Discrete pluri-Lagrangian systems and discrete pliriharmonic functions

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SFB TRR 109 Discretization in Geometry and Dynamics

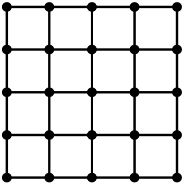
Variational principles in discrete differential geometry

- Variational principles of discrete integrability
- Convex variational principles
  - uniqueness (implied)
  - existence (helps)
  - construction (numerics by minimization)
- Beyond the integrability:
  - geometric interpretation (discrete complex analysis)
  - applications (texture mapping,...)

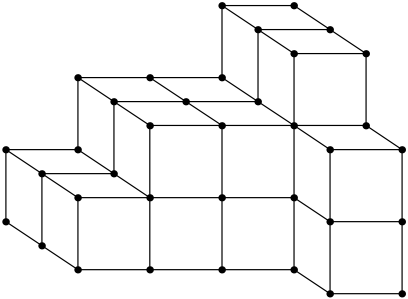
Papers

- Bobenko, Suris. Discrete pluriharmonic functions as solutions of linear pluri-Lagrangian systems. Comm. Math. Phys. 2015
- Bobenko, Skopenkov. Discrete Riemann surfaces: linear discretization and its convergence, J. reine angew. Math. 2016
- Bobenko, Günther, Discrete Riemann surfaces based on quadrilateral cellular decompositions, Adv. in Math., 2017

Quad-graph equation



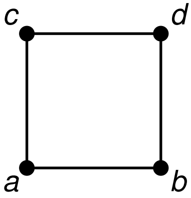
$\mathbb{Z}^2$ lattice



quad-graph

Integrability as Consistency

- Equation



$$f(a, b, c, d) = 0$$

- Consistency

Integrability as Consistency

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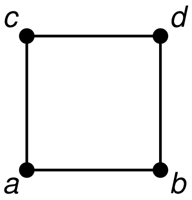

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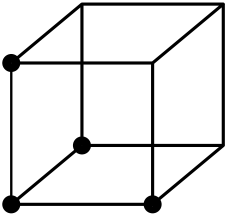

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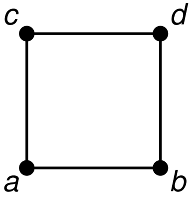

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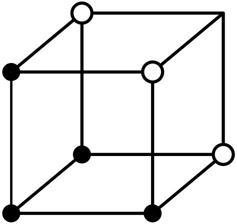

Integrability as Consistency

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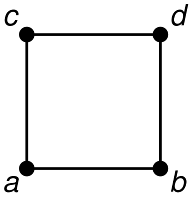

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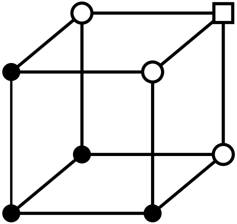
- Consistency

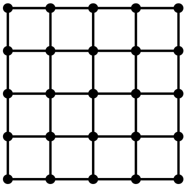


Can be derived from consistency

- Lax representation, Bäcklund-Darboux transformations
- Classification of 3D-consistent multi-linear systems [ABS '09]
- Master equation Q4

$$\operatorname{sn}(\alpha)\operatorname{sn}(\beta)\operatorname{sn}(\alpha-\beta)(k^2xx_ix_jx_{ij}+1)+\operatorname{sn}(\alpha)(xx_i+x_jx_{ij}) \\ -\operatorname{sn}(\beta)(xx_j+x_ix_{ij})-\operatorname{sn}(\alpha-\beta)(xx_{ij}+x_ix_j)=0.$$

Quad-graph equation



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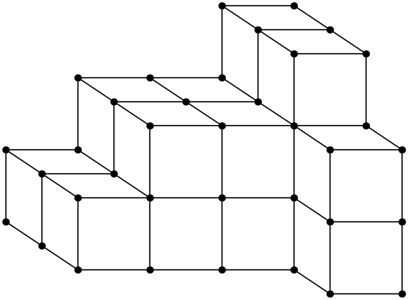


quad-graph

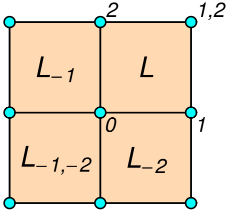
Variational equations are on stars and not on quads.
Integrability of variational equations?

Lagrangian systems on $\mathbb{Z}^2$

- Lagrangian equation

$$\partial_u(L + L_{-1} + L_{-2} + L_{-1,-2}) = 0,$$

for some function $L(u, u_1, u_2, u_{12})$ on the quad $\sigma^{12} = (n, n + e_1, n + e_1 + e_2, n + e_2)$.



pluri-Lagrangian systems

Definition

Let L be a discrete 2-form (Lagrangian), defined on the oriented 2-dimensional faces σ^{ij} of $\mathbb{Z}^d$, $L_{\sigma_{ij}} = -L_{\sigma_{ji}}$. It depends on fields $u : \mathbb{Z}^m \to \mathbb{R}$ at vertices. The action on a 2-dimensional surface Σ in $\mathbb{Z}^m$ is given by

$$S_\Sigma = \sum_{\sigma^{ij} \in \Sigma} L_{\sigma_{ij}}.$$

The field $u : \mathbb{Z}^m \to \mathbb{R}$ satisfying the Euler-Lagrange equations $\delta S_\Sigma(u) = 0$ for any 2-dimensional surface Σ in $\mathbb{Z}^d$ is called a solution of the pluri-Lagrangian problem.

Origins

- Pluriharmonic functions h.
  $h : \mathbb{C}^n \to \mathbb{R}$, for any analytic curve $C \subset \mathbb{C}^n$ the function $h_{|C}$ is harmonic.
  $h = \mathrm{Re}\, f$ of holomorphic $f : \mathbb{C}^n \to \mathbb{C}$
- Baxter's Z-invariance $\to$ quasiclassical limit [Bazhanov at al.]
- Lagrangian multi-form structure of discrete integrable equations [Lobb, Nijhoff]
- Laplace equations of quad-equations are pluri-Lagrangian [ABS]

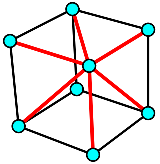
Discrete pluriharmonic functions as solution of linear pluri-Lagrangian systems.

Definition

Discrete pluriharmonic function is a solution of a discrete pluri-Lagrangian system, where the Lagrangian $L_{\sigma_{ij}}$ is a quadratic form of 4 variables at vertices $L_{\sigma_{ij}}(u, u_i, u_j, u_{ij})$. The action $S_\Sigma = \sum_{\sigma^{ij} \in \Sigma} L_{\sigma_{ij}}$ is discrete Dirichlet energy.

- linear EL equations but non-trivial example
- 10 coefficients per face
- $\to$ integrable systems and important discrete Laplace operator

Elementary Euler-Lagrange equations. 3D-corner equations



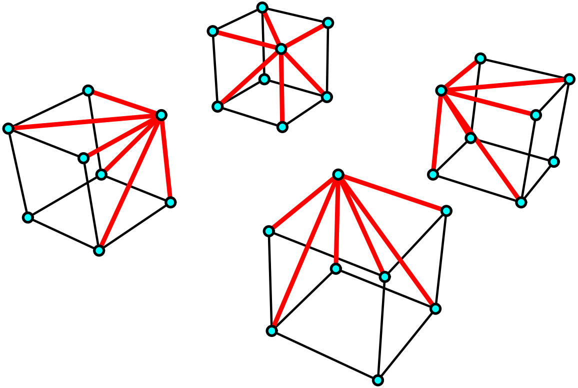
In particular

$$(dL)^{ijk} = \Delta_i L_{\sigma_{jk}} + \Delta_j L_{\sigma_{ki}} + \Delta_k L_{\sigma_{ij}}, \quad \Delta_i = T_i - 1$$

the action on an elementary 3-cube implies the Euler-Lagrange equations on 3D-corners. We call them elementary Euler-Lagrange equations.

From 3D-corners to a surface

The elementary Euler-Lagrange equations imply the Euler-Lagrange equations on an arbitrary 2-dimesional surface



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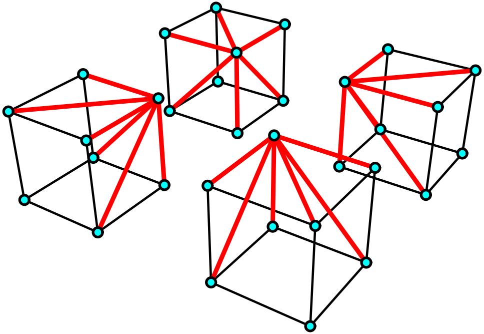

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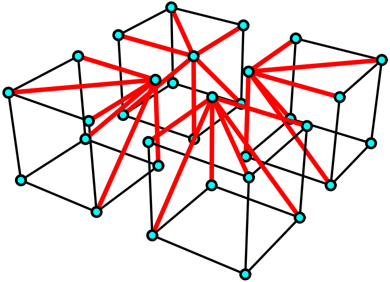

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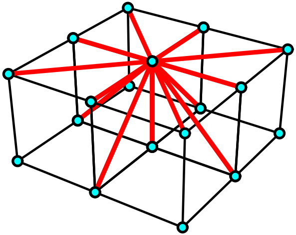

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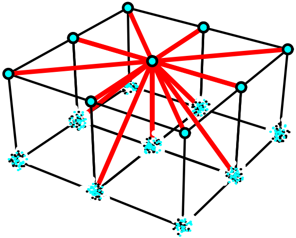

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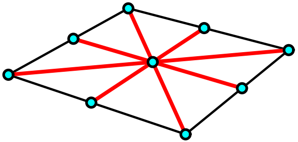

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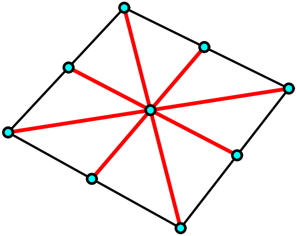

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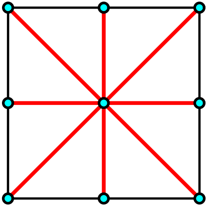


Action invariance

Theorem

The Dirichlet energy of a discrete pluriharmonic function over an elementary cube is zero

$$(dL)^{ijk} = \Delta_i L_{\sigma_{jk}} + \Delta_j L_{\sigma_{ki}} + \Delta_k L_{\sigma_{ij}} = 0.$$

Proof.

$$\delta dL = 0 \rightarrow dL = \mathrm{const} \rightarrow dL = 0,$$

evaluate on the zero solution.

Closed 2-form $L_{\sigma_{ij}}$ on pluriharmonic functions.

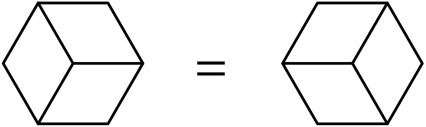
Action invariance

Theorem

The discrete Lagrangian 2-form $L_{\sigma_{ij}}$ is closed on pluriharmonic functions. The Dirichlet energy of a discrete pluriharmonic function on a 2-dimensional surface Σ with a fixed boundary is independent of the choice of the surface

$$S_{\Sigma}(u) = S_{\Sigma'}(u).$$

- flip-invariance



Diagonal quadratic Lagrangians

- Problem 1. Classify Dirichlet energies of discrete pluriharmonic functions (unsolved)
- Problem 2. Classify Dirichlet energies of discrete pluriharmonic functions with diagonal Lagrangians

$$L_{\sigma_{ij}} = \frac{1}{2}\alpha_{ij}(u_{ij} - u)^2 + \beta_{ij}(u_{ij} - u)(u_i - u_j) + \frac{1}{2}\gamma_{ij}(u_i - u_j)^2$$

Conjugate discrete pluriharmonic function

Equations

$$
\begin{aligned}
v_i - v_j &= -\frac{\partial L^{ij}}{\partial u} = \frac{\partial L^{ij}}{\partial u_{ij}} = \alpha^{ij}(u_{ij} - u) + \beta^{ij}(u_i - u_j) \\
v_{ij} - v &= -\frac{\partial L^{ij}}{\partial u_i} = \frac{\partial L^{ij}}{\partial u_j} = -\beta^{ij}(u_{ij} - u) - \gamma^{ij}(u_i - u_j)
\end{aligned}
$$

are consistent (by virtue of the corner equations for u) and define the function $v : \mathbb{Z}^m \to \mathbb{C}$, called *conjugate pluriharmonic function*.

Another representation (vector Moutard equation)

$$
\begin{pmatrix} u_{ij} - u \\ v_{ij} - v \end{pmatrix} = A^{ij} \begin{pmatrix} u_i - u_j \\ v_i - v_j \end{pmatrix}, \quad A^{ij} = \begin{pmatrix} b^{ij} & a^{ij} \\ c^{ij} & b^{ij} \end{pmatrix},
$$

$$
b^{ij} = -\frac{\beta^{ij}}{\alpha^{ij}}, \quad a^{ij} = \frac{1}{\alpha^{ij}}, \quad c^{ij} = \frac{(\beta^{ij})^2 - \alpha^{ij}\gamma^{ij}}{\alpha^{ij}}.
$$

Non-commutative star-triangle equation

Lemma

Vector Moutard equation is consistent if and only if the matrix coefficients A_i satisfy

$$A_i A_j^{-1} A_k = A_k A_j^{-1} A_i,$$

and then the matrices $\hat{A}_i$ are given by *non-commutative star-triangle relations*

$$-\hat{A}_i^{-1} = A_j + A_k + A_k A_i^{-1} A_j,$$

Notations: $A_i := A^{jk}$, and $\hat{A}_i = T_i A_i$ is shifted A_i (on the opposite face).

Solution

Matrices A^{ij} solve a consistent system of vector Moutard equations iff their entries satisfy

$$\lambda a^{ij} + \mu(-1)^{|n|} b^{ij} + \nu c^{ij} = 0$$

for some fixed triple (λ, μ, ν), where $|n| = n_1 + \ldots + n_m$, and their evolution is expressed through a solution of *coupled star-triangle relations*

$$\frac{1}{p_k^{ij}} = \frac{\lambda}{\nu} \cdot \frac{q^{ij}q^{jk} + q^{jk}q^{ki} + q^{ki}q^{ij}}{q^{ij}}, \quad \frac{1}{q_k^{ij}} = \frac{\lambda}{\nu} \cdot \frac{p^{ij}p^{jk} + p^{jk}p^{ki} + p^{ki}p^{ij}}{p^{ij}},$$

via the following relations:

$$p^{ij} = a^{ij} + \xi b^{ij}, \quad q^{ij} = a^{ij} + \eta b^{ij}, \quad p_k^{ij} = a_k^{ij} - \xi b_k^{ij}, \quad q_k^{ij} = a_k^{ij} - \eta b_k^{ij},$$

where $-\xi$, $-\eta$ are the two roots of the quadratic equation $\lambda\xi^2 + \mu\xi + \nu = 0$.

Classification theorem

Theorem [BS '15]

Lagrangians

$$L^{ij} = \frac{1}{2}\alpha^{ij}(u_{ij} - u)^2 + \beta^{ij}(u_{ij} - u)(u_i - u_j) + \frac{1}{2}\gamma^{ij}(u_i - u_j)^2.$$

define pluriharmonic functions, iff their coefficients satisfy

$$\lambda - \mu(-1)^{|n|}\beta^{ij} + \nu((\beta^{ij})^2 - \alpha^{ij}\gamma^{ij}) = 0$$

for some fixed triple (λ, μ, ν), where $|n| = n_1 + \ldots + n_m$, and their evolution is expressed through a solution of *the coupled star-triangle relations* with

$$p^{ij} = \frac{1 - \xi\beta^{ij}}{\alpha^{ij}},\ q^{ij} = \frac{1 - \eta\beta^{ij}}{\alpha^{ij}},\ p^{ij}_k = \frac{1 + \xi\beta^{ij}_k}{\alpha^{ij}_k},\ q^{ij}_k = \frac{1 + \eta\beta^{ij}_k}{\alpha^{ij}_k}.$$

Special case: discrete complex analysis

$\mu = 0$, $\lambda/\nu > 0$, wlog

$$(\beta^{ij})^2 - \alpha^{ij}\gamma^{ij} = -1.$$

The Lagrangians can be parametrized as

$$\alpha^{ij} = \frac{1}{\Re(c^{ij})}, \quad \beta^{ij} = \frac{\Im(c^{ij})}{\Re(c^{ij})}, \quad \gamma^{ij} = \frac{|c^{ij}|^2}{\Re(c^{ij})},$$

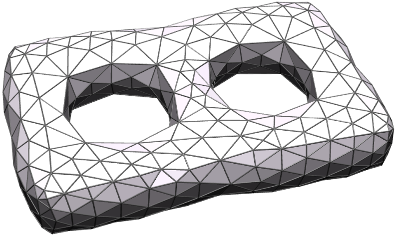
where c^{ij} is a complex-valued solution of the star-triangle relation

$$c_k^{ij} := T_k c^{ij} = \frac{c^{ij}}{c^{ij}c^{jk} + c^{jk}c^{ki} + c^{ki}c^{ij}}.$$

This is the case of discrete complex analysis based on a discretization of the Riemann-Cauchy equations.

Polyhedral surfaces as RS

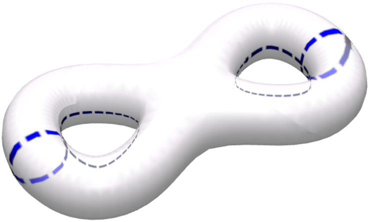
- polyhedral metric $\to$ RS
  z^a-coordinate at conical singularities, plane z-coordinate at regular points



- Every RS can be induced by an abstract polyhedral metric (flat metric with conical singularities). Troyanov ['86]
- Every abstract polyhedral metric can be realized as a polyhedral surface embedded in $\mathbb{R}^3$. Burago-Zalgaller ['60]
- Every RS can be realized as a polyhedral surface embedded in $\mathbb{R}^3$.

Period matrix

- compact genus g
- canonical homology basis $a_i, b_i,\ i = 1, \ldots, g$
- dual basis of holo differentials $\omega_i,\ i = 1, \ldots, g$, $\int_{a_i} \omega_j = \delta_{ij}$



- period matrix

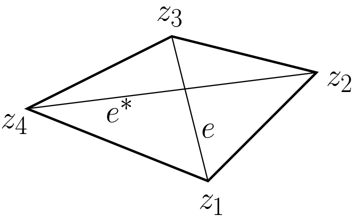
$$\Pi_{ij} = \int_{b_j} \omega_i, \quad \Pi = \Pi^T, \mathrm{Im}\, \Pi > 0$$

- Torelli theorem. Π determines its RS
- How to compute Π for a given RS?

Discrete Riemann Surfaces. Linear Theory. Cauchy-Riemann equations

[Mercat '01]

- Discrete Riemann Surface is a quad-graph D with a discrete complex structure that is $c : \{\text{diags of quads of } D\} \to \mathbb{C}$, such that $c(e^*) = \dfrac{1}{c(e)}$

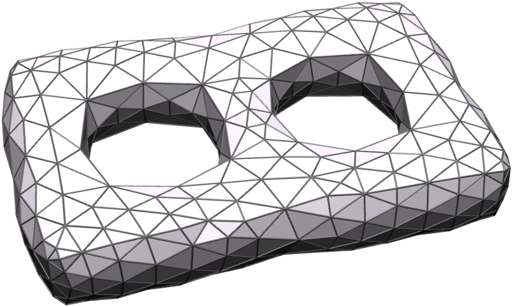


  - $f : V(D) \to \mathbb{C}$ discrete holomorphic if it satisfies discrete Cauchy-Riemann equations

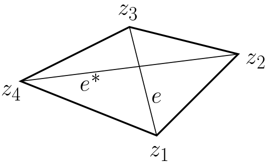
$$\frac{f(z_4) - f(z_2)}{f(z_3) - f(z_1)} = ic(e)$$

  - important special case - real c.

Discrete complex structure from a polyhedral surface



Quads are identified with planar quads in the complex plane $\mathbb{C}$



$$c(e) = -i\frac{z_4 - z_2}{z_3 - z_1}$$

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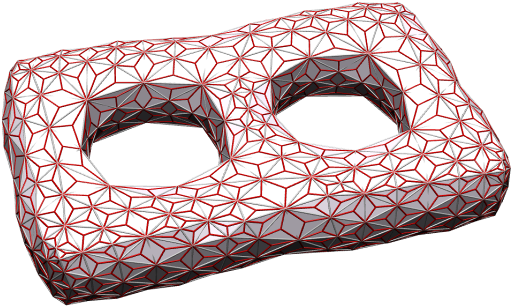

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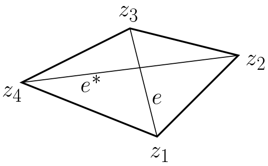

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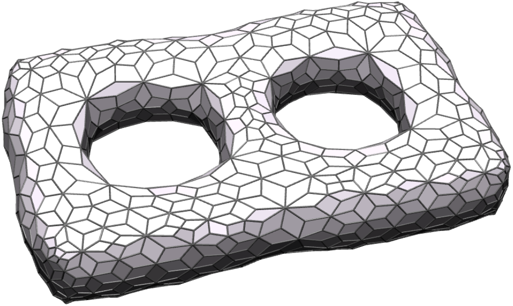

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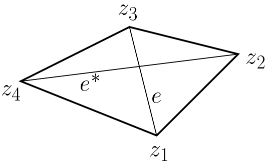

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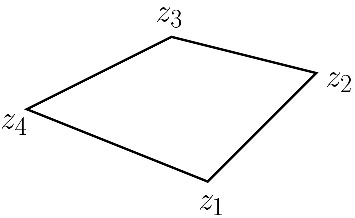
Discrete holomorphic and discrete harmonic



- $f : V(D) \to \mathbb{C}$ discrete holo if, Mercat ['08]

$$\frac{f(z_4) - f(z_2)}{z_4 - z_2} = \frac{f(z_3) - f(z_1)}{z_3 - z_1}$$

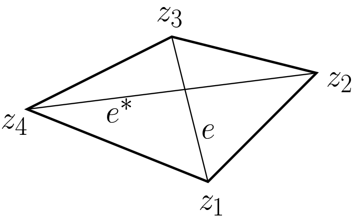
  - Real part $h = \mathrm{Re}\, f$ is discrete harmonic
  - $\Leftrightarrow$ discrete Laplace operator vanishes

$$\Delta h(z_1) = \sum \frac{1}{\mathrm{Re}\ c} \Big(|c|^2 (h(z_1) - h(z_3)) + \mathrm{Im}\ c(h(z_2) - h(z_4)) \Big) = 0,$$

  - $\Leftrightarrow$ critical for the (convex!) discrete Dirichlet energy

$$S(h) = \sum \tfrac{1}{\mathrm{Re}\ c} (|c|^2 (h(z_1) - h(z_3))^2 +$$
$$2\mathrm{Im}\ c(h(z_1) - h(z_3))(h(z_2) - h(z_4)) + (h(z_2) - h(z_4))^2).$$

Discrete holomorphic and discrete harmonic



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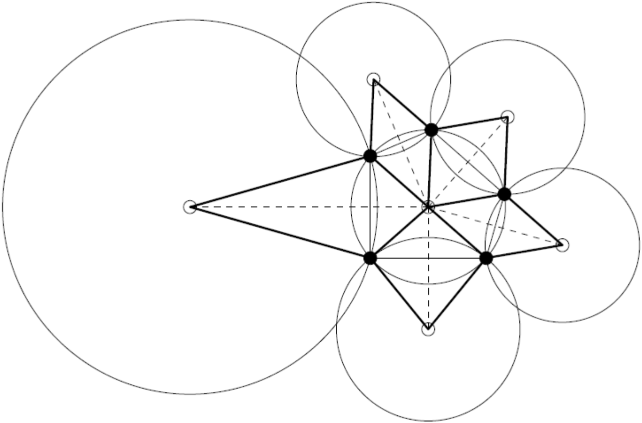
$$2\mathrm{Im}\, c(h(z_1) - h(z_3))(h(z_2) - h(z_4)) + (h(z_2) - h(z_4))^2).$$

Real weights

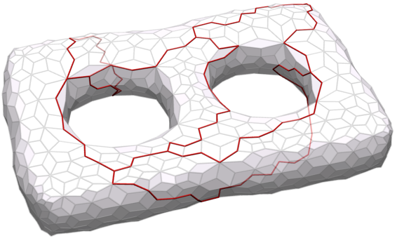
- $$\frac{f(z_4) - f(z_2)}{f(z_3) - f(z_1)} = ic(e), \quad c \in \mathbb{R}$$
- discrete Laplace operator
$$\Delta h(z) = \sum_{e=[z,w]} c(e)(h(z) - h(w)),$$
sum is over the diagonals incident to z,
- (convex!) discrete Dirichlet energy
$$S(h) = \sum_{e=[z,w]} c(e)(h(z) - h(w))^2,$$
sum over all diagonals e.
- $c \in \mathbb{R}$ - cotan-Laplace operator, $c = \mathrm{cotan}\alpha + \mathrm{cotan}\beta$

Delaunay tesselation



Orthogonal diagonals, real c

Multivalued functions with periods

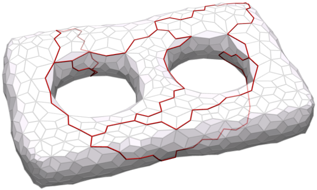


- canonical homology basis $a_1, b_1, \ldots, a_g, b_g$
- A multivalued function with periods $A_1, \ldots, A_g$, $B_1, \ldots B_g \in \mathbb{C}$ is a pair of functions $f = (\mathrm{Re} f : V \to \mathbb{R}$, $\mathrm{Im} f : F \to \mathbb{R})$ such that for any $x \in V, y \in F$

$$\mathrm{Re} f({}_{a_k}x) - \mathrm{Re} f(x) = \mathrm{Re} A_k, \quad \mathrm{Re} f({}_{b_k}x) - \mathrm{Re} f(x) = \mathrm{Re} B_k$$
$$\mathrm{Im} f({}_{a_k}y) - \mathrm{Re} f(y) = \mathrm{Im} A_k, \quad \mathrm{Im} f({}_{b_k}x) - \mathrm{Re} f(y) = \mathrm{Im} B_k,$$

where ${}_{a_k}x$ is a deck transformation of x

Discrete period matrix



multi-valued discrete holomorphic functions are called discrete Abelian integrals of the first kind.

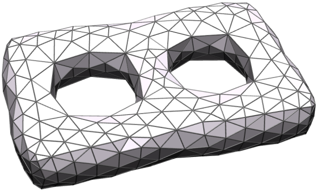
Theorem

There exist normalized discrete Abelian integrals of the first kind Ω_k^d and Ω_{*k}^d with $\Delta_{a_j}\Omega_k^d = \delta_{jk}$ and $\Delta_{a_j}\Omega_{*k}^d = i\delta_{jk}$.

Definition

The matrix $\Pi^D = \frac{1}{2}(\Pi^d + \Pi_*^d)$, $(\Pi^d)_{jk} = \Delta_{b_k}\Omega_j^d$, $(\Pi_*^d)_{jk} = -i\Delta_{b_k}\Omega_{*j}^d$ is called the discrete period matrix.

Convergence of discrete period matrix



Theorem [B., Skopenkov]

Consider a polyhedral surface R of genus g. For any $\delta < 0$ there exist two constants $\mathrm{Const}, \mathrm{const}$ (depending on R and δ only) such that for any Delaunay triangulation T of R which vertices include all conical singularities of R, and the maximal edge length $r < \mathrm{const}$, and with the minimal face angle $< \delta$ there holds

$$\|\Pi^D - \Pi\| < \mathrm{Const}\ r^a, \quad a = \min\{1, 4\pi/\Theta_i\},$$

where Θ_i are the conical angles at singularities.

Idea of the proof

- Consider (discrete) harmonic differentials u with prescribed periods A_i, B_i
- Minimize the Dirichlet energy E_d (convex) for given A_i, B_i

$$\sum_{[x_i x_j]=e\in E} c(e)(u_i-u_j)^2$$

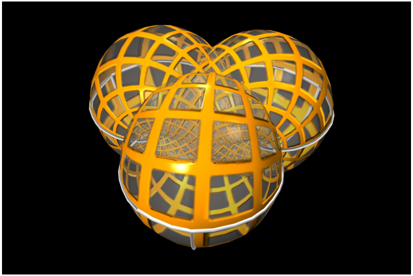
- $\min E_d$ is a quadratic form of A_i, B_i, coefficients give Π_d

$$\begin{pmatrix} \mathrm{Re}\,\Pi^d_*(\mathrm{Im}\,\Pi^d_*)^{-1}\mathrm{Re}\,\Pi^d+\mathrm{Im}\,\Pi^d & -(\mathrm{Im}\Pi^d_*)^{-1}\,\mathrm{Re}\,\Pi^d \\ -\mathrm{Re}\,\Pi^d_*(\mathrm{Im}\,\Pi^d_*)^{-1} & (\mathrm{Im}\,\Pi^d_*)^{-1} \end{pmatrix}$$

- same in the smooth case with $E=\int|\nabla u|^2$
- show that $\min E_d \to \min E$

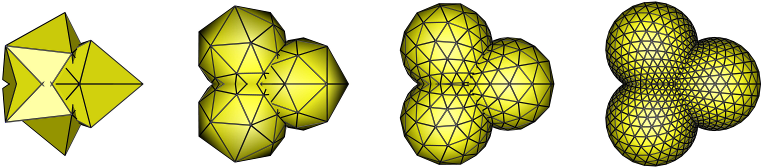
Computational RS.
Tori with constant mean curvature (CMC)

- first example. Wente ['86]
- all tori, description as integrable systems. Hitchin, Pinkall, Sterling ['89]
- explicit formulas in terms of RS (theta functions, Abelian integrals). Bobenko ['91]



Heil ['95]

Wente torus

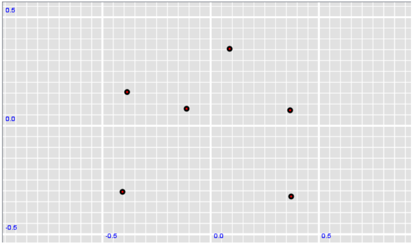


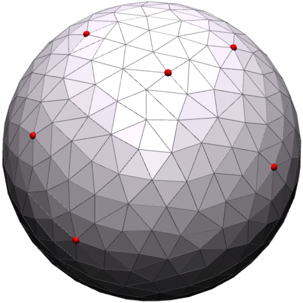
Grid	10×10	20×20	40×40	80×80
$\|\Pi_d - \Pi\|$	$5.69 \cdot 10^{-3}$	$2.00 \cdot 10^{-3}$	$5.11 \cdot 10^{-4}$	$2.41 \cdot 10^{-4}$

$$\Pi = 0.41300 + i0.91073$$

B., Mercat, Schmies ['11]

Hyperelliptic curve



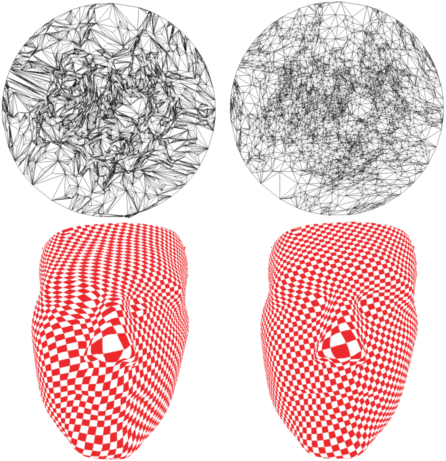


$$\Pi = \begin{pmatrix} -7.70 - i0.17 & 3.72 - i2.00 \\ 3.72 - i2.00 & -6.61 + i2.70 \end{pmatrix}$$

$$\Pi_d = \begin{pmatrix} -7.70 - i0.15 & 3.73 - i2.00 \\ 3.73 - i2.00 & -6.62 + i2.70 \end{pmatrix}$$

Knöppel, Sechelmann

Cotan discrete Laplace operator for texture mapping



Original and iDT (Beatuful Freack dataset): Texture plane image (Dirichlet boundary conditions) and resulting checker board mapping. [Fischer et al. '07]

Problem

Classification of linear pluri-Lagrangian systems

- discrete pluriharmonic functions (massive Laplacian ?, Boutiller et al.)
- possibly 3 parameters per face
- 2D linear pluri-Lagrangian systems $\to$ 3D nonlinear discrete integrable systems for coefficients
- new discrete Laplace operators for approximation theory and geometry processing?

Laplace version of conjugated nets

Theorem [BS]

The discrete Lagrangian 2-form

$$L(\sigma_{ij}) = \frac{1}{2s^{ij}c^{ij}c^{ji}}(u_{ij} - c^{ji}u_j - c^{ij}u_j - c^{ij}c^{ji}u_{ij})^2, s^{ij} = -s^{ji}$$

describes a pluri-Lagrangian system iff the coefficients satisfy the extended conjugate net equation

$$c^{ij}_k = \frac{1}{c^{kj}}(c^{ik}c^{ki} - c^{ik}c^{kj} - c^{ij}c^{ki}),$$
$$s^{ij}_k = c^{ki}c^{kj}s^{ij} + c^{ki}(c^{ij} - c^{ik})s^{jk} + c^{kj}(c^{ji} - c^{jk})s^{ki}.$$

3 coefficients c^{ij}, c^{ji}, s^{ij} per face.