Lectures of Summer School
Various aspects of mathematical physics, 8-11 August 2016

the Euler International Mathematical Institute, St.-Petersburg,
the St. Petersburg Branch of Steklov Mathematical Institute
St. Petersburg State University.

Anton Baranov, (Russia), *Spectral synthesis for operators and systems*

Spectral synthesis is the possibility of the reconstruction of any invariant subspace of a linear operator from generalized eigenvectors that it contains. Another version of the spectral synthesis problem is the reconstruction of a vector in a Hilbert space from its Fourier series with respect to some complete and minimal system.

It was a long-standing problem in the nonharmonic Fourier analysis whether any complete and minimal system of exponentials in $L^2(-\pi,\pi)$ has the spectral synthesis property. Namely, given a complete and minimal system of exponentials $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ with the biorthogonal system $\{g_\lambda\}$, is it true that any function $f \in L^2(-\pi,\pi)$ belongs to the closed linear span of its 'harmonics' $(f, g_\lambda)e^{i\lambda t}$. Recently, we answered this question in the negative. At the same time it was shown that the spectral synthesis for exponential systems always holds up to one-dimensional defect.

We also discuss the spectral synthesis problem for systems of reproducing kernels in some Hilbert spaces of entire functions (the exponential systems correspond to the classical Paley–Wiener space). These include de Branges spaces where the problem is equivalent to the spectral synthesis for rank one perturbations of compact selfadjoint operators, and the Bargmann–Fock space. The talk is based on joint works with Yurii Belov, Alexander Borichev, and Dmitry Yakubovich.

Alexander Kiselev, (USA) *Regularity and blow up in ideal fluid*

The incompressible Euler equation of fluid mechanics has been derived in 1755. It is one of the central equations of applied analysis, yet due to its nonlinearity and non-locality many fundamental properties of the solutions remain poorly understood. In particular, the global regularity vs finite time blow up question for incompressible three dimensional Euler equation remains open.

In two dimensions, it has been known since 1930s that solutions to Euler equation with smooth initial data are globally regular. The best available upper bound on the size of derivatives of the solution has been double exponential in time. I will describe a construction showing that such fast generation of small scales can actually happen, so that the double exponential bound is qualitatively sharp.

This work has been motivated by numerical experiments due to Hou and Luo who propose a new scenario for singularity formation in solutions of 3D Euler equation. The scenario is axi-symmetric. The geometry of the scenario is related to the geometry of 2D Euler double exponential growth example and involves hyperbolic points of the flow located at the boundary of the domain. If time permits, I will discuss some recent attempts to gain insight into the three-dimensional fluid behavior in this scenario.

Sergei Kuksin (France-Russia) *The long-time behaviour of small-amplitude solutions for multidimensional hamiltonian PDEs under periodic boundary conditions.*

I will discuss the problem of long-time behaviour of solutions for nonlinear Hamiltonian PDEs on a d-dimensional torus, $d \geq 1$. I will explain that in certain sense the behaviour of small-amplitude solutions for space-multidimensional equations (d>1) significantly differs from that of solutions for the 1d systems. Namely, the former are significantly more stochastic than the latter". The results are rigorously proven for the nonlinear beam equation, but the approach is rather general. The talk is based on my recent joint work with H. Eliasson and B.Grebert, arXiv 1604.01657

Evgeny Korotyaev, (Russia), *Resonances and entire functions*

We consider resonances 1-dim self-adjoint operators with compactly supported coefficients: Schrödinger operators, Dirac operators, Stark operators, Euler-Bernoulli operator. We discuss the relationship between resonances and zeros of entire functions. We determine the class of corresponding functions.

We describe asymptotics of a counting function of resonances in complex discs at large radius. Moreover, we consider the forbidden domain for resonances and obtain trace formulas in terms of resonances. We consider the problem to recover the potential using the resonances.

Mark Malamud, (Ukraine) *Schrodinger and Dirac operators with δ-interactions*

Consider one dimensional Schrodinger differential expression with δ-interactions

$$l_{X,\alpha} = -\frac{d^2}{dx^2} + \sum_{k=1}^{\infty} \alpha_k \delta(x - x_k). \tag{1}$$

Here $X = \{x_k\}_1^\infty \subset (0, \infty)$ is the set of point interactions, $\alpha = \{\alpha_k\}_1^\infty \subset R$ the set of intensities. With expression (1) one associates a (closed) minimal symmetric operator $H_{X,\alpha}$ in $L^2(R_+)$. It turns out that certain spectral properties of $H_{X,\alpha}$ correlate with that of a special Jacobi matrix $J_{X,\alpha}$ in $\ell^2(N)$. For instance, the negative part of the spectrum $\sigma(H_{X,\alpha})$ is bounded, discrete, finite, or empty if and only if so is the negative part of the spectrum $\sigma(J_{X,\alpha})$ of $J_{X,\alpha}$. The absolutely continuous and singular parts of $H_{X,\alpha}$ will also be discussed.

Similar connection between Dirac operator with δ-interactions and a certain class of Jacobi matrices will be discussed too. The talk is based on results of [1]-[4].

[1] A.S. Kostenko, M.M. Malamud, 1–D Schrödinger operators with local point interactions on a discrete set, J. Diff. Eq-s, v. 249, (2010), 253–304.

[2] M.M. Malamud, H. Neidhardt, On the unitary equivalence of absolutely continuous parts of self-adjoint extensions, J. Funct. Anal., v. 260, No 3 (2011), p. 613–638.

[3] M.M. Malamud, H. Neidhardt, Sturm-Liouville boundary value problems with operator potentials and unitary equivalence, J. Diff. Eq-s, v.252 (2012), p. 5875-5922.

[4] R. Carlone, M. Malamud, A. Posilicano, On the spectral theory of Gesztesy–Šeba realizations of 1–D Dirac operators with point interactions, J. Diff. Eq-s., **v. 254** (2013), p. 3835-3902.

Andrei Mironov (Russia), *Baker-Akhiezer function and the Korteweg - de Vries equation*

Many of soliton equations, resulting in geometry and mathematical physics, and integrated with assistance the Baker-Akhiezer function. The lecture will be explained as using the Baker-Akhiezer function defined on a singular algebraic curve construct N soliton solutions of the Korteweg - de Vries equation.

Jacob Möller, (Danmark), *Local Spectral Deformation*

The topic of the talk is analytic perturbation theory of embedded eigenvalues for self-adjoint operators. After briefly reviewing (part of) Kato's analytic perturbation theory of isolated eigenvaues, we shall see two examples illustrating that the usual picture may break down for embedded eigenvalues.

It is wellknown that the usefulness of dilation analyticity is intimately connected with the role of the generator of dilation in Mourre theory for Schrodinger operators. If one expands the dilated Hamiltonian in powers of a dilation parameter, the leading order correction to the Hamiltonian is the commutator between the Hamiltonian and the dilation generator. In principle, a Mourre estimate should therefore cause the non-normal dilated Hamiltonian to form a gap in its essential spectrum, locally where a Mourre estimate is valid. This will lay bare embedded eigenvalues and enable an application of Kato's analytic perturbation theory.

After reviewing the connection between complex dilations and Mourre's commutator method, we proceed to set up an abstract theory of analytic spectral deformations local in energy. If time permits, we shall apply the abstract theory to two-body dispersive system where we show that the non-threshold part of the embedded energy-momentum point spectrum form a semi-analytic subset of the complement of the threshold set in energy-momentum space.

The presentation is based on joint work with Matthias Engelman and Morten Grud Rasmussen.

Vladimir Peller, (Russia-USA), *Functions of operators and their perturbations*

I am planning to speak about a functional calculus $f \mapsto f(A)$ for self-adjoint operators A on Hilbert space and functions f on the real line. We consider the problem of the behaviour of such functions under perturbations of A. The problem becomes meaningful and interesting even if we restrict ourselves to the case of functions of operators on finite-dimensional spaces, or, in other words, functions of matrices.

In particular, we will consider the natural problem to study the class of functions f, for which

$$\|f(A) - f(B)\| \le C\|A - B\|.$$

Such functions are called *operator Lipschitz.*

I will also consider similar problems for functions of normal operators (matrices) and other generalizations.

Erik Skibsted, (Denmark), *Stationary scattering theory on manifolds*

We present a stationary scattering theory for the Schrödinger operator on Riemannian manifolds with the structure of *ends* each of which is equipped with an *escape function* (for example a convex distance function). This includes manifolds with ends modeled as cone-like subsets of the Euclidean space and/or the hyperbolic space. Our results include Rellich's theorem, the limiting absorption principle, radiation condition bounds, the Sommerfeld uniqueness result, and we give complete characterization/asymptotics of the generalized eigenfunctions in a certain Besov space and show asymptotic completeness.

The talk is based on our joint work with K. Ito (Kobe University).

Stanislav Smirnov (Swizerland-Russia) (TBA)

Gunter Stolz, (USA), *An introduction to the mathematics of the XY spin chain*

Quantum spin systems provide some of the few models of quantum many-body systems which allow rigorous mathematical investigations. They also pose numerous interesting open problems in spectral theory. Specifically, we will introduce the XY chain, the simplest example of a quantum spin system. The XY chain can be studied via the Jordan-Wigner transform, which reduces it to a free Fermion system and the analysis of an effective one-particle operator in the form of a Jacobi matrix. This leads to a rather complete understanding of the spectral and scattering theory of the XY chain. For other important examples of quantum spin systems, such as the Heisenberg model, many of these questions are wide open.

Tanya Suslina, (Russia), *Spectral Approach to Homogenization of Periodic Differential Operators*

The talk is devoted to the operator-theoretic (spectral) approach to homogenization suggested by M. Birman and T. Suslina.

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider matrix elliptic second order differential operators $A = A(\mathbf{x},\mathbf{D})$ admitting a factorization of the form $A = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$. Here an $(m \times m)$-matrix-valued function $g(\mathbf{x})$ is bounded, positive definite, and periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. Next, $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$, where b_l are constant $(m \times n)$-matrices. It is assumed that $m \geq n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{l=1}^d b_l \xi_l$ has rank n for any $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$.

We study the operator $A_\varepsilon := A(\mathbf{x}/\varepsilon, \mathbf{D})$ for small $\varepsilon > 0$. It turns out that the resolvent $(A_\varepsilon + I)^{-1}$ converges in the L_2-operator norm to the resolvent of the *effective operator* $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$. Here g^0 is a constant positive *effective matrix*. We prove that

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C\varepsilon. \tag{1}$$

Also, we obtain more accurate approximation of the resolvent:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C\varepsilon^2, \tag{2}$$

and approximation of the resolvent in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leq C\varepsilon. \tag{3}$$

Here $K(\varepsilon)$ and $K_1(\varepsilon)$ are the so called *correctors*; they contain rapidly oscillating factors and so depend on ε. Estimates (1)–(3) are order-sharp.

The method is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory. By the scaling transformation, the problem is reduced to approximation of the operator $\varepsilon^2(A + \varepsilon^2 I)^{-1}$. The operator A admits expansion in the direct integral of the operators $A(\mathbf{k})$ acting in $L_2(\Omega;\mathbb{C}^n)$ (where Ω is the cell of the lattice Γ). The operator $A(\mathbf{k})$ is given by the expression $b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k})$ with periodic boundary conditions. This operator family is studied by means of the analytic perturbation theory. It is possible to approximate the resolvent $(A(\mathbf{k}) + \varepsilon^2 I)^{-1}$ in terms of the spectral characteristics near the bottom of the spectrum. This shows that homogenization can be treated as a *spectral threshold effect*.

General results are applied to specific operators of mathematical physics.

Iskander Taimanov, (Russia), *The Darboux-Moutard transformations*

We expose the definitions of the Darboux–Moutard transformations and different applications of them to constructing integrable Schrodinger operators, two-dimensional Schrodinger and Dirac operators with interesting spectral properties, and blowup solutions of two-dimensional solitons equations.

Vitaly Tarasov , (Russia-USA), *Quantum integrable models: Heisenberg spin chain and around*

The talk is an introduction to the Bethe ansatz method for quantum integrable models. Basic examples are the XXX chain and the Gaudin model. We will discuss algebraic aspects of the Bethe ansatz as well as it relation to other areas.

Vyacheslav Yurko, (Russia), *Inverse spectral problems for differential operators*

The lecture is devoted to spectral theory for differential operators. We pay the main attention to nonlinear inverse spectral problems of recovering coefficients of differential operators (potentials) from the given spectral characteristics. Inverse problems for differential operators arise in various problems of mathematics as well as in applications in natural sciences and engineering. The main results and methods for this class of inverse problems will be presented.