Ballistic Diffusion in One-Dimensional Lattice Schrödinger Equation

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Outline

Central problem

For the lattice Schrödinger equation

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V_n q_n, \quad n \in \mathbb{Z},$$

with the potential $\{V_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ independent of time,

what is the growth rate of the “diffusion norm”(Bourgain-Wang)

$$\|q(t)\|_D := \left(\sum_{n\in\mathbb{Z}} n^2 |q_n(t)|^2\right)^{1/2},$$

with $q(0) \neq 0$ and $\|q(0)\|_D < \infty$?

Physical motivation

l^2-conservation law: $\sum_n |q(t)|^2$ is independent of time.

$\|q(0)\|_D < \infty$—the concentration on q_n with $|n|$ not so large

$\|q(t)\|_D$—measures the propagation into q_n, $|n| \gg 1$

Outline

Linear Schrödinger operator

$$\begin{aligned} H : \ell^2(\mathbb{Z}) &\rightarrow \ell^2(\mathbb{Z}), \\ (q_n) &\mapsto (Hq)_n = -(q_{n+1} + q_{n-1}) + V_n q_n \end{aligned}$$

- Decomposition for the spectrum:

$$\sigma(H) = \sigma_{pp} \cup \sigma_{sc} \cup \sigma_{ac}$$

- Decomposition for the Hilbert space:

$$\ell^2(\mathbb{Z}) = \ell^2_{pp} \oplus \ell^2_{sc} \oplus \ell^2_{ac}$$

Dynamical localization

Definition

The operator H exhibits dynamical localization if for any $q(0)$ with $|q_n(0)| \leq ce^{|n|\varrho}$, the solution of equation $\mathrm{i}\dot{q} = Hq$ satisfies

$$\sup_t \sum_{n\in\mathbb{Z}} |n|^s |q_n(t)|^2 < \infty, \quad \forall s > 0.$$

$D.L. \Rightarrow p.p.\ spectrum$, but $p.p.\ spectrum \not\Rightarrow D.L.$

Disordered potential

In general, for the Schrödinger operator H, if the potential $\{V_n\}_n$ is a disordered sequence, dynamical localization has been proven in many cases of pure point spectrum:

Theorem

H has dynamical localization, if

- *(De Bièvre-Germinet 1998, Damanik-Stollmann 2001, etc.)*
  Anderson model:
  $\{V_n\}_n$ is a family of random variables i.i.d., with prob.$= 1$
- *(Germinet-Jitomirskaya 2001, Bourgain 2007)*
  Harper model:
  $V_n = \lambda \cos 2\pi(\theta + n\alpha)$, α Diophantine, $|\lambda| > 2$, a.e. $\theta \in \mathbb{T}$.
- *(Bellissard-Lima-Scoppola 1983, Craig 1983)*
  Maryland model:
  $V_n = \tan \pi(\theta + n\alpha)$, α Diophantine, a.e. $\theta \in \mathbb{T}$.

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Well-ordered potential

If the potential is well ordered(for example, periodic or quasi-periodic but sufficiently small), normally there is the absolutely continuous spectrum.

For the equation $\mathrm{i}\dot{q} = Hq$, how about the diffusion norm of its solution?

A numerical result(Hiramoto-Abe 1988): "ballistic regime"

$$V_n = \lambda \cos 2\pi n\alpha, \quad \alpha = \left(\frac{\sqrt{5}+1}{2}\right)^{-1}, \quad q_n(0) = \delta_{n,0},$$

if $|\lambda| < \lambda_c$, $\|q(t)\|_D \sim t$.

Outline

The theorem

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V_n q_n, \quad n \in \mathbb{Z},$$

with $\{V_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ $p-$periodic, $p \in \mathbb{Z}_+$: $V_{n+p} = V_n, \forall n \in \mathbb{Z}$

Theorem

There exists a constant $0 < C < \infty$, *depending on* $q(0)$ *and* $\{V_n\}_{n=1}^{p}$, *such that*

$$\lim_{t\to\infty} t^{-1} \|q(t)\|_D = C.$$

Outline

Case $p = 1$

Fourier transform: $(q_n)_{n\in\mathbb{Z}} \mapsto g(\theta) = \sum_{n\in\mathbb{Z}} q_n e^{\mathrm{i}n\theta}$, $\theta \in \mathbb{T}$.

$$\begin{aligned} \mathrm{i}\partial_t g(\theta, t) &= \sum_{n\in\mathbb{Z}} [-(q_{n+1}(t) + q_{n-1}(t)) + Vq_n(t)] e^{\mathrm{i}n\theta} \\ &= [-(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}) + V] g(\theta, t) \end{aligned}$$

$\Rightarrow g(\theta, t) = g(\theta, 0) e^{-\mathrm{i}(2\cos\theta + V)t}$.

Parseval's equality:

$$\sum_n n^2 |q_n(t)|^2 = \int_{\mathbb{T}} |\partial_\theta g(\theta, t)|^2 d\theta \sim t^2.$$

Case $p \geq 2$

- Fourier transform: $(q_n)_{n\in\mathbb{Z}} \mapsto g(\theta) = \sum_{n\in\mathbb{Z}} q_n e^{\mathrm{i}n\theta}$, $\theta \in \mathbb{T}$
- Decompose $g(\theta, t) = \sum_{j=1}^{p} g_j(\theta, t)$, with

$$g_j(\theta, t) := \sum_{k\in\mathbb{Z}} q_{kp+j}(t) e^{\mathrm{i}(kp+j)\theta} --- \rightarrow \mathcal{G}(\theta, t) = (g_j(\theta, t))_{j=1}^{p}$$

- $\mathrm{i}\partial_t \mathcal{G}(\theta, t) = A(\theta)\mathcal{G}(\theta, t)$ with

$$A(\theta) := \begin{pmatrix} V_1 & -e^{-\mathrm{i}\theta} & & -e^{\mathrm{i}\theta} \\ -e^{\mathrm{i}\theta} & \ddots & \ddots & \\ & \ddots & \ddots & -e^{-\mathrm{i}\theta} \\ -e^{-\mathrm{i}\theta} & & -e^{\mathrm{i}\theta} & V_N \end{pmatrix},$$

- $A(\theta)$ is Hermitian $\Rightarrow \mathcal{G}(\theta, t) = e^{-\mathrm{i}A(\theta)t}\mathcal{G}(\theta, 0)$.

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Outline

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V(\theta + n\omega)q_n, \quad n \in \mathbb{Z}$$

- $V : \mathbb{T}^d \to \mathbb{R}$ analytic in $\{z \in \mathbb{C}^d : |\mathrm{Im} z| < r \leq 1\}$,
- $\omega \in DC(\gamma, \tau) \subset \mathbb{R}^d$, $\gamma > 0$, $\tau > d - 1$, i.e.,

$$\inf_{j \in \mathbb{Z}} \left| \frac{\langle k, \omega \rangle}{2} - j\pi \right| > \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.$$

Theorem (Eliasson 1992; Hadj Amor 2009; Avila 2008)

There exists $\varepsilon_* = \varepsilon_*(\gamma, \tau, r)$ such that, if $|V|_r < \varepsilon_*$ then for all $\theta \in \mathbb{T}^d$, the spectrum of H is purely absolutely continuous.

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V(\theta + n\omega)q_n, \quad n \in \mathbb{Z}$$

Theorem (with H. Eliasson)

If $|V|_r < \varepsilon_*$, then for any $\theta \in \mathbb{T}^d$, there exist two constants depending on $|V|_r$, θ and $q(0)$, with $0 < C_1 \le C_2 < 2\|q(0)\|_{\ell^2(\mathbb{Z})}$ and $C_2 - C_1 \to 0$ as $|V|_r \to 0$, such that

$$\liminf_{t\to\infty} t^{-1}\|q(t)\|_D \ge C_1, \quad \limsup_{t\to\infty} t^{-1}\|q(t)\|_D \le C_2.$$

Rough idea

$\psi(E) = (\psi_n(E))$, $E \in \sigma(H)$, — generalized eigenvector of H

For $g(E,t) := \sum_n q_n(t)\psi_n(E)$,

$$\mathrm{i}\partial_t g(E,t) := Eg(E,t) \quad \Rightarrow \quad g(E,t) = e^{-\mathrm{i}Et} g(E,0).$$

- if $\psi_n(E)$ is well derived and well estimated, then

$$\sum_n q_n(t)\psi_n'(E) = \partial_E g(E,t) \sim t.$$

- if $\psi_n'(E) = n \cdot (bdd)$, as $(e^{\mathrm{i}n\theta})' = n \cdot (\mathrm{i}e^{\mathrm{i}n\theta})$ for the Fourier transform, then

$$\left\| \sum_n q_n(t)\psi_n'(E) \right\|^2 \sim \sum_n n^2 |q_n(t)|^2.$$

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Outline

Classical spectral transformation

$Hq = Eq$ corresponds to the Schrödinger co-cycle

$$\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} = (A_0(E) + F_0(\theta + n\omega)) \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix},$$

with $A_0(E) = \begin{pmatrix} -E & -1 \\ 1 & 0 \end{pmatrix}$ and $F_0(\theta) = \begin{pmatrix} V(\theta) & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ two sequence $(u_n(E,\theta))_{n\in\mathbb{Z}}$, $(v_n(E,\theta))_{n\in\mathbb{Z}}$ generated by Schrödinger co-cycle $(\omega, A_0(E) + F_0(\theta))$ with

$$\begin{pmatrix} u_1 & v_1 \\ u_0 & v_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For each $\theta \in \mathbb{T}^d$, let $(g_1, g_2) := (\sum_n q_n u_n, \sum_n q_n v_n)$.

Theorem (Pastur-Figotin 1992)

There is a Hermitian matrix of measure $d\mu = (d\mu_{jk})_{j,k=1,2}$*, supported on* $\sigma(H)$*, such that*

$$\int_{\sigma(H)} \sum_{j,k=1,2} g_j \bar{g}_k d\mu_{jk} = \|q\|^2_{\ell^2(\mathbb{Z})},$$

and $d\mu_{11}$*,* $d\mu_{22}$ *are a.c. if* $\sigma(H)$ *is a.c..*

Define $\mathcal{L}^2(\mu) := \left\{(g_1, g_2) : \sum_{j,k=1,2} \int_{\sigma(H)} g_j \bar{g}_k d\mu_{jk}\right\}$

Classical spectral transformation for H:

$$\begin{array}{rccc} \mathcal{S}: & \ell^2(\mathbb{Z}) & \rightarrow & \mathcal{L}^2(\mu) \\ & (q_n)_{n\in\mathbb{Z}} & \mapsto & \left(\sum_{n\in\mathbb{Z}} q_n u_n, \sum_{n\in\mathbb{Z}} q_n v_n\right) \end{array} \quad ------ \text{unitary}$$

Outline

Rotation number of QP Schrödinger cocycle

Bloch-wave: $H\psi(E) = E\psi(E)$, $E \in \sigma(H)$, with

$$\psi_n(E) = e^{in\rho(E)} f(\theta + n\omega, E),$$

- $f : \mathbb{T}^d \times \sigma(H) \to \mathbb{C}$ analytic on $\mathbb{T}^d$
- $\rho = \rho_{(\omega, A_0+F_0)}$ is a function of $E \in \mathbb{R}$
  - continuous
  - increasing between $[0, \pi]$ in the spectrum of H (On the gaps of spectrum, ρ is a constant.)
  - $2\sin\rho \cdot \rho' \geq 1$.

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Reducibility of Schrödinger co-cyle

Theorem (Eliasson 1992; Hadj Amor 2008)

If $|V|_r < \varepsilon_*$, then there exists a subset $\Sigma \subset \sigma(H)$ with $Leb(\sigma(H) \setminus \Sigma) = 0$, such that there are

- $Z : (2\mathbb{T})^d \times \Sigma \to SL(2, \mathbb{R})$ *analytic on* $(2\mathbb{T})^d$
- $B : \Sigma \to SL(2, \mathbb{R})$ *with eigenvalues* $e^{\pm \mathrm{i}\rho}$ *(*$Bv_\pm = e^{\pm \mathrm{i}\rho} v_\pm$*),*

with $Z(\theta + \omega, E)\, B(E)\, Z^{-1}(\theta, E) = A_0(E) + F_0(\theta)$.

Construction of Bloch-wave

Choose $\begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = Z(\theta)v_+$, then we have

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \prod_{j=1}^{n}(A_0 + F_0(\theta + j\omega)) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = e^{in\rho} Z(\theta + n\omega)\, v_+$$

$$\Rightarrow \psi_n = e^{in\rho} f(\theta + n\omega)$$

$$\Rightarrow \begin{cases} u_n = \dfrac{\mathrm{Im}\left[e^{in\rho} f(\theta + n\omega)\overline{f(\theta)}\right]}{\mathrm{Im}\left[e^{i\rho} f(\theta + \omega)\overline{f(\theta)}\right]}, \\ v_n = -\dfrac{\mathrm{Im}\left[e^{i(n-1)\rho} f(\theta + n\omega)\overline{f(\theta + \omega)}\right]}{\mathrm{Im}\left[e^{i\rho} f(\theta + \omega)\overline{f(\theta)}\right]} \end{cases} \quad \text{for } E \in \Sigma$$

Singularities of Bloch-waves

Disadvantage: u_n and v_n are not well differentiated

For free Schrödinger operator $(Hq)_n = -(q_{n+1} + q_{n-1})$,

$$u_n = \frac{\sin n\rho}{\sin \rho}, \quad \rho(E) = \cos^{-1}(-\frac{E}{2})$$

$$\Rightarrow u_n' = \rho' \left(\frac{n \cos n\rho}{\sin \rho} - \frac{\sin n\rho \cdot \cos \rho}{\sin^2 \rho} \right), \rho' = \frac{1}{2 \sin \rho}$$

When ρ approaches 0 and π, ...

Outline

Details for the reducibility

There exists a decomposition $\Sigma = \bigcup_{j\geq 0} \Sigma_j$ with $|\rho(\Sigma_{j+1})| \leq \varepsilon_j^{\frac{\sigma}{2}}$, and a sequence $\{k_j\}_{j\geq 0} \subset \mathbb{Z}^d$ with $k_l = 0$ on Σ_j if $l \geq j$, and

$$0 < |\rho - \frac{1}{2}\sum_{l\geq 0}\langle k_l, \omega\rangle| < \varepsilon_j^{\sigma} \;\; on \;\; \Sigma_{j+1} \left(\Rightarrow \xi := \rho - \frac{1}{2}\sum_{l\geq 0}\langle k_l, \omega\rangle \right)$$

such that

- B, Z are $\mathcal{C}^1_W$ on Σ_0, and

$$|Z - Id.|_{\mathcal{C}^1_W(\Sigma_0),(2\mathbb{T})^d}, \;\; |B - A_0|_{\mathcal{C}^1_W(\Sigma_0)} \leq \varepsilon_0^{\frac{1}{3}},$$

- $\sin^4\xi \cdot B$, $\sin^4\xi \cdot Z$ are $\mathcal{C}^1_W$ on each Σ_{j+1}, $j \geq 0$, and

$$|\sin^{2+2\nu}\xi \cdot Z|_{\mathcal{C}^{\nu}_W(\Sigma_{j+1}),(2\mathbb{T})^d}, \;\; |\sin^{2+2\nu}\xi \cdot B|_{\mathcal{C}^{\nu}_W(\Sigma_{j+1})} \leq \varepsilon_j^{\frac{2\sigma}{3}}, \;\; \nu = 0, 1.$$

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Modified spectral transformation

On Σ, define

- $K_n(E) := \mathrm{Im}(e^{in\rho} h_n \bar{h}_0)$, $J_n(E) := \mathrm{Re}(e^{in\rho} h_n \bar{h}_0)$ with

$$h_n := \begin{cases} f_n, & E \in \Sigma_0 \\ \sin^5 \xi \cdot f_n, & E \in \Sigma_{j+1} \end{cases},$$

  and

$$G(E) = (G_1(E), G_2(E)) := \left(\sum_{n \in \mathbb{Z}} q_n K_n(E), \sum_{n \in \mathbb{Z}} q_n J_n(E) \right)$$

- the matrix of measure

$$d\varphi := \frac{1}{\pi} \begin{pmatrix} (\partial \rho)^{-1} & 0 \\ 0 & (\partial \rho)^{-1} \end{pmatrix} dE$$

Lemma

For any $q \in \ell^2(\mathbb{Z})$ *with* $\|q\|_D < \infty$,

$$1 - \varepsilon_0^{\frac{\sigma^2}{16}} \leq \frac{\|(\sum_n q_n K_n'(E), \sum_n q_n J_n'(E))\|^2_{\mathcal{L}^2(d\varphi)}}{\sum_n (n^2+1)|q_n|^2} \leq 1 + \varepsilon_0^{\frac{\sigma^2}{16}}$$

$(q_n(t))$—solution of $\mathrm{i}\dot{q} = Hq$. Then

$G(E,t) = (\sum_n q_n(t)\, K_n(E), \sum_n q_n(t)\, J_n(E))$

$\Rightarrow \mathrm{i}\partial_t\, G(E,t) = E \cdot G(E,t) \Rightarrow G(E,t) = e^{-\mathrm{i}Et}\, G(E,0)$

Lemma

For any $q(0)$ *with the support finite,*

$$\left(\sum_n q_n(t) K_n'(E), \sum_n q_n(t) J_n'(E)\right) = \partial_E G(E,t) \ \ a.e. \ \ on \ \ \Sigma$$

So for all $q(0)$ with the support finite, we have

$$\frac{\|G(E,0)\|^2_{\mathcal{L}^2(\varphi)}}{1+\varepsilon_0^{\frac{\sigma^2}{16}}} \leq \frac{\sum_n n^2|q(t)|^2}{t^2} \leq \frac{\|G(E,0)\|^2_{\mathcal{L}^2(\varphi)}}{1-\varepsilon_0^{\frac{\sigma^2}{16}}}, \quad t \to \infty.$$

Here, $\|G(E,0)\|^2_{\mathcal{L}^2(\varphi)}$ is almost $\sum_n |q_{n+2} - q_n|^2$.

For any $q(0)$ such that $\sum_n n^2|q_n(0)|^2 < \infty$, it can be approximated by the initial datum with finite support.

Outline

Question

Purely absolutely continuous spectrum $\Rightarrow$ Ballistic motion?

Question

For QP Schrödinger operator

$$H : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}),$$
$$(q_n) \mapsto (Hq)_n = -(q_{n+1} + q_{n-1}) + \lambda \cos 2\pi(\theta + n\omega) q_n$$

and the corresponding equation $\mathrm{i}\dot{q} = Hq$,

Operator H	Equation $\mathrm{i}\dot{q} = Hq, \quad q_n(0) = \delta_{n,0}$
$\lvert\lambda\rvert > 2$, $\ell^2(\mathbb{Z}) = \ell^2_{pp}$	$\lvert\lambda\rvert > \lambda_c$, $\sup_t \lVert q(t)\rVert_D < \infty$
$\lvert\lambda\rvert = 2$, $\ell^2(\mathbb{Z}) = \ell^2_{sc}$	$\lvert\lambda\rvert = \lambda_c$, $\lVert q(t)\rVert_D \sim t^\alpha$, $\alpha \approx \frac{1}{2}$?
$\lvert\lambda\rvert < 2$, $\ell^2(\mathbb{Z}) = \ell^2_{ac}$	$\lvert\lambda\rvert < \lambda_c$, $\lVert q(t)\rVert_D \sim t$?