmor frequency. They invoked finer *analytic* methods for obtaining finer results on ensemble controllability.

They also advocated a possibility to apply the methods to the study of infinite-dimensional systems, which involve operators with continuous spectra.

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It is worth mentioning large amount of publications (starting at least from 1980's) regarding design of "robust control for the *systems with uncertainties*".

The methods involved in those publications are mainly "direct", based on estimates of the "funnels of trajectories", Lyapunov functions etc.

Ensemble - Control System in Infinite Dimensions

We may rewrite the ensemble of control systems as a dynamic equation for the function $x_\theta(t)$ (with the initial condition added):

$$\partial_t x_\theta(t) = f_\theta(x_\theta(t), u(t)), \quad x_\theta(0) = \alpha(\theta). \qquad (1')$$

The equation describes the controlled dynamics in the "state space" of functions $y(\theta)$, which is infinite- dimensional, whenever Θ is an infinite set.

The equation is L_p-approximately controllable in time-T, if for any target function $\omega(\theta)$ and $\forall \varepsilon > 0$ there exists a control $u(\cdot)$, such that for the corresponding solution of (1') there holds

$$\|x_\theta(T) - \omega(\theta)\|_{L_p} < \varepsilon.$$

Below $p = 2$

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Particular cases

- In this presentation we mainly consider the case where the parameter (the 'dispersion') only enters the r.-h. side of the control system $\dot{x}_\theta = f_\theta(x_\theta, u)$, while the initial point α and the target ω are fixed. We seek for a control which "compensates for dispersion of the parameter".
- The opposite case, where the parameter enters just the initial and target data, results in a interesting control system in the space of parameterized surfaces or curves.
- Finally one can consider a controllability problem on the group Diff$M$ of diffeomorphisms.

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Lie Algebraic Methods for Control in Infinite Dimension

- Recently some successful attempts have been made to extend the differential geometric/Lie algebraic approach to control theory onto the area of NON LINEAR infinite-dimensional control systems and controlled non linear PDE.
- Nagano theorem, valid in finite dimensions, shows that two control systems are equivalent if they satisfy the same *Lie relations*.
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Following this way of reasoning Lie algebraic controllability criteria have been established recently for some nonlinear PDE such as

Navier-Stokes and Euler equations of fluid motion, Burgers equation, linear and cubic Schrödinger equation

cf. A.Agrachev, AS, S.Rodrigues, A.Shirikyan, V. & H. Nersesyan, U.Boscain, M.Sigalotti, T.Chambrion and others.

The core of the approach used is method of Lie extensions - enriching the control system by nice or *compatible Lie brackets* of the control vector fields and avoiding bad Lie brackets or *obstructions*.

We wish to develop Lie algebraic methods for studying ensemble controllability.

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Elementary Case: Controlling Finite Ensembles

If the parameter set Θ being finite then often the fact of exact controllability can be established in pretty the same way as its counterpart for a single system.

Definition. Finite ensemble

$$\frac{dx_\theta}{dt} = f_\theta(x_\theta, \bar{u}(t)), \ \theta \in \{1, \dots, N\}$$

is exactly controllable if for any two n-tuples $(\alpha_\theta), (\omega_\theta)\}$ there exists $T > 0$ and a single θ-independent control $\bar{u}(t)$, which steers in time T θ-th system of the ensemble from α_θ to ω_θ. $\square$

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Controllability property for control-linear ensembles is generic

For a **single control-linear system**

$$\frac{dx}{dt} = f(x)u(t) + g(x)v(t), \; r \geq 2$$

controllability property follows from bracket generating condition for the couple of vector fields (f, g) (Rashevsky-Chow theorem).

C.Lobry has established that the bracket generating property is generic - it holds for each couple (f, g) from an opens and dense in C^N-metric subset of $\mathrm{Vect}(M) \times \mathrm{Vect}(M)$.

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This fact can be extended in a rather straightforward way onto **finite control-linear ensembles.**

Proposition. Consider a finite ensemble of control-linear systems on a connected C^∞ manifold M

$$\dot{x}_\theta = f_\theta(x_\theta)u(t) + g_\theta(x_\theta)v(t), \ x_\theta \in M, \ \theta \in \{1, \ldots, N\},$$

driven by **common** bi-dimensional control $(u(t), v(t))$. Then the controllability property holds for a generic $2N$-tuple of vector fields $f_1, \ldots, f_N, g_1, \ldots, g_N$. $\square$

An interesting fact is the possibility to control a finite ensemble of pairwise distinct points x_θ by a single control **for a single generic system**

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Controlling finite ensemble of rigid bodies by a single control

Consider control of attitude motion - angular velocity & orientation - of an *asymmetric* rigid body by a *single controlled torque* applied along a generic direction.

It is described by (forced) Euler system

$$\dot{Q} = Q\widehat{JK}, \ \dot{K} = K \times JK + Lu$$

and is known to be controllable for generic J, L. The proof can be accomplished by verification of some Lie bracket generating condition.

We are able to prove similar result for a finite ensemble of controlled rigid bodies, described by Euler equations

$$\dot{K}^\theta = K^\theta \times J^\theta K^\theta + Lu, \ \theta = 1, \ldots, N, \ u \in U, \ \text{int conv}(U) \ni 0.$$

with the *scalar* control torque $u(t)$ applied along one and the same direction L.

Proposition.

For each nonzero L a finite ensemble of rigid bodies, characterized by inertia tensors $J^1, \ldots, J^N$ is (exactly) controllable by means of *common control signal* $u(\cdot)$, applied via the torque along L, for a generic N-tuple of inertia tensors $((\mathcal{I}_1, L_1), \ldots, (\mathcal{I}_N, L_N))$, *lying outside proper algebraic variety*. $\square$

Note on the Complexity of the Ensemble Control

- As we have shown, controllability of an ensemble of N systems can often be established by Lie algebraic methods, but the number and the order of the Lie brackets, involved into the corresponding Lie rank condition, increases with the growth of N and tends to infinity as $N \to \infty$.
- In fact the dimension of the cartesian product of N state spaces tends to infinity with N.
- According to 'geometric control wisdom' this means that the 'complexity' of the control strategy which realizes exact controllability of an ensemble of N systems also grows with N.

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Continual ensembles: discretization of the parameter set

We saw that in many cases one can establish controllability of any finite ensemble.

It seems natural, when dealing with an ensemble, parameterized by a continuum compact Θ, to 'discretize' Θ by taking a finite ε-net $\mathcal{N}_\varepsilon = \{\theta_1, \ldots, \theta_N\} \subset \Theta$.

Arranging a control $u_\varepsilon(t)$ which drives the finite ensemble of systems $\partial_t x(t, \theta_j) = f_{\theta_j}(x(t, \theta_j), u(t)), \ \theta_j \in \mathcal{N}_\varepsilon$ exactly to the target, one may hope, that for sufficiently small $\varepsilon > 0$ the whole ensemble, parameterized by all $\theta \in \Theta$ will be driven to the target approximately.

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... and its drawbacks

But MIND THE PREVIOUS SLIDE: the complexity of the control $u_\varepsilon(t)$ grows as $\varepsilon \to 0$ and $\mathcal{N}_\varepsilon \to \infty$. For θ, which ε-close to θ_j the control u_ε may drive the corresponding system far from the target.

The better we approximate Θ, the more complex are the controls, which arrange controllability, and larger are the deviations between the trajectories with ε-close initial points.

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Ensemble as infinite-dimensional systems

Leaving the idea of discretization out we come back to viewing ensemble as a system in infinite-dimensional space of functions, defined on the parameter set Θ.

We will seek for an infinite-dimensional variant of the *method of Lie extensions* and of *Lie rank controllability criteria*.

The classical geometric control Lie algebraic methods deal with the vector fields, which are sections of the tangent bundle TM.

We will consider instead fiber bundles over the base M with infinite-dimensional fibers $L_2(\Theta, T_xM)$ over each $x \in M$.
Instead of vector fields, we consider sections of the new fiber bundle.

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Geometric control for ensembles

We introduce kind of Lie structure for the sections $X : M \to L_2(\Theta, TM)$ of the 'fiber bundle' by taking Lie brackets of the vector fields on M for each $\theta \in \Theta$. We can iterate the Lie brackets and seek for an analogue of *Lie rank condition*.

Therefore, when approaching controllability of an ensemble we employ the iterated Lie brackets (Taylor expansions) in $x \in M$, and functional expansions (e.g. Fourier series) in θ.

Note that **if Θ is finite then the fiber is just a Carthesian product of a finite number of copies of T_xM** and we come back to the above described approach to finite ensembles.

We invoke large but finite number of iterated Lie brackets, and aim at **approximate controllability**.

In infinite dimension the notions of rank, dimension and linear independence need additional care.

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Technical Assumptions

Assumption 1 (Uniform analyticity in x) Let $M_{\mathbb{C}}$

The function $f^{\theta}(x), \ x \in M$ can be extended for each $\theta \in \Theta$ to an analytic function $f^{\theta}(z), \ z \in B_{\rho}(x)$, where $B_{\rho}(x)$ is ρ-neighborhood of x in complexification of M. □

Assumption 2 (Uniform L_2 boundedness in θ)

For each $zV_{\rho}(M)$ the map $\theta \to f^{\theta}(x)$ is measurable and each compact $A \subset B_{\rho}(\mathbb{R})$ there is a function $m_A(\theta)$ square integrable on Θ, such that

$$\sup_{z \in A} \|f^{\theta}(z)\| \leq m_A(\theta).$$

Frames in Hilbert space

We still do it in practical computations Fourier series in θ are handy, in formulations of criteria the notion of *frame in Hilbert space* allows to avoid choosing a specific basis and spares indices.

Definition. Elements $a_s, \ s = 1, \ldots$ of a Hilbert space $\mathcal{H}$ form a *frame* in $\mathcal{H}$ if:

$$\exists M, N > 0 : \ \forall h \in \mathcal{H} : \ M\|h\|^2 \leq \sum_{s=1}^{\infty} |\langle h, a_s \rangle|^2 \leq N\|h\|^2;$$

M, N are called a *lower and an upper frame bounds*.

Proposition

Let $(a_s)_{s=1}^{\infty}$ be a frame with the frame operator $F : h \to \sum_{s=1}^{\infty} \langle h, a_s \rangle a_s$. Then

$$\forall h \in \mathcal{H} : \ h = \sum_{s=1}^{\infty} \langle h, F^{-1} a_s \rangle a_s.$$

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Frames in Hilbert spaces ctd.

We cite few results, which demonstrate, that frames are *overcomplete* versions of *Riesz bases* in $\mathcal{H}$.

A sequence $\{a_s\}_{s=1}^{\infty}$ is called a Riesz basis, if:

- it is a basis, i.e. each $h \in \mathcal{H}$ is uniquely representable as a sum of an unconditionally converging series $h = \sum_{s=1}^{\infty} \alpha_s a_s$;
- $0 < \inf_s \|a_s\| \leq \sup_s \|a_s\| < \infty$.

Lemma

A frame $(a_s)_{s=1}^{\infty}$ is a Riesz basis in $\mathcal{H}$ if any one of the following conditions hold:

- *the frame is exact, i.e. ceases to be a frame when any of its elements is removed;*
- *the frame is minimal, i.e. $\forall j \in \mathbb{N} \quad a_j \notin \text{span}\{a_s\}_{s \neq j}$;*
- *the frame is ω-independent sequence, i.e. whenever $\sum_s \alpha_s a_s$ converges and equal 0, then $\alpha_s = 0, \ \forall s$.*

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Frames in Hilbert spaces ctd.

We use in the presentation a corollary of this proposition.

Corollary (approximative properties of frames)

For a frame $(a_s)_{s=1}^{\infty}$ *there exists a constant* C *such that for each* $h \in \mathcal{H}$ *and each* $\delta > 0$ *there exists a finite linear combination* $\sum_{s \in S} \alpha_s a_s$ *such that*

$$\left\| h - \sum_{s \in S} \alpha_s a_s \right\|_{\mathcal{H}} < \delta, \quad \left(\sum_{s \in S} |\alpha_s|^2 \right)^{1/2} < C\|h\|. \quad \square$$

Toy Model of Control-Linear Ensemble

We will look for an approach to ensemble controllability, using following model in $\mathbb{R}^3$ with 2 inputs:

$$\dot{x} = u, \ \dot{y} = v, \partial_t z_\theta(t) = f^\theta(x)v, \ \theta \in \Theta,$$
$$x(0) = y(0) = z(\theta, 0) = 0,$$

This is an ensemble of control-linear systems with the right-hand sides, spanned by the vector fields

$$X = \frac{\partial}{\partial x}, \ Y^\theta = \frac{\partial}{\partial y} + f^\theta(x)\frac{\partial}{\partial z_\theta}.$$

We set the following *time T approximate ensemble controllability problem*: given a target function $\hat{z}(p) \in L_\infty(\Theta)$ and $\varepsilon > 0$ find θ-independent controls $u(\cdot), v(\cdot) \in L_\infty[0, T]$, such that:

$$x(T) = y(T) = 0, \ \int_\Theta \|z_\theta(T) - \hat{z}(\theta)\|^2 d\theta \leq \varepsilon.$$

for the trajectory, driven by $u(\cdot), v(\cdot)$.

Note that we ask for exact controllability in x, y.

Lie algebraic frame condition and controllability criterion

Take Taylor expansion of $f^\theta(x)$ in x (recall that f is analytic) at 0:

$$f^\theta(x) = \sum_{m=1}^{\infty} f_m(\theta)x^m, \; f_m(\theta) = \frac{1}{m!} \left. \frac{\partial^m f^\theta}{\partial x^m} \right|_{x=0}.$$

Lie algebraic frame condition

Functions $f_m(\theta) = \frac{1}{m!} \left. \frac{\partial^m f^\theta}{\partial x^m} \right|_{x=0}$, $m = 1, \ldots$ *form a frame in* $L_2(\Theta)$.

We coin it **Lie algebraic** condition as far as $f_m(\theta)$ are z_θ-components of the iterated Lie brackets $\frac{1}{m!}\left((\mathrm{ad}X)^m Y_\theta\right)$, evaluated at $x = 0$.

Controllability Criterion

Let uniform analyticity and boundedness assumptions hold.
Toy ensemble is time-T approximately controllable for each $T > 0$
if the Lie algebraic frame condition is satisfied.
and only if $span\{f_m(\theta), \; m = 1, \ldots\} = L_2(\Theta)$.

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Example.

Consider controlled ensemble

$$\dot{x} = u, \dot{y} = v,$$

$$\dot{z} = f^{\theta}(x)v, \ f^{\theta}(x) = \frac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2}, \ \theta \in [0, \pi], \ |x| < 1.$$

It is known, that

$$\frac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2} = \sum_{m=0}^{\infty} x^m \cos m\theta.$$

Obviously $\{\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}\cos m\theta, m = 1, \ldots\}$ form an orthonormal basis in $L_2[0, \pi]$ and hence Lie algebraic frame condition is satisfied.

Control design

Fix $T = 1$.

For this simple model the 'output' $z_\theta(T)$ can be computed explicitly:

$$z_\theta(1) = \int_0^1 f^\theta(U(t))dV(t) = \int_0^1 f^\theta(U(t))v(t)dt, \qquad (*)$$

where $U(t) = \int_0^t u(s)ds,\ V(t) = \int_0^t v(s)ds$.

We impose $U(1) = V(1) = 0$ and wish to construct functions $U(t), v(t)$ such that $z^\theta(1)$ in (*) would approximate in $L_2(\Theta)$ the target function $\hat{z}(\theta)$.

We proceed by a variant of moment's method.
Choosing function $U(t)$ of small magnitude we get:

$$f^\theta(U(t)) = \sum_{m=1}^{\infty} f_m(\theta)(U(t))^m,$$

$$z^\theta(1) = \sum^{\infty} f_m(\theta) \int^1 (U(t))^m v(t)dt.$$

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Control design ctd.

We seek $v(t)$ as a linear combination: $v(t) = \sum_{r=1}^{R} y_r v_r(t)$, ($R$ to be specified later), the magnitude of v will be large.
Now

$$z^\theta(1) = \sum_{m=1}^{\infty} f_m(\theta) \sum_{r=1}^{R} \gamma_{mr} y_r, \quad \gamma_{mr} = \int_0^1 (U(t))^m v_r(t) dt, \ \Gamma = (\gamma_{mr}).$$

Our goal is to choose $R, U(t), v_r(t)$ so that infinite linear system

$$\sum_{m=1}^{\infty} \left(\sum_{r=1}^{R} \gamma_{mr} y_r \right) f_m(\theta) = \sum_{m=1}^{\infty} \alpha_m f_m(\theta) \Leftrightarrow \Gamma y = A$$

would be approximately solvable w.r.t. y.

Controllability of a Control-Linear Ensemble. Rashevsky-Chow-type theorem

Consider an ensemble of control-linear systems

$$\partial_t x_\theta(t) = f_1^\theta(x_\theta)u_1(t) + \cdots + f_r^\theta(x_\theta)u_r(t). \qquad (**)$$

For the sake of brevity we restrict our attention to the case where the initial point $\tilde{x}$ and the target $\hat{x}$ are fixed (θ-independent).

Definition.
Ensemble (**) is time-T approximately steerable from $\tilde{x}$ to $\hat{x}$, if for each $\varepsilon > 0$ there exists a θ-independent control $u(\cdot)$, which steers in time T ensemble (**) from $\tilde{x}$ to $x(T, \theta)$, so that:

$$\int_\Theta \left(\mathrm{dist}(x_\theta(T), \hat{x})\right)^2 d\theta < \varepsilon^2.$$

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Lie algebraic frame condition and controllability criterion

Sufficient criterion for ensemble controllability will be again formulated formulated in terms of
Lie algebraic frame condition,

$\forall z \in V_\rho(\mathbb{R})$ the evaluations at z of the iterated Lie brackets

$$X^\theta_\alpha(z) = [f^\theta_{\alpha_1}(z), [f^\theta_{\alpha_2}(z), [\ldots, f^\theta_{\alpha_N}(z)]\ldots]](z), \ \theta \in \Theta,$$

form a frame in the Hilbert space $L_2(\Theta, T_x M)$, and the frame bounds can be chosen uniform for all z from a compact subset.

Rashevsky-Chow-type theorem for control-linear ensembles

Theorem

Let uniform analyticity and uniform boundedness assumptions as well as Lie algebraic frame condition be satisfied for ensemble (**). Then for each couple $(\tilde{x}, \hat{x})$ and $T > 0$ (**) is time-T approximately steerable from $\tilde{x}$ to $\hat{x}$. $\square$

Remark. For Θ being finite ($|\Theta| = N$), the Hilbert space $L_2(\Theta, T_x M)$ is finite-dimensional, isomorphic to $T_y M^N$, and the Lie algebraic frame condition is equivalent to the bracket generating condition on M^N. Then stronger result on *exact* ensemble controllability holds.

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Steering by means of extended control

Proposition

Under the assumptions of the Theorem there exist $\delta, C > 0$ and a finite set of multi-indices $A = \{(\alpha_1, \ldots, \alpha_N)\}$, such that the extended ensemble

$$\frac{d}{dt}x_\theta(t) = \sum_{\alpha \in A} X_\alpha^\theta(x)v_\alpha(t), \tag{2}$$

where X_α^θ are the iterated Lie brackets from Lie algebraic frame condition, can be approximately steered from 0 to $\hat{x}$ in time δ by an extended control $(v_\alpha(t)),\ \alpha \in A$:

$$\|x_\theta(\delta) - \hat{x}\|_{L_2(\Theta)} < C\delta^2. \quad \square \tag{3}$$

Steering by means of small-dimensional control. Lie extensions

Once the possibility to steer ensemble (**) by a high-dimensional *extended* control is established, we will show that the same goal can be achieved by means of lower-dimensional control. This is done via so-called Lie extensions.

The following result shows, that the control-linear ensemble

$$\frac{d}{dt} x_\theta(t) = X^\theta u(t) + Y^\theta v(t), \tag{4}$$

and the extended ensemble

$$\frac{d}{dt} x_\theta(t) = X^\theta u_e(t) + Y^\theta v_e(t) + [X^\theta, Y^\theta] w_e(t). \tag{5}$$

have 'almost the same steering capacities'.

Proposition

If extended ensemble (5) *can be steered from* $\tilde{x}$ *to* $\hat{x}$ *approximately in time* T*, then the same holds for ensemble* (4). □