On the integrability of the sine-Gordon equation

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Motivation: pseudosphere

$\Sigma \subset \mathbb{R}^2$ open. $\exists$ one to one correspondence (Hilbert):

(i) Immersion $f : \Sigma \to \mathbb{R}^3, (x,t) \mapsto f(x,t)$ with Gaussian curvature $\equiv -1$ (pseudosphere) and $|\partial_x f| = |\partial_t f| = 1$.

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"$(i) \Rightarrow (ii)$" the angle $\angle(\partial_x f, \partial_y f) =: u(x,y)$ is a solution of (1).

"$(ii) \Rightarrow (i)$" $\angle(\partial_x, \partial_y) = u(x,y)$ with $|\partial_x| = |\partial_t| = 1$ characterizes metric on Σ; integrate Gauss-Mainardi-Codazzi equations to get pseudosphere f.

sine-Gordon Equation

$$(IVP)\begin{cases} \partial_t\partial_x u(x,t) = \sin(u(x,t)) & t \in \mathbb{R},\ x \in \mathbb{T} = \mathbb{R}/\mathbb{Z} \\ u(0,x) = u_0(x) & u(x,t) \in \mathbb{T}_{2\pi} = \mathbb{R}/2\pi\mathbb{Z} \end{cases}$$

$u \in H^m(\mathbb{T}, \mathbb{T}_{2\pi})$, $m \geq 1$: $\exists\ \mathring{u} \in H^m(\mathbb{T}, \mathbb{R})$, $k \in \mathbb{Z}$ s.t.

$$u(x) = \mathring{u}(x) + 2\pi k x \pmod{2\pi}.$$

Involution: $u \mapsto \breve{u} + \pi$, $\breve{u}(x,t) = u(-x,t)$

$$u \text{ solution } \Leftrightarrow \breve{u} + \pi \text{ solution}$$

The phase space

observation: Any C^1 solution $u : t \mapsto u(t,\cdot) \in H^m(\mathbb{T}, \mathbb{T}_{2\pi})$ satisfies

$$\int_{\mathbb{T}} \sin(u)\,\mathrm{d}x = \int_{\mathbb{T}} \partial_t \partial_x u\,\mathrm{d}x = \partial_t 2\pi k = 0.$$

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Introduce for $m \geq 1$:

$$M^m := \left\{ u \in H^m(\mathbb{T},\mathbb{T}_{2\pi}) \ : \ \int_{\mathbb{T}} e^{\mathrm{i}u}\,\mathrm{d}x \in \mathbb{R} \right\}$$

$$M^m = M^m_+ \cup M^m_- \cup M^m_0$$

$$M^m_0 \subset M^m : \ \int_{\mathbb{T}} e^{\mathrm{i}u}\,\mathrm{d}x = 0$$

$$M^m_\pm \subset M^m : \ \pm \int_{\mathbb{T}} e^{\mathrm{i}u}\,\mathrm{d}x \in \mathbb{R}_{>0}$$

The phase space

$\forall\ u \in H^m(\mathbb{T}, \mathbb{T}_{2\pi})$ consider the line $u + \mathbb{T}_{2\pi}$ and note,

$$\int_{\mathbb{T}} e^{\mathrm{i}(u+c)}\,\mathrm{d}x = e^{\mathrm{i}c}\int_{\mathbb{T}} e^{\mathrm{i}u}\,\mathrm{d}x \qquad \forall\ c \in \mathbb{T}_{2\pi}.$$

case 1: $\int e^{\mathrm{i}u}\,\mathrm{d}x = 0$

$$u + c \in M_0^m \qquad \forall\ c \in \mathbb{T}_{2\pi}$$

case 2: $\int_{\mathbb{T}} e^{\mathrm{i}u}\,\mathrm{d}x \neq 0$

$$\exists\,!\, c_+ \in \mathbb{T}_{2\pi} \text{ s.t. } u + c_+ \in M_+^m$$

$$\exists\,!\, c_- \in \mathbb{T}_{2\pi} \text{ s.t. } u + c_- \in M_-^m$$

where $c_+ = c_- + \pi = -\arg\left(\int_{\mathbb{T}} e^{\mathrm{i}u}\,\mathrm{d}x\right) \pmod{2\pi}$

parametrization of $M_\pm^m$

Introduce new variable $v := \frac{1}{2}u_x$
note that $u \in H^m(\mathbb{T}, \mathbb{T}_{2\pi}) \rightsquigarrow v \in H^{m-1}(\mathbb{T}, \mathbb{R})$

Introduce for $l \geq 0$

$$W^l = \left\{ v \in H^l(\mathbb{T}, \mathbb{R}) \ : \ \int_{\mathbb{T}} \exp(2\mathrm{i}\partial_x^{-1} v)\,\mathrm{d}x \neq 0, \int_{\mathbb{T}} v\,\mathrm{d}x \in \pi\mathbb{Z} \right\}$$

where ∂_x^{-1} is the mean zero anti-derivative.

Definition

$$\Psi^+ : W^l \to M_+^{l+1},\ v \mapsto 2\partial_x^{-1} v + \gamma(v)$$
$$\Psi^- : W^l \to M_-^{l+1},\ v \mapsto \left(2\partial_x^{-1} v\right)\breve{} + \gamma(v) + \pi$$

where $\breve{v}(x) = v(-x)$ *and* $\gamma(v) = -\arg\left(\int_{\mathbb{T}} e^{2i\partial_x^{-1} v}\,\mathrm{d}x\right)$

Theorem

For any $l \geq 0$, the following holds:

(i) Ψ^+ $[\Psi^-]$ *parametrizes* M_+^{l+1} $[M_-^{l+1}]$ *real analytically. Equation* (1) *in* v*-coordinates:*

$$v_t = \frac{1}{2}\sin(2\partial_x^{-1}v + \gamma(v)) \quad \text{on } W^l. \tag{2}$$

(ii) *Equation* (2) *is Hamiltonian with respect to the Gardner bracket, i.e. is equivalent to* $v_t = \partial_x \partial_v H$*, with Hamiltonian*

$$H : W^l \to \mathbb{R}, v \mapsto \frac{1}{4}\int_{\mathbb{T}} \cos(2\partial_x^{-1}v + \gamma(v))\,\mathrm{d}x. \tag{3}$$

(iii) *The Hamiltonian H is in the Poisson algebra of the focusing mKdV equation and hence* (2) *is an integrable PDE on* W^l.

Birkhoff coordinates

Corollary

For any $l \geq 0$, equation (2) *admits Birkhoff coordinates near 0. In particular, near 0, the Hamiltonian H is a function of the actions alone.*

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sketch of the proof

(i) [KST08]: global real analytic Birkhoff coordinates for defocusing mKdV

(ii) $\Rightarrow$ existence of Birkhoff coordinates near 0 for focusing mKdV

(iii) all Hamiltonians in the Poisson algebra of focusing mKdV, when expressed in Birkhoff coordinates are in normal form.

Analysis of M_0^m

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Lemma

$\forall\ m \geq 1$ M_0^m is a codimension 2 submanifold of $H^m(\mathbb{T}, \mathbb{T}_{2\pi})$

Definition

For all $k \in \mathbb{Z}$ introduce

$$\begin{aligned} M_0^{m,k} = \{\, u \in M_0^m) :\ \exists\ \mathring{u} \in H^m(\mathbb{T}, \mathbb{R})\ s.t. \\ u(x) = \mathring{u}(x) + 2\pi k x \pmod{2\pi} \,\} \end{aligned}$$

Then

$$M_0^m = \bigcup_{k\in\mathbb{Z}} M_0^{m,k} \quad \text{(disjoint union)}$$

Theorem

$\forall\ k \in \mathbb{Z}\ \ \exists\ U \subset M_0^{1,k}$ open s.t. $\forall\ u_0 \in U,\ \ \forall\ T > 0$ the initial value problem

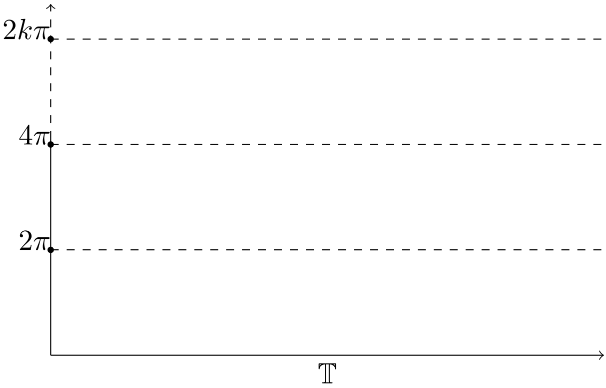
$$\begin{cases} \partial_t \partial_x u = \sin(u) \\ u(0) = u_0 \end{cases}$$

admits no C^1 solution u in $C^1([0,T], M^1)$.

Remark.

Note that there are elements in U which are C^∞.

Examples of elements in M_0^1
$(M_0^1 = \{u \in H^1(\mathbb{T}, \mathbb{T}_{2\pi}), \int_{\mathbb{T}} e^{iu}\,\mathrm{d}x = 0\})$



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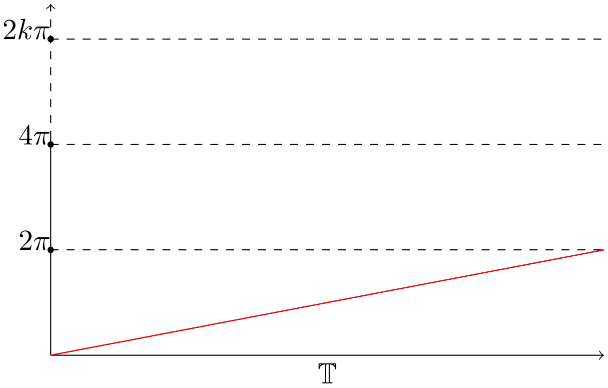

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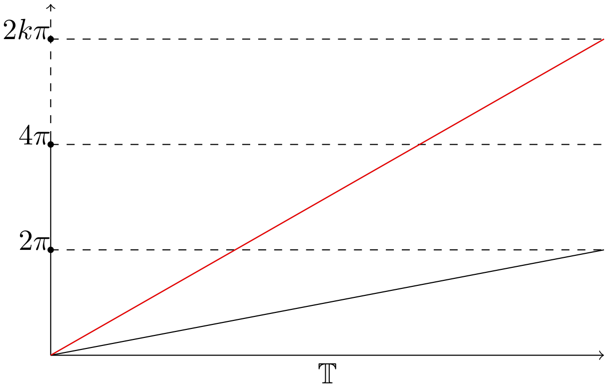

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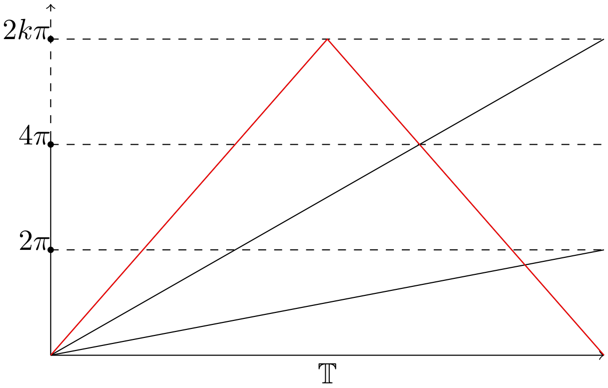

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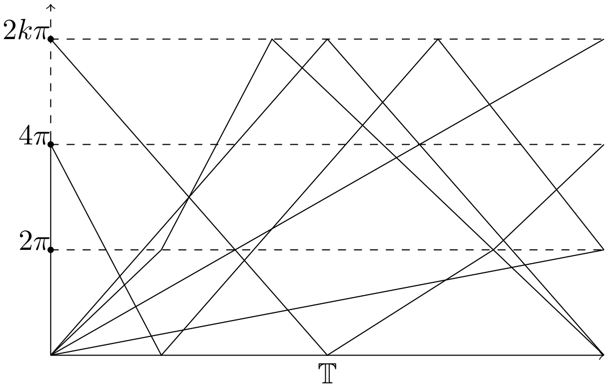

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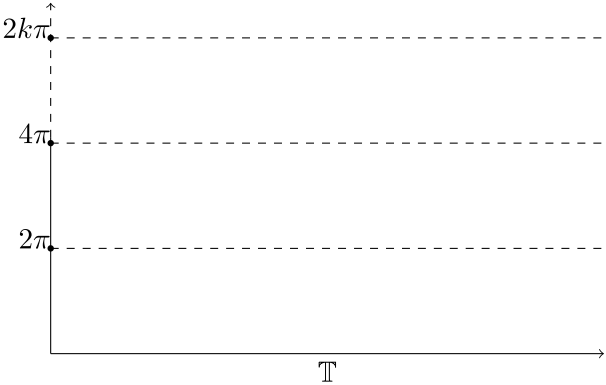
$$0 = \int_{\mathbb{T}} \cos(u)\partial_x^{-1}\sin(u)\,\mathrm{d}x$$

When does $0 = \int_{\mathbb{T}} \cos(u)\partial_x^{-1}\sin(u)\,\mathrm{d}x$ hold?



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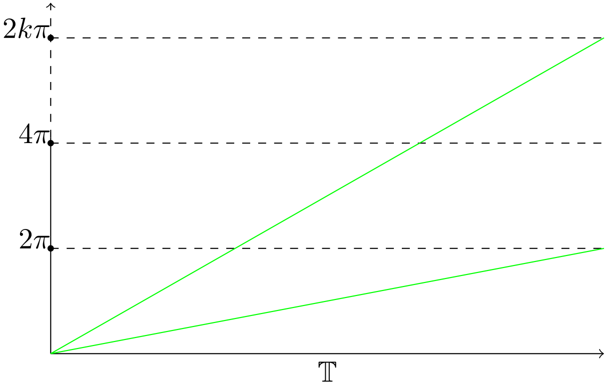

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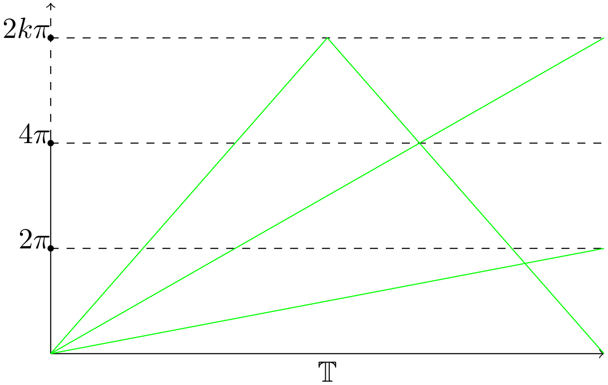

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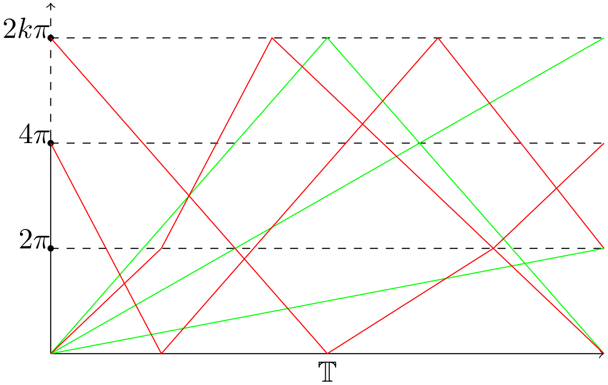

References

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