On conformally Hamiltonian vector fields

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An autonomous system of differential equations

$$\dot{x}_i = X_i(x_1, \dots, x_n)$$

is geometrically interpreted in terms of a vector field X in a n-dimensional manifold M with a local expression

$$X = X_1 \frac{\partial}{\partial x_1} + \dots + X_n \frac{\partial}{\partial x_n}.$$

The integral curves of X are the solutions of equations of motion.

Integrating the system amounts to determining its general solution. In particular, integrability by quadratures means that you can determine the solutions by means of a finite number of algebraic operations and integrations of known functions.

Lie theorem

A classical Lie theorem says that the vector field X is integrable by quadratures if there are n linearly independent at each point vector fields

$$X_1 = X,\ X_2, \dots, X_n$$

which generate a solvable Lie algebra with respect to the commutation operation $[.,.]$, and $[X, X_i] = \lambda_i X_i$.

Vector fields X_i are not necessary symmetries of the dynamics.

Some modifications and generalizations may be found in

V.V. Kozlov, The Euler-Jacobi-Lie integrability theorem, 2013.

J.F. Carinena, F. Falceto, J. Grabowski, M.F. Ranada, Geometry of Lie integrability by quadratures, 2015.

Vector field X is integrable by quadratures if there are k functional independent integrals of motion

$$H_1\,,\ldots,H_k$$

and $n-k$ linearly independent vector fields

$$X_1=X,\;X_2,\ldots,X_{n-k}$$

which generate a solvable Lie algebra, such that

$$\mathcal{L}_{X_i}\,H_j=0\,.$$

The above statement follows immediately from the Lie theorem applied to restriction on the integral manifold

$$N_c=\{x\in M\,,\qquad H_1=c_1\,,\ldots,H_k=c_k\}\,.$$

Poisson manifolds

If M is a Poisson manifold endowed with a Poisson bivector P, then integrals of motion

$$H_1\,,\ldots,H_k$$

give rise to m nontrivial Hamiltonian vector fields

$$Y_1=P\mathrm{d}H_1,\ldots,\,Y_m=P\mathrm{d}H_m$$

in addition to the $n-k$ generators of the solvable algebra

$$X_1=X,\,X_2,\ldots,X_{n-k}\,.$$

For Y_i we can apply the Hamilton-Jacobi method etc.

What can we say about integration of X_i?

Hamiltonian mechanics

In this simplest case vector fields X_i are Hamiltonian

$$X_i = Y_i = P\mathrm{d}H_i\,,$$

If integrals of motion $H = H_1, \ldots, H_k$ are in involution

$$\{H_i, H_j\} = 0\,,$$

we have commutative algebra and Liouville theorem.

If X_i form a solvable algebra, then we have noncommutative integrability:

Mishenko A.S., Fomenko A.T., Generalized Liouville method of integration of Hamiltonian systems, 1978

Fernandes R.L., Laurent-Gengoux C., Vanhaecke P., Global Action-Angle Variables for Non-Commutative Integrable Systems, 2015

Non Hamiltonian mechanics

In this case integrals of motion $H_1, \ldots, H_k$ give rise to linearly independent vector fields

$$Y_i = P\mathrm{d}H_i\,,$$

such that generators of the solvable algebra

$$X_i = \sum_{i=1}^{n-k} \mathrm{g}_{ij}\, Y_j$$

are linear combinations of Hamiltonian vector fields.

In generic case Lie (1912) and Bianchi (1918) found invariant measure as function on coefficients g_{ij} and we know nothing more.

Conformally Hamiltonian fields:

Let

- (M, P) is a smooth Poisson manifold;
- H is s Hamilton function;
- g is a smooth function on an open dense subset M_0 of M.

We want to study conformally Hamiltonian vector field on M_0

$$X = \mathrm{g}\, Y = \mathrm{g}\, P\mathrm{d}H\,,$$

with a conformal factor g (usually nonvanishing function).

J.M. Marle, A property of conformally invariant Hamiltonian vector fields; application to the Kepler problem, J. Geom. Mech., 2012.

A.V. Borisov, I.S. Mamaev, A.V. Tsiganov, Non-holonomic dynamics and Poisson geometry, Russian Math. Surveys, 2014.

P. Guha, Nonholonomic deformation of coupled and supersymmetric KdV equations and Euler-Poincaré-Suslov method, Rev. Math. Phys., 2015.

Change of time

Conformally Hamiltonian vector field $X = \mathrm{g}\, Y$ is defined by equations

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n)\,.$$

Formally, using change of independent variable

$$dt = \mathrm{g}^{-1}(x_1, \ldots, x_n) d\tau$$

one gets Hamiltonian equations

$$\frac{dx_i}{d\tau} = \, Y_i(x_1, \ldots, x_n)\,.$$

For a general smooth Hamiltonian vector field Y and a general smooth nowhere vanishing function g, there may be no globally defined function $\tau = \sigma(x, t)$.

Kepler problem

Let us consider equations of motion

$$\frac{dq_i}{dt} = \frac{p_i}{m}\,, \qquad \frac{dp_i}{dt} = \frac{km\, q_i}{r^3}\,, \qquad r = \sqrt{q_1^2 + q_2^2 + q_3^2}\,.$$

The corresponding vector field is Hamiltonian with respect to the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{mk}{r}\,.$$

and canonical Poisson bivector

$$P = \left(\begin{array}{cc} 0 & Id \\ -Id & 0 \end{array}\right)\,.$$

Regularization

Let τ be a Levi-Civita parameter

$$\tau = \frac{1}{mk}\Big((p,q) - 2H\,t\Big)\,.$$

so that the system of equations

$$\frac{dx_i}{d\tau} = X_i(x_1\,\ldots,x_6) = \mathrm{g}\,Y_i$$

is no more Hamiltonian, but rather conformally Hamiltonian.

Initial vector field $Y = P\mathrm{d}H$ is complete, g satisfies to some special conditions, so this conformally Hamiltonian vector field

$$X = P'\mathrm{d}H'$$

is Hamiltonian with respect to another Poisson bivector.

Thus, there is a Poisson map

$$P \to P'$$

from the phase space of the Kepler problem restricted to negative (positive) values of the energy, onto an open dense subset of the cotangent bundle to a three-dimensional sphere (hyperboloid):
Györgyi (1968), Ligon and Schaaf (1976) , Cushman and Duistermaat (1997).

Inverse map was proposed by Fock (1935) and studied by Moser (1970), Anosov (2002) and Milnor (1983).

However, it is very special example because we can explicitly integrate equation

$$dt = \mathrm{g}^{-1} d\tau \,.$$

Euler equations

In classical mechanics the Euler-Poisson equations

$$\dot{M} = M \times \omega + \gamma \times \frac{\partial V(\gamma)}{\partial \gamma}, \qquad \dot{\gamma} = \gamma \times \omega$$

describe rotation of a rigid body with a fixed point.

Here

- ω is the angular velocity;
- $\mathbb{I}$ is a tensor of inertia;
- $M = \mathbb{I}\omega$ is the angular momentum;
- γ is a unit Poisson vector;
- $V(\gamma)$ is a potential field.

All the vectors are expressed in the so-called body frame.

The Euler-Poisson equations define Hamiltonian vector field

$$Y = P\mathrm{d}H$$

where

$$H = \frac{1}{2}(M, \mathbb{I}^{-1}M) + \frac{1}{2}\,V(\gamma),$$

and Poisson bracket

$$\{M_i\,, M_j\,\} = \varepsilon_{ijk} M_k\,, \quad \{M_i\,, \gamma_j\,\} = \varepsilon_{ijk}\gamma_k\,, \quad \{\gamma_i\,, \gamma_j\,\} = 0\,,$$

has two Casimir functions

$$C_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2\,, \qquad C_2 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3\,,$$

Vector field Y is integrable in the Euler, Lagrange, Clebsch and Kowalewski cases.

Nonholonomic deformation

Let us impose nonholonomic Suslov constraint on the angular velocity

$$f = (\omega, a) = const\,,$$

where a is a fixed unit vector in the rotating (body) frame, or the so-called Veselova constraint

$$f = (\omega, \gamma) = const\,,$$

where γ is a fixed unit vector in the stationary frame.

In the both cases we have the **Euler-Poincaré-Suslov** equations

$$\dot{M} = M \times \omega + \gamma \times \frac{\partial V(\gamma)}{\partial \gamma} + \lambda n, \qquad n = a, \gamma,$$

where λ is a Lagrange multiplier, which has to be found from

$$\dot{f} = 0.$$

Conformally Hamiltonian vector field

Because

$$\lambda = -\frac{\left(M \times \omega + \gamma \times \dfrac{\partial V(\gamma)}{\partial \gamma}, \mathbb{I}^{-1} n\right)}{(n, \mathbb{I}^{-1}\, n)} \qquad n = a, \gamma,$$

equations of motion

$$\dot{M} = M \times \omega + \gamma \times \frac{\partial V(\gamma)}{\partial \gamma} + \lambda n, \qquad \dot{\gamma} = \gamma \times \omega$$

have the following form

$$X = P \mathrm{d} H + \lambda N = \Big[(\mathrm{Id} - \Pi_n) P \Big] \mathrm{d} H .$$

So, we have a conformally Hamiltonian vector field

$$X = \mathrm{g} P' \mathrm{d} H$$

Conformally Hamiltonian vector field

In the Suslov case $n = a$ conformal factor is singular.

In the Veselova case $n = \gamma$ conformal factor is a smooth nowhere vanishing bounded function

$$\mathrm{g} = \sqrt{(\gamma, \mathbb{I}^{-1}\gamma)}\,, \quad \mathbb{I} = \mathrm{diag}(\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3)\,, \quad \mathbb{I}_k > 0\,.$$

If

- configuration space is a Lie group G;
- the Euler-Poincaré equations - $Y = P\mathrm{d}H$;
- kinetic energy is a left invariant metric on G;

then we can get similar conformally Hamiltonian vector fields for the LL and LR systems with invariant with respect to left (respectively right) translation constraints.

Lagrange top

Let us consider a heavy symmetric top (Lagrange, 1788)

$$\mathbb{I}_1 = \mathbb{I}_2\,, \qquad V(\gamma) = \gamma_3\,.$$

Integrals of motion

Hamiltonian	conf. Hamiltonian
$(\gamma,\gamma) = 1$	$(\gamma,\gamma) = 1$
$C = (\gamma, M)$	$C = (\gamma, \mathbb{I}^{-1} M)$
$H = (M, M) + \mathbb{I}_3\gamma_3$	$H = (M, M) + \mathbb{I}_3\gamma_3 - (\gamma, M)^2$
$K = M_3$	$K = \mathrm{g} M_3$

$\mathrm{g} = \sqrt{\mathbb{I}_1^{-1}(1-\gamma_3^2) + \mathbb{I}_3^{-1}\gamma_3^2}$ - conformal factor.

Solution

In Hamiltonian case we have the classical Lagrange linearization on an elliptic curve

$$\mathbb{I}_1^2\,\dot\gamma_3^2 = (1-\gamma_3^2)(h - c^2 - \mathbb{I}_1\gamma_3) - (k - c\gamma_3)^2$$

In conformally Hamiltonian case we have genus tree hyperelliptic curve

$$\mathbb{I}_1^2\,\dot\gamma_3^2 = (1-\gamma_3^2)(h - \mathbb{I}_1\gamma_3) - \left(\sqrt{\mathbb{I}_1^{-1}(1-\gamma_3^2) + \mathbb{I}_3^{-1}\gamma_3^2}\,k + c\gamma_3\right)^2$$

In both cases we can get solutions in terms of the Weierstrass elliptic function, because in the second case the Prym variety becomes the Jacobian of elliptic curve.

Lax representation

In Hamiltonian case we have the standard Lax pair

$$\frac{d}{dt}\left(\mathbf{A}u+\mathbf{M}+\frac{\Gamma}{u}\right)=\left[\mathbf{A}u+\mathbf{M}+\frac{\Gamma}{u},\,u\mathbf{A}+\Omega\right]$$

In conformally Hamiltonian case we have the Lax triad

$$\frac{d}{dt}\left(\mathbf{A}u+\mathbf{M}+\frac{\Gamma}{u}\right)=\left[\mathbf{A}u+\mathbf{M}+\frac{\Gamma}{u},\,u\mathbf{A}+\Omega\right]+\lambda\Gamma\,.$$

Here u is a spectral parameter and vectors were replaced by matrices

$$[z_1,z_2,z_3]\to\begin{pmatrix}0 & z_3 & -z_2\\ -z_3 & 0 & z_1\\ z_2 & -z_1 & 0\end{pmatrix}.$$

Yu. B. Suris, On the bi-Hamiltonian structure of Toda and relativistic Toda lattices, 1993.

NLS

In Hamiltonian case real solutions of the Lagrange top give one-gap solutions of NLS$\pm$ equations

$$v_{xx} = iv_t \pm |v|^2 \, v \, .$$

L. Gavrilov, A. Zhivkov, The Complex Geometry of Lagrange Top, 1998.

In conformally Hamiltonian case we can also get one-gap solutions of some new integrable equations.

Infinite-dimensional analogue of the Euler-Poincaré- Suslov deformations is discussed in

P. Guha, Nonholonomic deformation of coupled and supersymmetric KdV equations and Euler-Poincaré-Suslov method, Rev. Math. Phys., 2015.

Deformation of KdV is a sixth-order equation by Karasu-Kalkani et al (2008) and by Kupershmidt (2008).