The bounded 19-vertex Model

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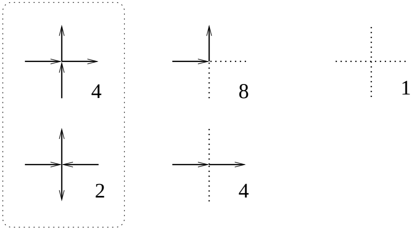
The 19-vertex Model

Definition (19-vertex Model on the square lattice)

Assign arrows between a vertex and its nearest neighbors in $\mathbf{Z}^2$ *according to the 19-vertex rule:* **there is the same number of incoming and outgoing arrows**. *The model consists of all configurations in which the rule is satisfied at every lattice point.*



Dotted line segment means no arrow. Frame encloses the legal Ice vertices; the 19-vertex Model generalizes the Ice/6-vertex Model.

For preceding work on the 19-vertex Model, see e.g. Izergin & Korepin, Fateev & Zamolodchikov (-81), Batchelor (with twisted boundary condition, -91), Inami & Odake & Zhang (-96), Pant and Wu (knot invariant, -97), recently Garbali, Hagendorf and others.

We will study the Model on a simple bounded domain with a special boundary condition that will make comparison to Ice Model/Alternating Sign Matrix and still earlier dimer results transparent.

In Ice one has a fully packed loop soup and all entropy arises from the reversal of unidirectional (off-boundary) loops. In 19-vertex one relaxes packing and allows a **diluted loop soup**. What does that imply in terms of limit shapes, entropy geometry etc.?

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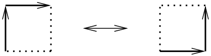
Dynamical Model

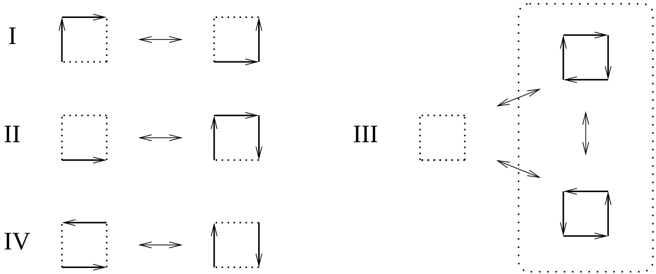
Any unidirectional loop or unidirectional infinite path can be reversed to generate a new configuration.

For an efficient algorithm to compute the configurations one should use the smallest legal perturbations, **actions/local moves/flips**.

- The move on the right (and all its rotations and reflections) works with 19-vertex configurations.
- Local moves need to be argued with some care. E.g. the one on the right does not work with other Ice-type zero flux rules like $0/1$, $0/2$ or $1/2$ arrows in and out.



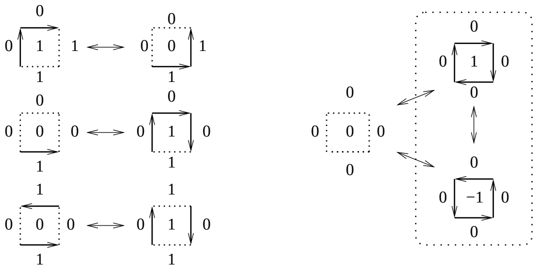
The minimal local moves for the 19-vertex rule (up to rotation and reflection):



In the frame the only legal minimal Ice-move (and it generates all Ice configurations).

Height

Height is a function defined on the dual lattice $\mathbf{Z}^2 + (1/2, 1/2)$. It works as in the Ice-context: upon crossing a configuration arrow pointing left/right, the height increases/decreases by 1. If no arrow encountered, height stays constant. This defines it uniquely up to an additive constant.



All local configurations are exhausted above.

Connectedness of the configurations

Theorem (Irreducibility of the actions)

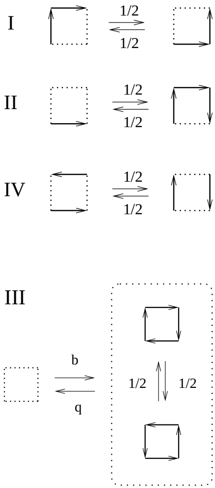
Given a 19-vertex configuration on a bounded domain, any other legal configuration with the same boundary condition can be generated from the former using a finite sequence of elementary actions I-IV. A strict subset of actions will not suffice.

Most direct proof is by height. By using a suitable sequence of flips, one can connect any given configuration/height surface to the maximal one (highest) with the same boundary. Through that (or minimal one just as well) the configurations can be connected. In a bounded domain the number of flips needed is of course finite. Last statement by counterexamples.

PCA

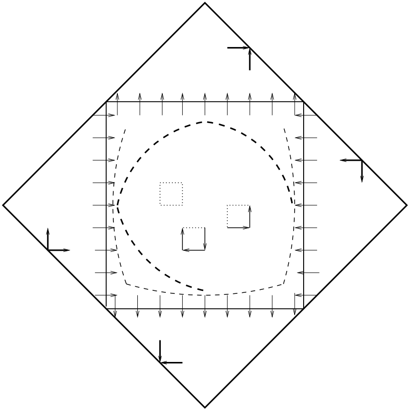
To generate the configurations with a fixed boundary from a seed configuration:

- Act alternatively on even and odd sublattice squares (black and white on chessboard). Update the shared arrows on the neighboring squares. Boundary arrows stay fixed.
- Cycle through the randomized actions I-IV on each sublattice.
- The full Markov Chain is irreducible and aperiodic once $0 < b, q < 1$.



Boundary condition

On a bounded domain all Ice-legal boundaries are 19-vertex legal (and then more). Here we will restrict to the Izergin-Korepin **Domain Wall Boundary Condition** (on the enclosed square):



DWBC (which is just “maximal tilt boundary segments without cul-de-sacs”) can be generalized to more complicated domain shapes yielding (more complicated) limit shape results to Ice/19-vertex.

On the boundary the arrow density is 1, but

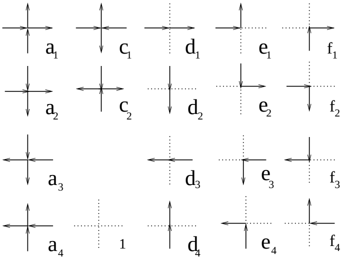
Proposition (Interior arrow density)

The arrow density of a 19-vertex configuration over a domain with DWBC is always at least $1/2$.

We do not know of any boundary condition with density of boundary arrows strictly below 1 yielding non-trivial limit shapes.

Weights

Subsequently we'll use the dynamic parameters (which can be converted to weights via somewhat complicated formulas) since under the detailed balance we have:



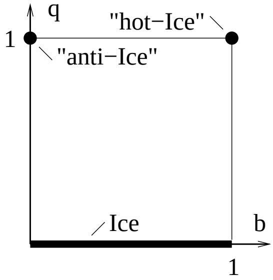
Proposition (Weight relations)

$$
\begin{array}{lll}
a_1 = d_1 d_4 = e_1 f_1 & & e_1 e_2 e_3 e_4 = 1 \\
a_2 = d_1 d_2 = e_2 f_2 & c_1 = e_1 e_3 = f_2 f_4 & f_1 f_2 f_3 f_4 = 1 \\
a_3 = d_2 d_3 = e_3 f_3 & c_2 = e_2 e_4 = f_1 f_3 & d_1 d_3 = 1 \\
a_4 = d_3 d_4 = e_4 f_4 & & d_2 d_4 = 1
\end{array}
$$

The big picture

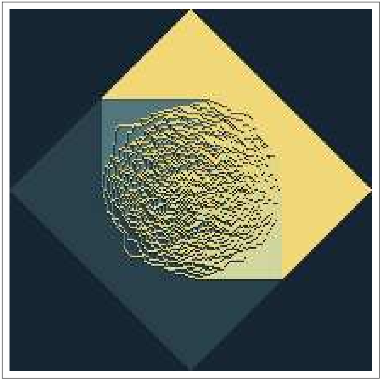
All samples are with DWBC, driven to equilibrium with the PCA.

- Bottom interval $[0, 1]$ is all Ice since no arrow vacancy can come about (and b makes no difference there).
- "Anti-Ice" means that no trace of Ice-action since no minimal oriented loops are reversed.
- $b + q$ is the directed 1-loop creation-annihilation rate, "Ice-temperature".
- Normalized Ice weights are 1 (1-enumeration point of ASM in the disordered phase).

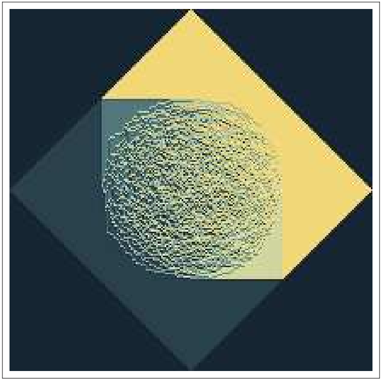


From Ice to hot-Ice

Configurations after 40.000 iterates with DWBC on the maximal square inside 106×106 diamond. 1-square arrow arrangements color coded.

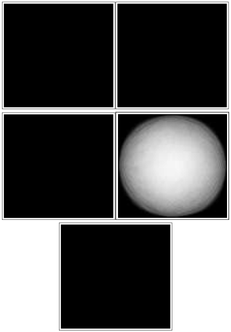


Ice $((b, q) = (0, 0))$

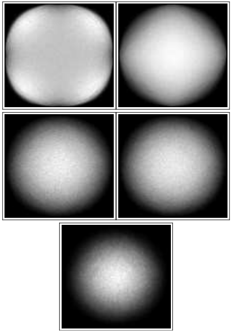


hot-Ice $((1, 1))$

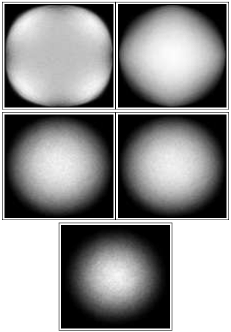
Actions along the rising diagonal $q = b$:



$(b, q) = (0, 0)$ (Ice)

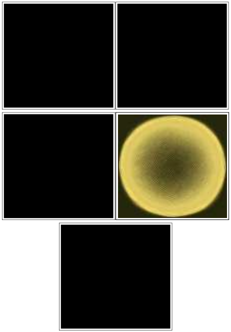


$\left(\frac{1}{2}, \frac{1}{2}\right)$



$(1, 1)$

Same, a bit enhanced:



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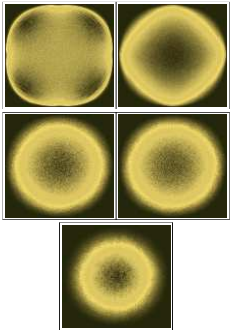

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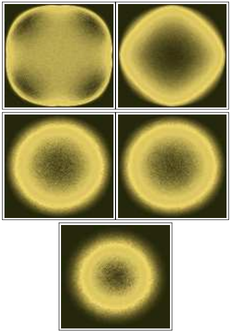
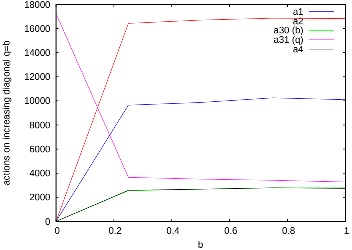
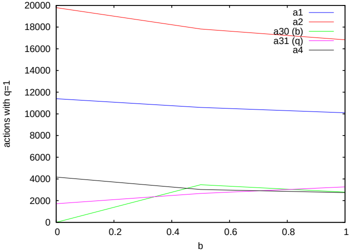


$(1, 1)$

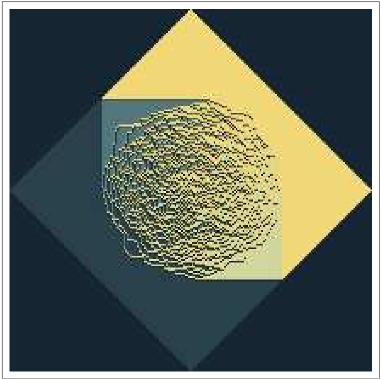
- Distribution maxima of actions I-IV along the diagonal $q = b$ and on the top $q = 1$ (The graphs in the former case should be constant for $b > 0$, since on the diagonal all weights are 1.)
- Annihilation of a directed 1-cycles likely first supplies action 2, this in turn action 1 etc. hence the shown domination.



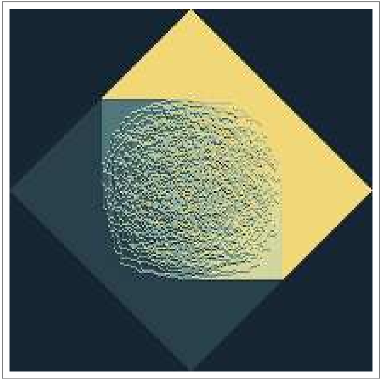


From Ice to Anti-Ice

Configurations after 40.000 iterates with maximal DWBC inside a 106×106 diamond:

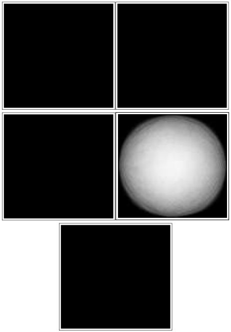


Ice $((1,0))$

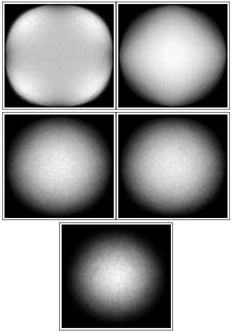


Anti-Ice $((0,1))$

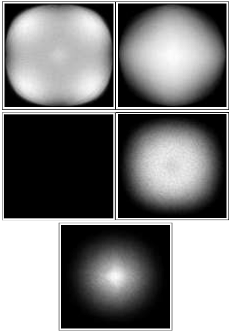
Actions from Ice to Anti-Ice along the diagonal $q = 1 - b,\ \ b : 1 \to 0$.
Note that now the vertex weights are not constant anymore.



$(b, q) = (1, 0)$ (Ice)

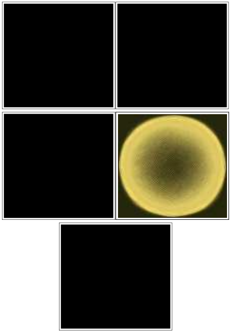


$\left(\frac{1}{2}, \frac{1}{2}\right)$



$(0, 1)$

A bit enhanced for details...



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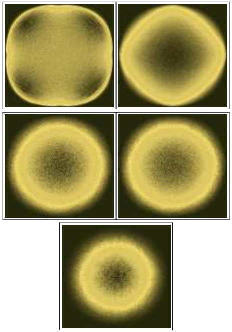

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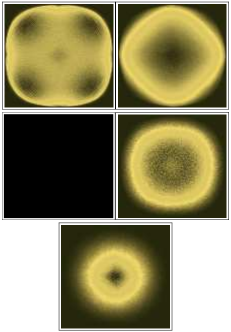
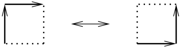
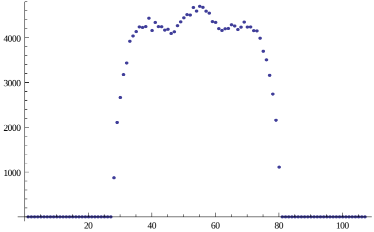


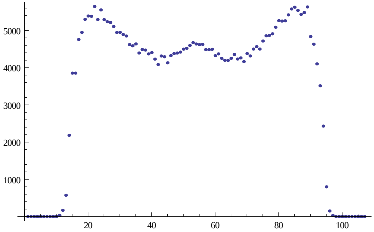
$(0, 1)$

Anti-Ice action I

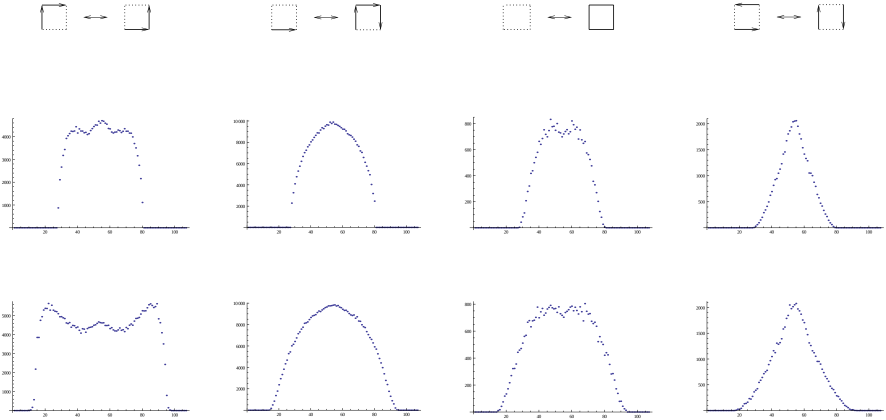


horizontal and diagonal profiles:





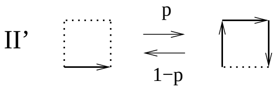
Horizontal and diagonal sections for the Anti-Ice action distributions:



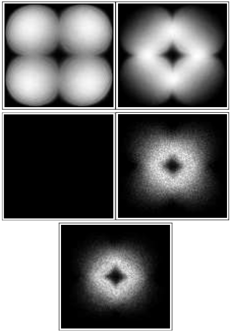
Skewing action II

Action II influences directly path/loop "smoothness" and their length. If it is tilted in favor one type of flip, macroscopic effects are to be expected.

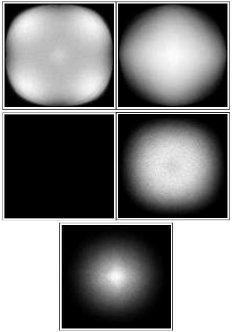
- Small p should lead to smoother paths/loops and decrease the frequencies of other actions.
- High p should lead to convoluted (at extreme "space filling") paths thereby evening out the action distributions.
- The effect should be most pronounced in the directions of the lattice axis.



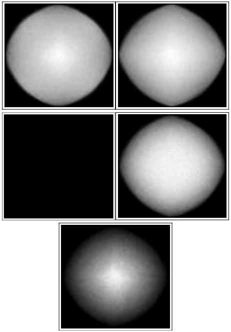
Weighted action II around Anti-Ice $\left((b,q)=(0,1),\ p=\frac{1}{2}\right)$:



$p=\frac{1}{8}$

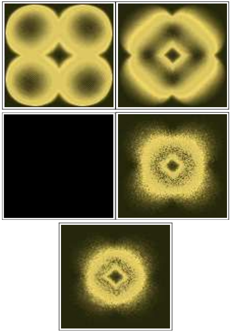


$p=\frac{1}{2}$



$p=\frac{7}{8}$

...enhanced:



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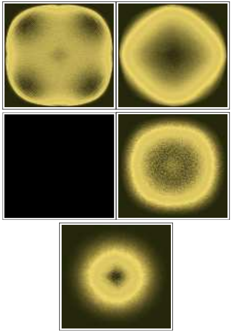
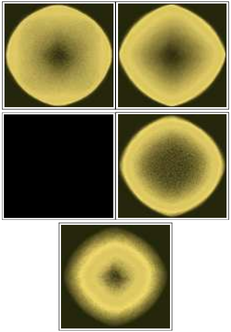
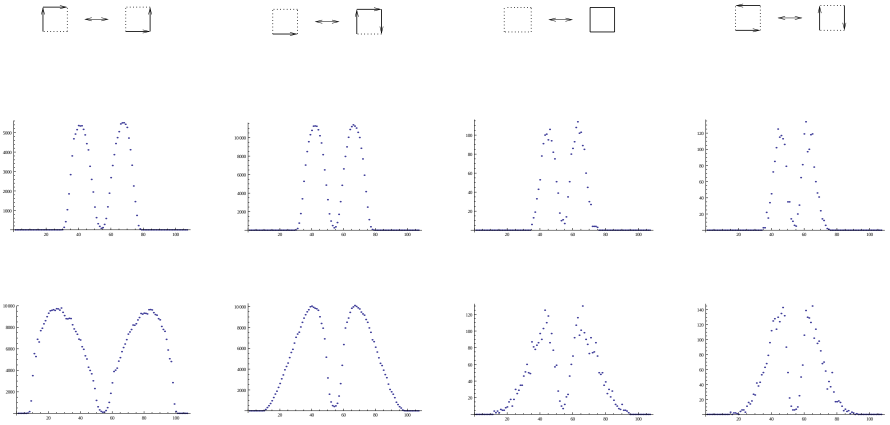


$p = \frac{1}{2}$



$p = \frac{7}{8}$

Horizontal and diagonal sections for the $p = \frac{1}{8}$ action distributions:



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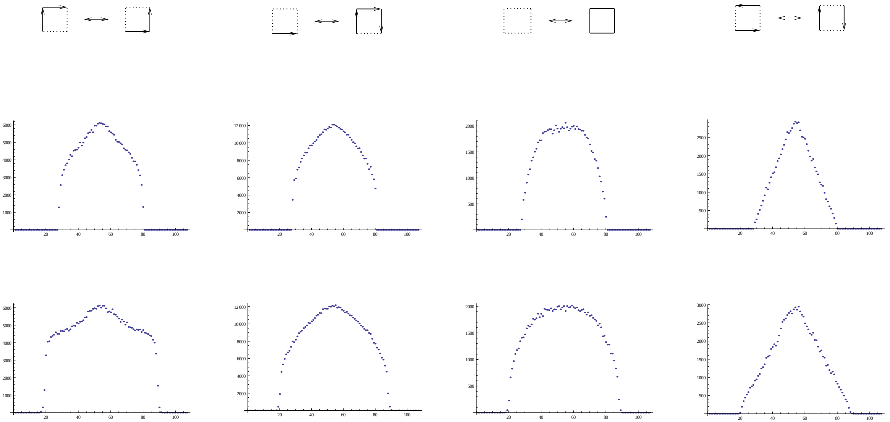
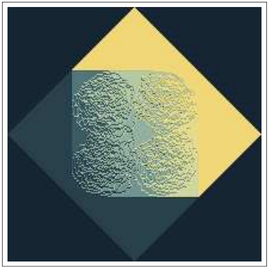


Thank you!



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