Quasi-Periodic Schrödinger Cocycles with Positive Lyapunov Exponent are not Open in the Smooth Topology

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$SL(2,\mathbb{R})$ Cocycles

Let X be a $\mathcal{C}^r$ compact manifold, $T : X \to X$ be an ergodic system with normalized invariant measure μ.

An example: $X = \mathbb{S}^1$, $T : x \to x + 2\pi\omega$ with irrational ω.

Given $A : X \to SL(2,\mathbb{R})$. The discrete dynamical system

$$(T, A) : \qquad (x, w) \to (Tx, A(x)w)$$

in $X \times \mathbb{R}^2$ is called a cocycle.

When A is $C^l, l = 0, 1, 2, \cdots, \infty, \omega$, we call (T, A) a C^l cocycle.

Schrödinger cocycles

$A : X \to SL(2, \mathbb{R})$ is of the form

$$A_E(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

which is equivalent to the eigenvalue equations of Schrödinger operators

$$L : l^2 \to l^2$$

$$(L_{v,x}u)_n = u_{n+1} + u_{n-1} + v(T^n x)u_n.$$

Physical background of Schrödinger operators

Conductivity and spectrum:

- Conductor $\longleftrightarrow$ purely absolutely continuous spectrum with bloch waves;
- Insulator $\longleftrightarrow$ pure point spectrum with exponentially decaying eigenfunctions (Anderson localization: Nobel prize).

Materials and potentials:

- Crystal $\longleftrightarrow V(x)$ is periodic;
- Crystal with impurity $\longleftrightarrow V(x)$ is random;
- Quasi-crystal(Nobel prize) $\longleftrightarrow V(x)$ is quasi-periodic.

Quasi-periodic $SL(2,\mathbb{R})$ cocycles

If the base system is a rotation on torus, i.e., $X = \mathbb{T}^n$,

$$T = T_\omega : x \to x + 2\pi\omega$$

with rational independent ω, we call (T_ω, A) a quasi-periodic cocycle.
$X = \mathbb{S}^1$ is the most special case.

Lyapunov Exponents, hyperbolicity

$$A^n(x) = A(T^{n-1}x) \cdots A(Tx)A(x)$$

and

$$A^{-n}(x) = A^{-1}(T^{-n}x) \cdots A^{-1}(T^{-1}x).$$

For fixed (X, T, μ), the (maximum) Lyapunov exponent of (T, A) is defined as

$$L(A) = \lim_{n\to\infty} \int \frac{1}{n} \log \|A^n(x)\| d\mu \in [0, \infty).$$

Hyperbolic: $L(A) > 0$. It is further divided into two cases: uniformly hyperbolic case and non-uniformly hyperbolic case.

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- It is one of the main issues in smooth dynamical systems. The problems is very subtle, which depend on the geometry of the manifold, the base dynamics and the smoothness of the matrix function.

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- It is one of the main issues in smooth dynamical systems. The problems is very subtle, which depend on the geometry of the manifold, the base dynamics and the smoothness of the matrix function.

- It has applications in spectral theory of Schrödinger operators. The LE of the Schrödingier cocycles coming from the eigenvalue equations of quasi-periodic Schödinger operators encodes enormous information on the spectrum. In fact, positive LE implies singular spectrum, often point spectrum and Anderson localization.

Problems: Positivity and regularity of Lyapunov Exponent

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People are interested in

- The positivity of the Lyapunov exponent.
- The regularity of the Lyapunov exponent: Continuity, Hölder continuity, smoothness.
- Is positivity of the LE stable? i.e., whether or not cocycles with positivity LE are open and dense?

Basic facts on Lyapunov Exponent

$L(A)$ is upper semi-continuous, thus it is continuous at generic A. Especially, it is continuous at A with $L(A) = 0$.

It is continuous at the uniformly hyperbolic cocycles.

Problem: the behavior of Lyapunov exponent $L(A)$ at non-uniformly hyperbolic A?

The problem is subtle when considering the quasi-periodic cocycles.

Positivity of LE for Schrödinger cocycles

- Herman 1983: Positivity of Schrödinger cocycles with the potential $2\lambda \cos x$ for $|\lambda| > 1$. Same result is true for λv when v is a triangle polynomial and λ is large enough.

- Sorets and Spencer 1991: The generalization to arbitrary one-frequency nonconstant real analytic potentials.

- Bourgain and Schlag 2000 and Goldstein and Schlag 2001: Same results for *Diophantine* multi-frequency.

Continuity of LE for Schrödingier cocycles

Goldstein and Schlag 2001: if ω is a Diophantine irrational number and $v(x)$ is analytic, then the Lyapunov exponent $L(E)$ is Hölder continuous.

Bourgain 2005, Goldstein and Schlag 2001: for underlying dynamics being a shift or skew-shift of a higher dimensional torus.

Bourgain and Jitomirskaya 2002: if ω is an irrational number and the potential $v(x)$ is analytic, then the Lyapunov exponent is jointly continuous on E and ω.

Jitomirskaya, Koslover and Schulteis 2009: the Lyapunov exponent is a continuous function of general (not necessarily $SL(2,\mathbb{R})$) analytic quasi-periodic cocycles.

Denseness result

Avila 2011: The set of quasi-periodic cocycles with positive LE is dense in the analytic topology, even in the infinite smooth topology.

Together with the continuity, one knows that the set of quasi-periodic cocycles with positive LE is open and dense in the analytic topology.

Applications:

- B-J's result implies that for $V(x) = 2\lambda \cos x$ which corresponds to Almost Mathieu operator, it holds that for

$$L(E, \lambda, \alpha) = \max\{\ln(\lambda), 0\}$$

  for each spectrum point E (Independent of E and α !)

  This formula has been crucial in later proofs of the *Ten Martini problem* (i.e., Cantor spectrum of Almost Mathieu operator) by **Avila and Jitomirskaya.**

- It is also important in Avila's global theory.

A Question

Whether or not the analyticity is necessary?

Discontinuity results: C^0 cocycles

Furman 1998 showed that $L(A) : \mathcal{C}^0(T^d, SL(2,\mathbb{R})) \to [0,\infty)$ is not continuous if and only if (ω, A) is non-uniformly hyperbolic.

Bochi: motivated by Mañé, proved that

$$\{A \in \mathcal{C}^0(T^d, SL(2,\mathbb{R})) : L(A) = 0\}$$

is dense in $\mathcal{C}^0(T^d, SL(2,\mathbb{R}) \backslash \{\text{hyperbolic cocycles}\}$.

These results suggests that discontinuity of L is very common among cocycles with low regularity.

Conclusions

- Lyapunov exponent $L(A)$ of quasi-periodic $SL(2,\mathbb{R})$ cocycles is alway discontinuous in $\mathcal{C}^0$ topology at non-uniformly hyperbolic cocycles. Moreover, any non-uniformly hyperbolic cocycle can not be an inner point.

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- Lyapunov exponent $L(A)$ is always continuous in $\mathcal{C}^\omega$ topology. Moreoever, all non-uniformly hyperbolic cocycles are inner points.
- We remark that, if the base dynamics is "chaotic", usually, the Lyapunov exponent $L(A)$ is quite regular even in lower topology as $\mathcal{C}^0$(Furstenburg, Viana, $\cdots$).

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- Question 3: How to control the LE in smooth category?

Answer to Question 1

$T : \mathbb{S}^1 \to \mathbb{S}^1, \quad x \to x + 2\pi\omega$ with irrational ω of bounded type.

Theorem (Wang-Y., Duke Math J. 2013)

For any $0 \le l \le \infty$, there exists a cocycle $A \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$ such that the Lyapunov exponent is discontinuous at $A(x)$ in $\mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$.

Theorem (Wang-Zhang, JFA 2015)

Lyapunov exponent is continuous at Schrodinger cocycles with $\mathcal{C}^2$ cosine-like quasi-periodic potentials.

The results show, concerning the continuity of Lyapunov exponent, $\mathcal{C}^l$-topology ($l = 1, 2, \cdots, \infty$) is different from C^ω-topology and C^0-topology.

Answer to Question 2

Theorem (Wang-Y., arXiv:1501.05380)

Suppose that α *is a fixed irrational number of bounded-type. For any* $0 < l \leq \infty$ *and* $\lambda \gg 1$, *there exists a* $\mathcal{C}^l$ *Schrödinger cocycle* $S_{\lambda V_l}$ *with a positive Lyapunov exponent and a series of* $\mathcal{C}^l$ *Schrödinger cocycles* $\{S_{\lambda V_k}\}_{k=1}^{\infty}$ *with zero Lyapunov exponents such that* $V_k \to V_l$ *in* C^l *topology.*

Remark. The result shows that there is an essential difference between analytic and smooth cases, since for analytic case, large coupling implies the positivity of LE.

Some observations

Observation : For $\lambda_1, \lambda_2 \gg 1$, the norm of the matrix

$$A(x) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & {\lambda_1}^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi(x) & -\sin\phi(x) \\ \sin\phi(x) & \cos\phi(x) \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & {\lambda_2}^{-1} \end{pmatrix}$$

is determined by the distance for the angle $\phi(x)$ from $\frac{\pi}{2} (\mathrm{mod}\ \pi)$.
For $\phi(x) = 0$, then $\|A(x)\| = \lambda_1\lambda_2$;
for $\phi(x) = \frac{\pi}{2}$, then $\|A(x)\| = \max\{\lambda_1\lambda_2^{-1},\ \lambda_1^{-1}\lambda_2\}$.
Remark.
The points of x such that $\phi(x) \approx \frac{\pi}{2}$ play a critical role in the growth of $\|A(x)\|$.

Product of the hyperbolic matrices

Let $A(x) = \Lambda R_{\phi(x)}$ where $\Lambda = \mathrm{diag}\{\lambda, \lambda^{-1}\}$. A^n is defined to be

$$\Lambda R_{\phi(x+(n-1)\omega)} \cdots \Lambda R_{\phi(x+\omega)} \Lambda R_{\phi(x)}.$$

The norm of A^n depends crucial the behavior of $\phi(x)$ at $\phi(x) = \frac{\pi}{2}$, which is called the critical points.

We start with

$$A(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi(x) & -\sin\phi(x) \\ \sin\phi(x) & \cos\phi(x) \end{pmatrix},$$

where $\lambda(x) > \lambda_0 \gg 1$, and $\phi(x)$ is infinitely smooth and weakly transversal at $\phi(x) = \frac{\pi}{2}$.

The initial $\phi(x)$

The picture of $\phi_0(x)$.

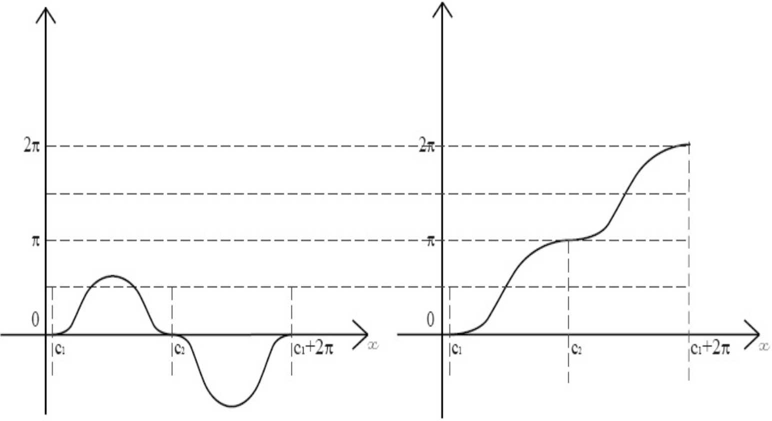


Fig1: homotopic to the Identity

Fig2: non-homotopic to the Identity

Outline of the proof: part I

1. A will be constructed by the limit of $\{A_n(x), n = N, N+1, \cdots\}$ in $\mathcal{C}^l(S^1, SL(2,\mathbb{R}))$.

2. All A_n are of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi_n(x) & -\sin\phi_n(x) \\ \sin\phi_n(x) & \cos\phi_n(x) \end{pmatrix}.$$

3. All $A_n(x)$ possess some kind of finite hyperbolic property, i.e., $\|A_n^{r_n^+}(x)\| \sim \lambda^{r_n^+}$ for most $x \in \mathbb{S}^1$ where $\lambda \gg 1$ and $r_n^+ \to \infty$ as $n \to \infty$, which gives a lower bound estimate $(1-\epsilon)\log\lambda$ of the Lyapunov exponent of $(\omega, A(x))$.

4. All $\phi_n(x), n = N, N+1, \cdots$ has a *degenerate critical point, i.e.,* $\phi_n(c) = \frac{\pi}{2}$*)*.

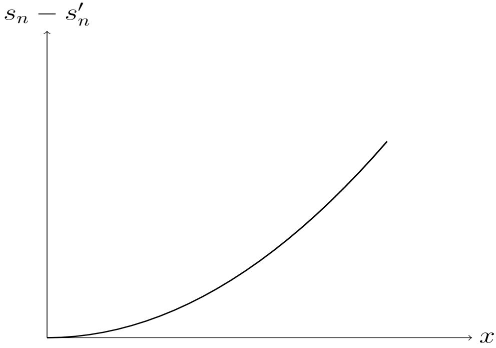
Outline of the proof: part II

1. Smoothly modifying $\phi_n(x)$ locally at the degenerate critical point c, we get $\tilde{\phi}_n(x)$ and thus $\{A_n(x)\}_{n=N}^{\infty}$, we construct another sequence $\{\tilde{A}_n(x)\}_{n=N}^{\infty}$ such that $\tilde{A}_n(x) \to A(x)$ in $\mathcal{C}^l$-topology as $n \to \infty$.

2. The most expanding direction of $\tilde{A}_n$ and the most expanding direction of $\tilde{A}_n^{-1}$ is orthogonal for x in a small interval.

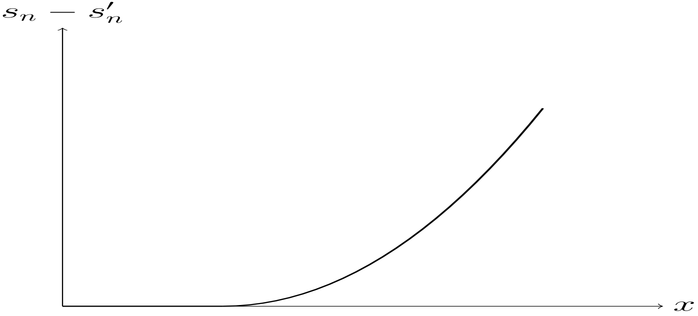
3. For each n, the Lyapunov exponent of $(\omega, \tilde{A}_n(x))$ is less than $(1-\delta)\log\lambda$ with $1 > \delta \gg \epsilon > 0$ independent on λ, which implies the discontinuity of the Lyapunov exponent at $(\omega, A(x))$.

The picture of $s_n - s_n'$



The local picture of $s_n - s_n'$ for $A_n^{r_n}$.

The picture of $\tilde{s}_n - \tilde{s}'_n$



The local picture of $\tilde{s}_n - \tilde{s}'_n$ for old $\tilde{A}_n^{r_n}$; The jump of LE is due to the platform.

Outline of the proof: part III

However, the construction did not tell us how small $L(A_1)$ can be!

For our purpose, we will further locally modify A_1 such that the modified cocycle A_2 satisfies $\|A_2 - A_1\|_{C^l} < \frac{1}{4}\delta$ and $L_{n_2}(A_2) < (1-\delta_1)L_{n_1}(A_1)$. Thus A_2 is in the δ-neighborhood of A and $L(A_2) < (1-\delta_1)^2 L(A)$.

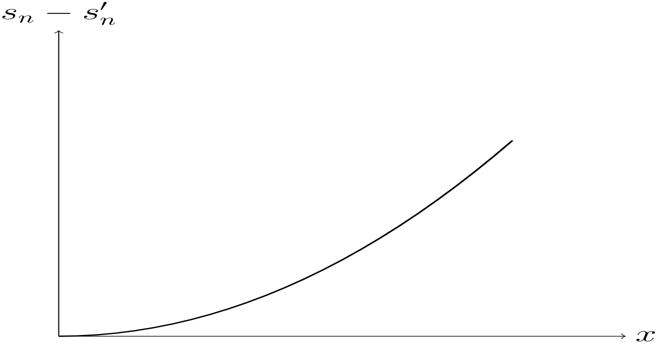
Inductively, we locally modify A_k such that the modified cocycle A_{k+1} satisfies $\|A_{k+1} - A_k\|_{C^l} < \frac{1}{2^k}\delta$ and $L_{n_{k+1}}(A_{k+1}) < (1-\delta_1)L_{n_k}(A_k)$, where $n_k \to \infty$ will be specified later.

Thus A_k are in the δ-neighborhood of A and $L(A_{k+1}) < (1-\delta_1)^k L(A)$. It is easy to see that A_k has a limit, say $\bar{A}$, with $L(\bar{A}) = 0$.

Outline of the proof

- To realize the above induction process, however, modification has to be made on the construction in the proof of discontinuity.
- Note that in the construction of A_1, a precondition is that $s_n(A) - s'_n(A)$ is locally in a good form, i.e., a higher order monomial $\phi_0(x)$. However, since it is not the case for A_1, the process cannot go forward.
- Thus a modification is needed such that the constructed A_1 satisfies the above condition while $L(A_1) < (1-\delta_1)L(A)$ still holds true.

The old construction



The local picture of $s_n - s'_n$ for $A_n^{r_n}$.

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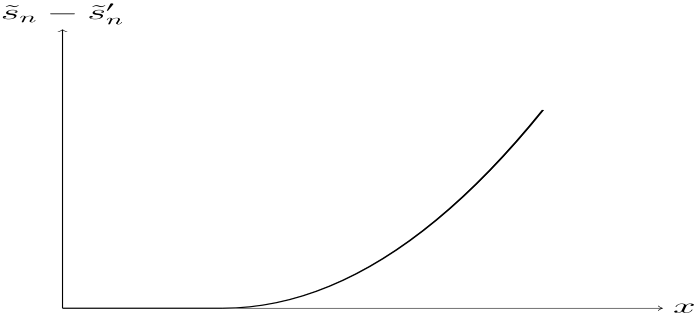
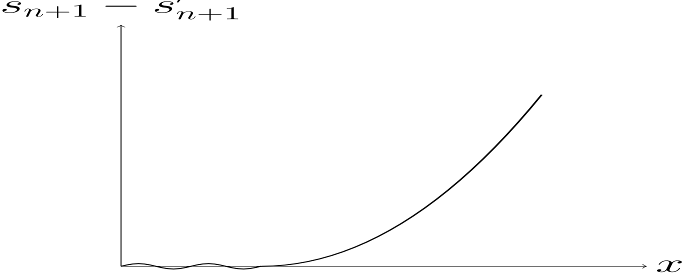


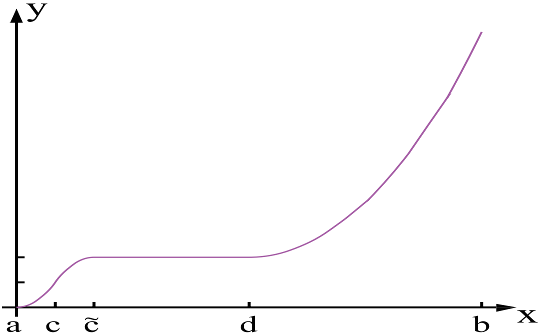
The local picture of $\tilde{s}_n - \tilde{s}'_n$ for old $A_1^{r_n}$; The jump of LE is due to the platform.

The shortcoming of the old construction



For old A_1, the number of critical points is out of control in the following steps.

The new construction



Local picture of $s_n - s_n'$ for new A_1.

The advantage of the new construction

The platform between $x = \tilde{c}$ and $x = d$ is 'low' but not 'too low'.

- On one hand, since the platform is low, we obtain a jump of LE although the jump is some smaller than before.
- On the other hand, since it is not too low, the number of critical points will not increase in the following induction steps.

Then as the induction goes forward, the size of critical intervals becomes smaller than $|ac|$ at some step m. Since the part of the curve $s_n - s_n'$ between $x = 0$ and $x = c$ is of the form of a monomial, so does the new curve $s_m - s_m'$.

Thus we can repeat the above procedure starting with new A_1 and step m.

Further problems

1. Discontinuity in parameters (Simon, Eliasson, Jitomirskaya)

2. Continuous results on LE in smooth topology?
The unique result in smooth case is obtained by Y. Wang and Z. Zhang on C^2 cos-like potentials which was defined by Sinai. If the Lyapunov exponent is continuous in general smooth situation is a question.