A KAM theorem for some quasi-linear PDEs

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Shallow Water wave . . .
Birkhoff norm form
KAM Theorem
Application to BBM . . .
Final remark

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1 Shallow Water wave Equations

- (KdV-mKdV)

$$u_t + \alpha u u_x + \beta u^k u_x + \gamma u_{xxx} = 0, \quad k = 1 \text{ or } k = 2$$

- (Burgers)

$$u_t - \nu u_{xx} + u u_x = 0$$

- (Camassa-Holm)

$$u_t - u_{xxt} + 2\kappa u_x + 3 u u_x = 2 u_x u_{xx} + u u_{xxx}$$

- (Benjamin-Bona-Mahony, BBM)

$$u_t - u_{xxt} + u_x + uu_x = 0$$

- (Hirota-Satsuma, HS)

$$u_t - u_{xxt} + u_x - 4uu_t - 2u_x\partial_x^{-1}u_t = 0$$

In this talk, we will focus on the last two equations.

1.1. BBM equation

Consider the BBM equation

$$u_t - u_{xxt} + u_x + uu_x = 0, \quad u(t,0) = u(t,T). \tag{1}$$

This equation can be written as a Hamiltonian system

$$u_t = J\nabla_u H(u) \tag{2}$$

with

$$J = -(1-\partial_{xx})^{-1}\partial_x \tag{3}$$

$$H(u) = \frac{1}{2}\int_0^T u^2 dx + \frac{1}{6}\int_0^T u^3 dx, \tag{4}$$

and

$$u \in \mathcal{H}_0^s = \{u \in \mathcal{H}^s(\mathbb{T} : \mathbb{R}) : \int_0^T u dx = 0\}.$$

Define $\bar{\mathbb{Z}} = \mathbb{Z} \setminus \{0\}$. Set $\tau = \frac{2\pi}{T}$. Introduce the expansion

$$u = \sum_{j \in \bar{\mathbb{Z}}} q_j \phi_j, \qquad \phi_j = \frac{1}{\sqrt{T}} e^{\mathbf{i}\tau j \cdot x}, \tag{5}$$

then

$$q \in \ell_s := \{\|q\|_s^2 = \sum_{j \in \mathbb{Z}} |q_j|^2 j^{2s} < \infty\}, \quad \bar{q}_j = q_{-j},$$

and (2) is changed into

$$\mathbf{i}\dot{q}_j = \frac{\tau j}{1+\tau^2 j^2} q_j + \frac{\tau j}{1+\tau^2 j^2} \frac{1}{2\sqrt{T}} \sum_{l+m=j} q_l q_m, \tag{6}$$

It is a Hamiltonian system:

$$\dot{q}_j = -\mathbf{i} \frac{\tau j}{1+\tau^2 j^2} \frac{\partial H}{\partial q_{-j}}, \tag{7}$$

with the symplectic structure

$$\sum_j -\mathbf{i} \frac{\tau j}{1+\tau^2 j^2} d\, q_j \wedge d\, q_{-j} \tag{8}$$

$$H(q) = \Lambda + R = \sum_{j \geq 1} |q_j|^2 + \frac{1}{6\sqrt{T}} \sum_{k+l+m=0} q_k q_l q_m. \tag{9}$$

- the frequencies

$$\lambda_j = \frac{\tau j}{1+\tau^2 j^2} = O(j^{-1}) \to 0, \quad j \to \infty,$$

- The perturbation is quasi-linear:

$$\tilde{\Lambda}^{-1} X_R : \ell_s \to \ell_s. \tag{10}$$

where

$$\tilde{\Lambda} = \mathrm{diag}\,(\lambda_j : j \in \bar{\mathbb{Z}}) \tag{11}$$

1.2. H-S equation

Consider the H-S equation:

$$u_t - u_{xxt} + u_x - 4uu_t - 2u_x\partial_x^{-1}u_t = 0, \quad u(t,0) = u(t,1) \tag{12}$$

and

$$u \in \mathcal{H}_0^s = \{u \in \mathcal{H}^s(\mathbb{T}:\mathbb{R}) : \int_0^1 u dx = 0\} \tag{13}$$

Remark: the dependence of the nonlinearity on u_t causes difficulty.

Introduce the expansion,

$$u = \sum_{j\in\mathbb{Z}\setminus\{0\}} q_j\phi_j, \quad \phi_j(x) = e^{\mathrm{i}2\pi jx}, \tag{14}$$

and $q \in \ell_s, \;\; q_{-j} = \bar{q}_j$. Eq (12) is changed into

$$\dot{q} = -\mathrm{i}Aq + B(q)\dot{q}, \tag{15}$$

$$A = \mathrm{diag}(\frac{2\pi j}{1+4\pi^2 j^2} : j \in \bar{\mathbb{Z}}), \tag{16}$$

$$B(q) = (B_{jl}(q))_{j,l\in\bar{\mathbb{Z}}} = (\frac{2(j+l)}{(1+4\pi^2 j^2)l} q_{j-l})_{j,l\in\bar{\mathbb{Z}}} \tag{17}$$

Lemma 1. For any $s > 0$, $B(q)$ defines a bounded operator from ℓ_{s+1} to ℓ_{s+1} with the norm

$$\|B(q)\|_{s+1,s+1} = O(\|q\|_s). \tag{18}$$

Consider (12) on a small neighborhood of the origin , then

$$\mathrm{i}\dot{q} = Aq + BAq + B^2Aq + \cdots := \Lambda + R. \tag{19}$$

Remark: Taking $-\mathrm{i}\sum_{j\geq 1}(2\pi j)^{-1} q_j \wedge q_{-j}$ as the symplectic structure, we can verify $Aq + BAq + B^2Aq$ is hamiltonian and give its explicit Hamiltonian function. We need to verify that the higher order $\cdots$ is also hamiltonian, but its explicit expression is not necessary for KAM theory.

For the H-S equation, we also get that $\lambda_j \to 0$.

Summary:

- Both BBM equation and H-S equation are hamiltonian system
- the frequencies λ_j has a finite limit point:

$$\lambda_j = O(j^{-1}) \to 0, \quad \text{as } j \to \infty,$$

- The perturbation is Quasi-linear:

$$\tilde{\Lambda}^{-1} X_R : \ell_s \to \ell_s. \tag{20}$$

where

$$\tilde{\Lambda} = \operatorname{diag}\,(\lambda_j : j \in \bar{\mathbb{Z}}) \tag{21}$$

Comparison with the perturbed KdV equation:

$$u_t = -u_{xxx} + f = 0, \quad f = 6uu_x + \ldots \tag{22}$$

where the frequencies

$$\lambda_j \sim j^3 \to \infty, \quad j \to \infty$$

- When $f = f(x, u, u_x)$, existence of KAM tori due to Kuksin (2000).
- When $f = f(x, u, u_x, u_{xx})$, existence of KAM tori can be obtained by Liu-Y method (2010,2011).
- When $f = f(x, u, u_x, u_{xx}, u_{xxx})$, existence of KAM tori due to Baldi-Berti-Montalto (2013),(2014). Quasi-Linear case!
- Procesi and her collaborators, Quasi-linear NLS (2015) with $\lambda_j = j^2$
- Berti and his collaborators, Water wave eqn. (quasi-linear with $\lambda_j = \sqrt{j(1+\kappa j^2)}$
- KdV is integrable in Liouville's sense, by Kappeler-Makarov(2001). provide a chance to show more results, e.g., Nekhoroshev estimate.

Key point in KAM is that the number of small divisors must be finite:

$$\leq K \approx 2^m$$

at the m-th Newton iteration. For this difficulty, the most complicated case is known as the second Melnikov's condition:

$$\langle k, \omega \rangle + \lambda_i - \lambda_j \tag{23}$$

For the KdV,

$$|\lambda_i - \lambda_j| \sim |i^3 - j^3| \sim |i-j||i^2+j^2|. \tag{24}$$

At the m-th step, we get $|k| \leq K \sim 2^m$, then when i or j is bigger than CK, (23) is not small, thus we only need to exclude resonances just for

$$(k, i, j) : |k| \leq K, i, j \leq CK, \tag{25}$$

and the number of (k, i, j) can be controlled by C^m. here I dropped the effect of the unbounded (quasi-linear) perturbation.

When $\lambda_j \to 0$, e.g.,

$$\lambda_j = \frac{1}{j},$$

the second Melnikov condition is

$$\langle k, \omega \rangle + \frac{1}{i} - \frac{1}{j}$$

No growth in i or j!

As in KdV, BBM equation and H-S equation do not contain parameters, hence we need to do Birkhoff normal form to extract parameters.

2 Birkhoff norm form

2.1. BBM equation

For convenience, we set $q_j = \gamma_j \tilde{q}_j$ with $\gamma_j = \sqrt{\frac{\tau|j|}{1+\tau^2 j^2}}$ and the symplectic structure changes into $-\mathrm{i}\sum_{j\geq 1}\sigma_j dq_j \wedge dq_{-j}$. Here $\sigma_j = \mathrm{sgn}(j)$, in the following, we still use q for simplicity and $q \in \ell_{s-1/2}$. The corresponding Hamiltonian function is

$$H(q) = \Lambda + R = \sum_{j\geq 1}\frac{\tau j}{1+\tau^2 j^2}|q_j|^2 + \frac{1}{6\sqrt{T}}\sum_{k+l+m=0}\gamma_k\gamma_l\gamma_m q_k q_l q_m. \tag{26}$$

Let $J = \{1 \leq j_1 < j_2 < \cdots < j_n = N : j_t \in \mathbb{N} \text{ for } 1 \leq t \leq n\}$. Set $(q_j)_{j\geq 1} = (\tilde{q}, \hat{q})$, and $\tilde{q} = (q_{j_1}, \cdots, q_{j_t})$, $\hat{q} = (q_j)_{j\in\mathbb{N}\setminus J}$.

Theorem 1. There exists a real analytic symplectic coordinate transformation defined on the neighborhood of the origin of $\ell_{s-\frac{1}{2}}$ which transform the Hamiltonian H defined by(26) into its Birkhoff normal form up to order four. More precisely,

$$H \circ \Phi = \Lambda + \bar{G} + \hat{G} + \tilde{R}, \tag{27}$$

where

$$\bar{G} = \sum_{k,l \geq 1} \bar{G}_{kl} |q_k|^2 |q_l|^2 \tag{28}$$

with

$$\bar{G}_{kl} = \begin{cases} -\frac{1}{T} \frac{\tau^2 kl}{[\tau^2(k^2+kl+l^2)+3][\tau^2(k^2-kl+l^2)+3]}, & k \neq l, \\ \frac{1}{12T} \frac{1}{\tau^2 k^2+1}, & \text{otherwise}, \end{cases} \tag{29}$$

and

$$|\hat{G}| = O(\|\hat{q}\|^3_{s-\frac{1}{2}}), \quad \|X_{\tilde{R}}\|_{s+\frac{1}{2}} = O(\|q\|^4_{s-\frac{1}{2}}) \tag{30}$$

2.2. H-S equation

Set $q_j = \gamma_j \tilde{q}_j$ with $\gamma_j = \sqrt{\frac{2\pi|j|}{1+4\pi^2 j^2}}$ and the symplectic structure changes into $-\mathrm{i}\sum_{j\geq 1}\sigma_j dq_j \wedge dq_{-j}$. Use q for simplicity in the following. Let $J = \{1 \leq j_1 < j_2 < \cdots < j_n = N : j_t \in \mathbb{N}$ for $1 \leq t \leq n\}$ and $(q_j)_{j\geq 1} = (\tilde{q}, \hat{q})$

Theorem 2. There exists a real analytic symplectic coordinate transformation defined on the neighborhood of the origin of $\ell_{s-\frac{1}{2}}$ which transform the Hamiltonian vector of (12) into its Birkhoff normal form up to order three. More precisely, the new Hamiltonian vector field is

$$\mathrm{i}\sigma_j \dot{q}_j = \frac{2\pi|j|}{1+4\pi^2 j^2} q_j + \bar{Q}_j^3 + \hat{Q}_j^3 + \tilde{Q}_j \tag{31}$$

where

$$\bar{Q}_j^3 = \sum_{k\geq 1} \bar{Q}_{jk}^3 |q_k|^2 q_j \tag{32}$$

with

$$\bar{Q}^3_{jk} = \begin{cases} \frac{4\cdot 2\pi|j|\cdot 2\pi k(4\pi^2 j^2\cdot 4\pi^2 k^2+3(4\pi^2k^2+4\pi^2j^2)+5)}{(1+4\pi^2j^2)^2(1+4\pi^2k^2)^2}, & k \neq |j|, \\ \frac{4(2\pi j)^4+18(2\pi j)^2+6}{(1+4\pi^2j^2)^3}, & \text{otherwise}, \end{cases} \tag{33}$$

and

$$|\hat{Q}_j| = O(\|\hat{q}\|^2_{s-\frac{1}{2}}), \quad \|\tilde{Q}\|_{s+\frac{1}{2}} = O(\|q\|^4_{s-\frac{1}{2}}) \tag{34}$$

Set $\lambda_j = \frac{\tau j}{1+\tau^2 j^2}$ (for H-S equation, $\tau = 2\pi$). Both the Birkhoff normal form theorems are based on the following three lemmas.

Lemma 1. Assume τ is some transcendental number. Let $\mathbb{N} \ni m \geq 1$. Then for any fixed $1 \leq j_1 < j_2 < \cdots < j_m$ and $r_1, r_2, \cdots r_m \in \bar{\mathbb{Z}}$ with $r_1 j_1 + \cdots + r_m j_m = 0$, we have

$$r_1\lambda_{j_1} + r_2\lambda_{j_2} + \cdots + r_m\lambda_{j_m} \neq 0. \tag{35}$$

Set $\Delta = \{(k, l, m, n) : k+l+m+n = 0, k+l, k+m, k+n \neq 0\}$. For $0 \leq i \leq 2$, set $\Delta_i = \{(k, l, m, n) \in \Delta :$ there exist i elements of $\{|k|, |l|, |m|, |n|\}$ not in $J\}$ and denotes the other cases by Δ_3.

Lemma 2. (BBM) Assume $\frac{1}{\tau} > 2N\sqrt{n}$, then for any $(k, l, m, n) \in \Delta_0 \cup \Delta_1 \cup \Delta_2$, we have

$$|\lambda_k + \lambda_l + \lambda_m + \lambda_n| \geq C(\tau, n, N). \tag{36}$$

Lemma 3. (H-S) Assume $\tau = 2\pi$, then for any $(k, l, m, n) \in \Delta_0 \cup \Delta_1 \cup \Delta_2$, we have

$$|\lambda_k + \lambda_l + \lambda_m + \lambda_n| \geq C(n, N). \tag{37}$$

3 KAM Theorem

Denote by $\mathcal{P}^p = \mathbb{T}^n \times \mathbb{R}^n \times \ell_p \times \ell_p \ni (x, y, z, \bar{z})$ and $\mathcal{T}_0 = \mathbb{T}^n \times \{0, 0, 0\}$. For $s, r > 0$, introduce the complex $\mathcal{T}_0$ neighborhoods in $\mathcal{P}^p$:

$$D(s, r) = |\text{Im}x| < s, |y| < r^2, \|z\|_p + \|\bar{z}\|_p < r.$$

For $W = (X, Y, U, V) \in \mathcal{P}^p$, define $\|W\|_{r,\bar{p}} = |X| + \frac{1}{r^2}|Y| + \frac{1}{r}\|U\|_{\bar{p}} + \frac{1}{r}\|V\|_{\bar{p}}$. Consider an infinite dimensional Hamiltonian in the parameter dependent normal form

$$N = \langle\omega(\xi), y\rangle + \sum_{j\in\mathbb{Z}^d} \Omega_j(\xi) z_j \bar{z}_j + \langle B^{zz}(x;\xi)z, z\rangle + \langle B^{z\bar{z}}(x;\xi)z, \bar{z}\rangle + \langle B^{\bar{z}\bar{z}}(x;\xi)\bar{z}, \bar{z}\rangle, \tag{38}$$

which has the invariant torus $\mathcal{T}_0$. We prove the persistence of the torus under small perturbation P. The following assumptions are required.

(A) Assume that $\omega(\xi)$, $\Omega(\xi)$ and $B^{uv}(\xi)$ are continuously differentiable in $\xi \in \Pi$ in the sense of Whitney;

(B) the map $\xi \to \omega(\xi)$ is a homeomorphism between Π and its image. Moreover, there exists $c_1 > 0$ such that $\sup_{\xi\in\Pi} |\partial_\xi^j \omega(\xi)| \le c_1, j = 0, 1;$

(C)there exists constants $c_2, c_3, c_4, c_5 > 0$ and a constant $\kappa > 0$ such that $(\Omega_j)^\sharp \le c_2|j|^{c_3}$ and

$$c_4|j|^{-\kappa} \le |\Omega_j| \le c_5|j|^{-\kappa}. \tag{39}$$

Set $\hat{\Omega} = \text{diag}\ (\partial_{w_t}\Omega_j - \frac{1}{\omega_t}\Omega_j\ :\ j \in \mathbb{Z}^d)$ for some $1 \le t \le n$ with $\omega_t \ne 0$, we assume that $\hat{\Omega}$ is a positive operator in ℓ^2.

(D) (quasi-linear)

Set $\Lambda = \text{diag}\ (\Omega_j; j \in \mathbb{Z}^d)$

$$\Lambda^{-1}X_P : D(s,r) \subset \mathcal{P}^p \to \mathcal{P}^p; \tag{40}$$

(E) B^{uv} is a bounded operator from ℓ_p to $\ell_{p+\kappa}$, and the operator norm has the estimates

$$\sup_{\xi\in\Pi}\|B^{uv}(x;\xi)\|_{p,p+\kappa},\quad \sup_{\xi\in\Pi}\|\partial_\xi B^{uv}(x;\xi)\|_{p,p+\kappa}\ll 1; \tag{41}$$

(F) the perturbation P and B satisfy the real condition. That is,

$$\overline{P(x,y,z,\bar{z};\xi)} = P(x,y,z,\bar{z};\xi) \tag{42}$$

for real (x,y) and for real x,

$$\overline{B^{zz}(x,\xi)} = B^{\bar{z}\bar{z}}(x,\xi),\quad \overline{B^{z\bar{z}}(x,\xi)} = (B^{z\bar{z}}(x,\xi))^T. \tag{43}$$

Theorem 3. Suppose $H = N + P$ satisfies assumption (A)-(F) and

$$\epsilon := |X_P|_{r,p+\kappa;D(s,r)\times\Pi} + \alpha|\partial_\xi X_P|_{r,p+\kappa;D(s,r)\times\Pi} \tag{44}$$

for some $0 < \alpha < 1$, then there exist some constant $\gamma = \gamma(p, \kappa, d, n) > 1$ and $\eta = \eta(n, s, r)$ sufficiently small such that for

$$\epsilon < \alpha^\gamma \eta, \tag{45}$$

there exists a subset $\Pi_\alpha \subset \Pi$ with

$$\text{Meas } \Pi_\alpha \geq (\text{Meas } \Pi)(1 - O(\alpha)),$$

and for every $\xi \in \Pi_\alpha$, there exist a family of torus embedding $\Phi : \mathbb{T}^n \times \Pi_\alpha \to \mathcal{P}^p$ and a map $\omega^* : \Pi_\alpha \to \mathbb{R}^n$ such that Φ restricted to $\mathbb{T}^n \times \{\xi\}$ ia an embedding of a torus with frequencies $\omega^*(\xi)$.

Proof. Decompose the perturbation

$$P = R^0 + R^1 + R^2 + O(|y|^2 + |y|\|z\| + \|z\|^3),$$

we need to find the symplectic transformation to eliminate lower order terms. Notice that $\lambda_j \sim j^{-\kappa}$, we can not control the number of the small divisor:

$$\langle k, \omega \rangle + \lambda_i - \lambda_j,$$

thus, instead of eliminating the second order terms , we put it into the normal form, then we need to solve the linearized equation with variable coefficients

$$\mathrm{i}\partial_\omega F^z + (\Omega + B^{zz}(x) + \cdots)F^z = R^z, \tag{46}$$

thus we needs to investigate the inverse of a big matrix of the form

$$\hat{\Lambda} = \text{diag}\,(\langle k, \omega \rangle + \Omega_j : |k| \le K, j \in \mathbb{Z}^d) + \hat{B}(k - l) : |k|, |l| \le K) \tag{47}$$

By excluding some parameters, we get

$$|\langle k,\omega\rangle| \geq \frac{\alpha}{K^{n+1}}, \quad 0 \neq |k| \leq K, \tag{48}$$

then

$$|\langle k,\omega\rangle + \Omega_j| \geq \frac{\alpha}{2K^{n+1}} \tag{49}$$

as long as $|\Omega_j| \leq c|j|^{-\kappa} \leq \frac{\alpha}{2K^{n+1}}$, that is $|j| \geq M = (\frac{K^{c(n)}}{\alpha})^{\frac{1}{\kappa}}$. Write the matrix above as

$$\begin{pmatrix} \hat{\Lambda}_1 & 0 \\ 0 & \hat{\Lambda}_1 \end{pmatrix} + \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{pmatrix} \tag{50}$$

It is enough to control the inverse of $\hat{\Lambda}_1 + \widehat{\mathcal{B}}_{11}$.

Lemma 4. (key lemma) There exists a subset $\Pi^+ \subset \Pi$ with Meas $(\Pi \setminus \Pi^+) = O(\frac{\alpha}{K})$ such that

$$\|(\hat{\Lambda}_1 + \widehat{\mathcal{B}}_{11})^{-1}\|_{p+\kappa,p+\kappa} \prec (\frac{1}{\alpha})^{c_1(n)} K^{c_2(n)} \tag{51}$$

Linear Stability: For the perturbed KdV, the linearized equation along the KAm tori can be reduced to

$$\sqrt{-1}\dot{q}_j = \tilde{\lambda}_j q_j, \quad j \in \bar{\mathbb{Z}}.$$

The KAM tori is linear stable in Sobolev space $\mathcal{H}_0^s$ (or ℓ_s).

Lemma 5. For BBM and H-S, the linearized equation along the KAm tori can be reduced to

$$\sqrt{-1}\dot{q} = A(t)q$$

where A is real and self-adjoint for any t. The obtained KAM tori is linear stable in L^2 space.

Proof. Since the linearized equation has L^2 energy conservation.

4 Application to BBM equation and H-S equation

4.1. Application to BBM equation

Theorem 4. Consider the BBM equation (1) on the ϵ-neighborhood of the origin of $\mathcal{H}_0^s$ written in the Hamiltonian form

$$u_t = -(1-\partial_{xx})^{-1}\partial_x \nabla_u H(u), \tag{52}$$

where the Hamiltonian H is defined by (4). For any given integer $n \in \mathbb{N}$ and each index set $J = \{1 \leq j_1 < j_2 < \cdots < j_n = N\} \subset \mathbb{N}$, Suppose $\tau = \frac{2\pi}{T}$ be some transcendental number with $\frac{1}{\tau} > 2N\sqrt{n}$, then there exists an $\epsilon_0 > 0$ depending only on J, τ and s such that for $\epsilon < \epsilon_0$, the equation has many KAM tori (quasi-periodic solutions) with frequency vector close to $(\frac{\tau j_1}{1+\tau^2 j_1^2}, \cdots \frac{\tau j_n}{1+\tau^2 j_n^2})$.

- Introduce symplectic polar and real coordinates $(x, y, z, \bar{z})$ by setting

$$\begin{cases} q_{j_t} = \sqrt{\xi_t + y_t} e^{-\mathrm{i} x_t}, & q_{-j_t} = \sqrt{\xi_t + y_t} e^{\mathrm{i} x_t}, \quad 1 \le t \le n, \\ q_j = z_j, & q_{-j} = \bar{z}_j, \quad j \in \mathbb{N} \setminus J, \end{cases} \tag{53}$$

  to (27), we get $\tilde{H} = N + P$ with a symplectic structure $\sum_{1 \le t \le n} dy_t \wedge dx_t - \mathrm{i} \sum_{j \notin J} dz_j \wedge d\bar{z}_j$, where

$$N = \sum_{1 \le t \le n} \omega_t^0(\xi) y_t + \sum_{j \notin J} \Omega_j^0(\xi) z_j \bar{z}_j \tag{54}$$

  The frequency was defined as follows.

$$\omega = \lambda^n + U\xi, \quad \Omega = \lambda^\infty + T\xi, \tag{55}$$

  where

$$\lambda^n = (\lambda_{j_t} : 1 \le t \le n), \quad \lambda^\infty = \{\lambda_j : j \notin J\}, \tag{56}$$

  and U and T are matrix.

- Check assumption (A) to (F) as well as the smallness condition (45), we can finish the proof of the above theorem.

Remark.

We choose $\tau = \frac{2\pi}{T}$ to be some transcendental number in order to ensure that the above theorem holds true for arbitrary n and $J = \{j_1 < j_2 < \cdots < j_n\}$. Otherwise, one need to prove the theorem under some extra conditions of n and J.

4.2. Application to H-S equation

Theorem 5. Consider the BBM equation (12) on the ϵ-neighborhood of the origin of $\mathcal{H}_0^s$ with the vector field defined by (19). For any given integer $n \in \mathbb{N}$ and the index set $J = \{1, 2, \cdots, n\}$ then there exists an $\epsilon_0 > 0$ depending only on n and s such that for $\epsilon < \epsilon_0$, the equation has many quasi-periodic solutions with frequency vector close to $(\frac{2\pi j}{1+4\pi^2 j^2})_{1\leq j\leq n}$.

The proof of Theorem 5 can be obtained by introducing symplectic polar and real coordinates and checking (A) to (F) and the smallness condition (45). All the assumptions are easily to check except assumption (C). We have the following lemma.

Lemma 5. (H-S) For any fixed $1 \le t \le n$, $\partial_{w_t}\Omega_j - \frac{1}{\omega_t}\Omega_j$ is not equal to zero with the sign keeping the same for every $j \ge n+1$.

Proof of Lemma 5.

After we introduce symplectic polar and real coordinates, we get

$$\omega = \lambda^n + U\xi, \quad \Omega = \lambda^\infty + T\xi, \tag{57}$$

where

$$\lambda^n = (\lambda_t : 1 \le t \le n), \quad \lambda^\infty = \{\lambda_j : j \ge n+1\}, \tag{58}$$

and

$$U_{kl} = \begin{cases} \frac{4\cdot 2\pi k\cdot 2\pi l(4\pi^2k^2\cdot 4\pi^2l^2+3(4\pi^2k^2+4\pi^2l^2)+5)}{(1+4\pi^2k^2)^2(1+4\pi^2l^2)^2}, & k\neq l, \\ \frac{4(2\pi k)^4+18(2\pi k)^2+6}{(1+4\pi^2k^2)^3}, & \text{otherwise}, \end{cases} \tag{59}$$

$$T_{jl} = \frac{4\cdot 2\pi j\cdot 2\pi l(4\pi^2j^2\cdot 4\pi^2l^2+3(4\pi^2l^2+4\pi^2j^2)+5)}{(1+4\pi^2j^2)^2(1+4\pi^2l^2)^2}. \tag{60}$$

We have

$$\partial_{w_t}\Omega_j - \frac{1}{\omega_t}\Omega_j = \frac{1}{\omega_t^2}(\lambda^\infty - TU^{-1}\lambda^n). \tag{61}$$

Let $\Lambda = \mathrm{diag}(\frac{2\pi j}{1+4\pi^2j^2} : j\geq n+1)$, $T=\Lambda\tilde{T}$, and $U=\Lambda\tilde{U}$, we can obtain

$$\tilde{U}_{kl} = \begin{cases} u(l)+\frac{1}{4\pi^2k^2}v(l), & k\neq l, \\ u(k)+\frac{1}{4\pi^2k^2}v(k)-s(k), & \text{otherwise} \end{cases} \tag{62}$$

$$\tilde{T}_{jl} = u(l)+\frac{1}{4\pi^2j^2}v(l). \tag{63}$$

Set $w(m) = \sum_{1\le i\le n} \tilde{U}^{-1}_{mi}$, hence we can check that

$$
\begin{aligned}
& \left(\frac{2\pi j}{1+4\pi^2 j^2}\right)^{-1}(\lambda^\infty - TU^{-1}\lambda^n)_j \\
= \;& 1 - (\tilde{T}\tilde{U}^{-1}1^n)_j \\
= \;& \Big(1 - \sum_{1\le m\le n} u(m)w(m)\Big) - \frac{1}{4\pi^2 j^2}\sum_{1\le m\le n} v(m)w(m).
\end{aligned}
\tag{64}
$$

By some calculations, we get for every $j \ge n+1$,

$$
\left(\frac{2\pi j}{1+4\pi^2 j^2}\right)^{-1}(\lambda^\infty - TU^{-1}\lambda^n)_j = \left(a + b\frac{1}{4\pi^2 j^2}\right)c \tag{65}
$$

where

$$
a > 0, \quad b > 0, \quad a - \frac{b}{1+4\pi^2 n^2} > 0, \quad c \ne 0,
$$

and thus the sign keep the same.

5 Final remark

The KAM theorem with $\lambda_j \to 0$ should be valid for higher spatial dimensional version of BBM and H-S, e.g. BBM:

$$u_t - \triangle_x u_t + u_x + u \cdot u_x = 0, \quad x \in \mathbb{T}^d, \quad d > 1$$

In this case, new difficulty comes from normal form.

The KAM theorem with $\lambda_j \to \infty$ should be valid for higher spatial dimensional version of PDEs remains open!

THANK YOU