On the integrability of the sine-Gordon equation

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Motivation: pseudosphere

\[ \Sigma \subset \mathbb{R}^2 \text{ open.} \exists \text{ one to one correspondence (Hilbert):} \]

(i) Immersion \( f : \Sigma \to \mathbb{R}^3, (x,t) \mapsto f(x,t) \) with Gaussian curvature \( \equiv -1 \) (pseudosphere) and \( |\partial_x f| = |\partial_t f| = 1 \).

(ii) solution \( u : \Sigma \to \mathbb{R}, (x,t) \mapsto u(x,t) \) to

\[ u_{xt} = \sin(u). \quad (1) \]
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"(i) \Rightarrow (ii)" the angle \( \angle(\partial_x f, \partial_y f) =: u(x, y) \) is a solution of (1).

"(ii) \Rightarrow (i)" \( \angle(\partial_x, \partial_y) = u(x, y) \) with \( |\partial_x| = |\partial_t| = 1 \)

characterizes metric on \( \Sigma \); integrate Gauss-Mainardi-Codazzi equations to get pseudosphere \( f \).
sine-Gordon Equation

\[ (IVP) \left\{ \begin{array}{l} \partial_t \partial_x u(x, t) = \sin(u(x, t)) \quad t \in \mathbb{R}, \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z} \\ u(0, x) = u_0(x) \quad u(x, t) \in \mathbb{T}_{2\pi} = \mathbb{R}/2\pi\mathbb{Z} \end{array} \right. \]

\[ u \in H^m(\mathbb{T}, \mathbb{T}_{2\pi}), \ m \geq 1: \ \exists \ \tilde{u} \in H^m(\mathbb{T}, \mathbb{R}), \ k \in \mathbb{Z} \text{ s.t.} \]

\[ u(x) = \tilde{u}(x) + 2\pi kx \pmod{2\pi}. \]

**Involution:** \( u \mapsto \tilde{u} + \pi, \ \tilde{u}(x, t) = u(-x, t) \)

\[ u \text{ solution } \Leftrightarrow \tilde{u} + \pi \text{ solution} \]
observation: Any $C^1$ solution $u : t \mapsto u(t, \cdot) \in H^m(\mathbb{T}, \mathbb{T}_{2\pi})$ satisfies

$$\int_{\mathbb{T}} \sin(u) \, dx = \int_{\mathbb{T}} \partial_t \partial_x u \, dx = \partial_t 2\pi k = 0.$$
The phase space

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$$

**Introduce for $m \geq 1$:**

$$
M^m := \left\{ u \in H^m(\mathbb{T}, \mathbb{T}_{2\pi}) : \int_{\mathbb{T}} e^{iu} \, dx \in \mathbb{R} \right\}
$$

$$
M^m = M_{+}^m \cup M_{-}^m \cup M_{0}^m
$$

$$
M_{0}^m \subset M^m : \int_{\mathbb{T}} e^{iu} \, dx = 0
$$

$$
M_{\pm}^m \subset M^m : \pm \int_{\mathbb{T}} e^{iu} \, dx \in \mathbb{R}_{>0}
$$
The phase space

∀ u ∈ \( H^m(\mathbb{T}, \mathbb{T}_{2\pi}) \) consider the line \( u + \mathbb{T}_{2\pi} \) and note,

\[
\int_{\mathbb{T}} e^{i(u + c)} \, dx = e^{ic} \int_{\mathbb{T}} e^{iu} \, dx \quad \forall \ c \in \mathbb{T}_{2\pi}.
\]

case 1: \( \int e^{iu} \, dx = 0 \)

\( u + c \in M^m_0 \quad \forall \ c \in \mathbb{T}_{2\pi} \)

case 2: \( \int_{\mathbb{T}} e^{iu} \, dx \neq 0 \)

\( \exists! c_+ \in \mathbb{T}_{2\pi} \) s.t. \( u + c_+ \in M^m_+ \)

\( \exists! c_- \in \mathbb{T}_{2\pi} \) s.t. \( u + c_- \in M^m_- \)

where \( c_+ = c_- + \pi = - \arg \left( \int_{\mathbb{T}} e^{iu} \, dx \right) \) (mod 2\( \pi \))
parametrization of $M^m_{\pm}$

Introduce new variable $v := \frac{1}{2} u_x$

note that $u \in H^m(\mathbb{T}, \mathbb{T}_{2\pi}) \rightsquigarrow v \in H^{m-1}(\mathbb{T}, \mathbb{R})$

Introduce for $l \geq 0$

$$W^l = \left\{ v \in H^l(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} \exp(2i \partial^{-1}_x v) \, dx \neq 0, \int_{\mathbb{T}} v \, dx \in \pi \mathbb{Z} \right\}$$

where $\partial^{-1}_x$ is the mean zero anti-derivative.

Definition

$$\Psi^+ : W^l \to M^{l+1}_+, v \mapsto 2\partial^{-1}_x v + \gamma(v)$$

$$\Psi^- : W^l \to M^{l+1}_-, v \mapsto (2\partial^{-1}_x v)^\sim + \gamma(v) + \pi$$

where $\tilde{v}(x) = v(-x)$ and $\gamma(v) = -\arg \left( \int_{\mathbb{T}} e^{2i \partial^{-1}_x v} \, dx \right)$
Theorem
For any \( l \geq 0 \), the following holds:

(i) \( \Psi^+ [\Psi^-] \) parametrizes \( M^{l+1}_+ [M^{l+1}_+] \) real analytically.

Equation (1) in \( v \)-coordinates:

\[
v_t = \frac{1}{2} \sin(2\partial_x^{-1}v + \gamma(v)) \quad \text{on } W^l.
\] (2)

(ii) Equation (2) is Hamiltonian with respect to the Gardner bracket, i.e. is equivalent to \( v_t = \partial_x \partial_v H \), with Hamiltonian

\[
H : W^l \rightarrow \mathbb{R}, \, v \mapsto \frac{1}{4} \int_T \cos(2\partial_x^{-1}v + \gamma(v)) \, dx.
\] (3)

(iii) The Hamiltonian \( H \) is in the Poisson algebra of the focusing \( mKdV \) equation and hence (2) is an integrable PDE on \( W^l \).
Corollary

For any $l \geq 0$, equation (2) admits Birkhoff coordinates near 0. In particular, near 0, the Hamiltonian $H$ is a function of the actions alone.
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Sketch of the proof

(i) \([\text{KST08}]\): global real analytic Birkhoff coordinates for defocusing mKdV

(ii) \( \Rightarrow \) existence of Birkhoff coordinates near 0 for focusing mKdV

(iii) all Hamiltonians in the Poisson algebra of focusing mKdV, when expressed in Birkhoff coordinates are in normal form.
Analysis of $M_0^m$

$$M_0^m = \left\{ u \in M^m : \int_T \cos(u) \, dx = 0 \right\}$$
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M_0^m = \left\{ u \in M^m : \int_{\mathbb{T}} \cos(u) \, dx = 0 \right\}
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**Lemma**

\[ \forall \ m \geq 1 \ M_0^m \text{ is a codimension 2 submanifold of } H^m(\mathbb{T}, \mathbb{T}_{2\pi}) \]
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**Lemma**

$\forall \ m \geq 1 \ M_0^m$ is a codimension 2 submanifold of $H^m(\mathbb{T}, \mathbb{T}_{2\pi})$

**Definition**

For all $k \in \mathbb{Z}$ introduce

$$M_0^{m,k} = \{ u \in M_0^m : \exists \hat{u} \in H^m(\mathbb{T}, \mathbb{R}) \text{ s.t.} \quad u(x) = \hat{u}(x) + 2\pi kx \ (\text{mod} \ 2\pi) \}$$

Then

$$M_0^m = \bigcup_{k \in \mathbb{Z}} M_0^{m,k} \quad (\text{disjoint union})$$
Second main result

**Theorem**
\[ \forall k \in \mathbb{Z} \ \exists U \subset M_0^{1,k} \text{ open s.t. } \forall u_0 \in U, \ \forall T > 0 \text{ the initial value problem} \]
\[
\begin{cases}
\partial_t \partial_x u = \sin(u) \\
u(0) = u_0
\end{cases}
\]

admits no \( C^1 \) solution \( u \) in \( C^1([0,T], M^1) \).

**Remark.**
Note that there are elements in \( U \) which are \( C^\infty \).
Examples of elements in $M^1_0$
($M^1_0 = \{ u \in H^1(\mathbb{T}, \mathbb{T}_{2\pi}), \int_{\mathbb{T}} e^{iu} \, dx = 0 \}$)
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Examples of elements in $M_0^1$
($M_0^1 = \{ u \in H^1(T, \mathbb{T}_{2\pi}), \int_T e^{iu} \, dx = 0 \}$)
4. Illposed: \textit{proof}

\[ u : t \mapsto u(t, \cdot) \in M^1 C^1 \text{-smooth solution of (1)}. \]
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\[ u_t = \partial_x^{-1} \sin(u) + \partial_t [u], \text{ with } [u] = \int_T u \, dx \text{ independent of } x \]
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On \( M^m_{\pm} \) (4) is trivially solved when \( u = \Psi^{\pm}(v) \) with \( v \) a solution of (2).
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On \( M^m_{\pm} \) (4) is trivially solved when \( u = \Psi^{\pm}(v) \) with \( v \) a solution of (2). On \( M^m_0 \) (4) reduces to

\[ 0 = \int_T \cos(u) \partial_x^{-1} \sin(u) \, dx \]
When does $0 = \int_{\mathbb{T}} \cos(u) \partial_x^{-1} \sin(u) \, dx$ hold?
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\[2k\pi\]
\[4\pi\]
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