Bethe states and separation of variables for SU(N) Heisenberg spin chain

Nikolay Gromov

King’s College London & PNPI

based on

1610.08032 [NG, F. Levkovich-Maslyuk, G. Sizov]

Below: FLM = F. Levkovich-Maslyuk

N.Gromov = NG

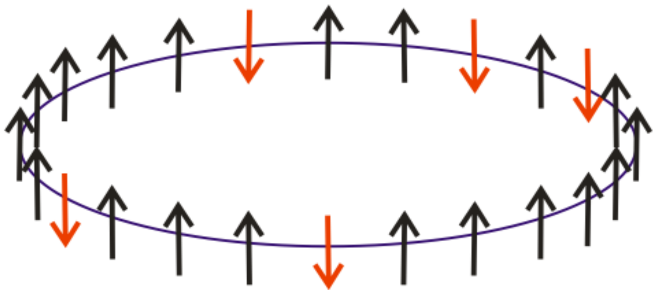
Motivation from AdS/CFT

Motivation from AdS/CFT

Planar N=4 SYM / strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ – integrable at any coupling

single trace operators $\Longleftrightarrow$ spin chain eigenstates

$$\mathrm{Tr}(\Phi_1(x)\Phi_2(x)\Phi_2(x)\Phi_1(x)\ldots)$$



spectrum of conformal dimensions Δ $\Longleftrightarrow$ energy levels

3-point structure constants C_{123} $\Longleftrightarrow$ scalar products of wavefunctions

Need an efficient description of spin chain eigenstates

XXX spin chain reformulation

Starting point: Baxter equation

$$T(u)Q(u) + (u+i/2)^L Q(u-i) + (u-i/2)^L Q(u+i) = 0$$

Two solutions: twisted polynomial $Q_1 \sim \lambda^{iu} u^N$

second solution (for the same T) $Q_2 \sim \lambda^{-iu} u^{L-N}$

Equivalent description:

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

Easy to generalize SU(3):

$$\begin{vmatrix} Q_1(u+i) & Q_2(u+i) & Q_3(u+i) \\ Q_1(u) & Q_2(u) & Q_3(u) \\ Q_1(u-i) & Q_2(u-i) & Q_3(u-i) \end{vmatrix} = u^L$$

Can use instead of the nested Bethe ansatz

[Kulish, Reshetikhin 1983]

Two main ingredients:

- QQ-relations

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

$$sl(2) \to psu(2,2|4)$$

$$(Q_1,\ Q_2) \quad \longrightarrow \quad (\underbrace{\mathbf{P}_1,\ \mathbf{P}_2,\ \mathbf{P}_3,\ \mathbf{P}_4}_{S^5}|\underbrace{\mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3,\ \mathbf{Q}_4}_{AdS_5})$$

- Analyticity

Q_1 - Twisted polynomial

Q_2 - Twisted polynomial

In N=4 polynomials are replaced with an analytic functions with cuts + monodromy condition (ask me later)

QQ-relations+monodromy = Quantum Spectral Curve (QSC)

Gromov,Kazakov,Leurent,Volin 2013

Motivation from AdS/CFT

Unlike 3-point functions, exact spectrum is under excellent control
Key tool – Quantum Spectral Curve

Gromov,Kazakov,Leurent,Volin 2013

Quantum Spectral Curve = QQ relations + analyticity constraints

Many new results

all-loop

Gromov, Kazakov, Leurent, Volin 14
Gromov, FLM, Sizov 13,14

numerical

Gromov, FLM, Sizov 15

perturbative

Marboe, Volin 14, 16, 17
Marboe, Velizhanin, Volin 14

BFKL

Alfimov, Gromov, Kazakov 14
Gromov, FLM, Sizov 15
Alfimov, Gromov, Sizov to appear

other models

Cavaglia, Fioravanti, Gromov, Tateo 14
Gromov, Sizov 14
Anselmetti, Bombardelli, Cavaglia, Tateo 15
Cavaglia, Cornagliotto, Mattelliano, Tateo 15
Bombardelli,Cavaglia, Fioravanti, Gromov,Tateo 17

Recently extended to quark-antiquark potential and ABJ theory

Gromov, FLM 15, 16
Cavaglia, Gromov, FLM 16

Can we link it with correlation functions?

Separation of Variables (SoV)

In Sklyanin's separated variables the wavefunction should factorize

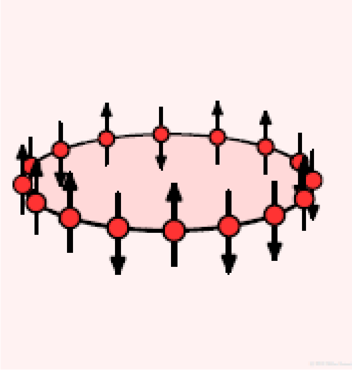
$$\Psi \sim Q(x_1)Q(x_2)\ldots Q(x_n)$$

The building blocks are Q-functions

Quantum Spectral Curve provides them at all loops !

Could give a non-perturbative formulation for 3-pt functions

E.g. for SU(2) XXX spin chain $Q(u) = \prod_i (u - u_i)$



SoV was already used efficiently for 3-point functions
But only in essentially rank-1 sectors like SU(2)

Kazama, Komatsu 11, 12
Kazama, Komatsu, Nishimura 13, 14, 15, 16
Sobko 13
Jiang, Komatsu, Kostov, Serban 15

- Need to extend SoV to full $PSU(2,2|4)$
- Need better descriptions for spin chain states

Our results

New and simple construction of SU(N) spin chain eigenstates

Explicitly describe SoV for these spin chains

I will present two results linked by a common theme:

Sklyanin’s separation of variables in integrable systems (SoV)

Sklyanin 91, 92

In separated variables the wavefunction factorizes

$$\Psi \sim Q(x_1)Q(x_2)\dots Q(x_n)$$

Powerful method, many aplications:

spin chains, sigma models, AdS/CFT, …

SoV should give access to 3-point functions
in dipole CFT and original N=4 SYM

So far used only in
rank-1 sectors like SU(2)

Kazama, Komatsu 11, 12
Kazama, Komatsu, Nishimura 13 - 16
Sobko 13
Jiang, Komatsu, Kostov, Serban 15

$$\Psi \sim Q(x_1)Q(x_2)\dots Q(x_n)$$

Q's are known to all loops in N=4 SYM
from the Quantum Spectral Curve

Gromov, Kazakov, Leurent, Volin 13

Could give a formulation for 3-pt functions
alternative to [Basso, Komatsu, Vieira 2015]

- Need to extend SoV to full $PSU(2,2|4)$
- Need better descriptions for spin chain states

Our results

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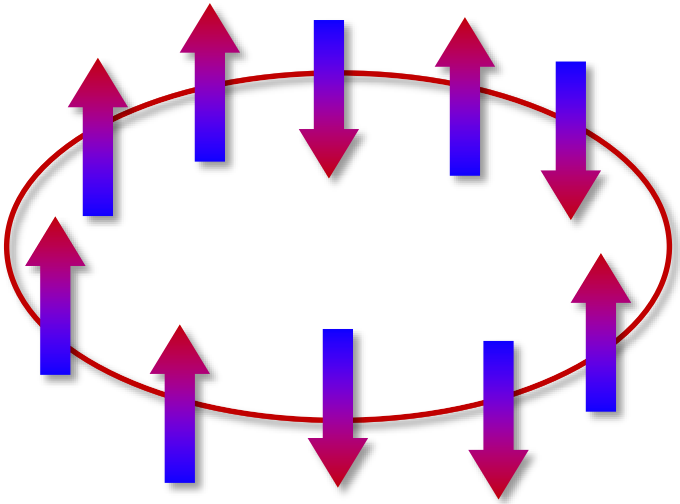
Integrable SU(N) spin chains

At each site we have a $\mathbb{C}^N$ space



Hamiltonian: $H = \sum_{n=1}^{J} (1 - P_{n,n+1})$

(+ boundary twisted terms)

Related to rational R-matrix

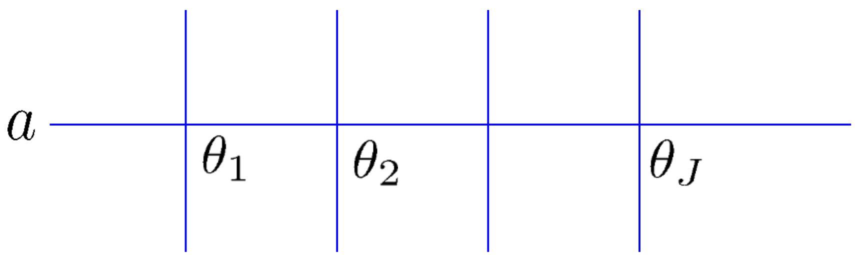
$$R_{12}(u) = (u - \tfrac{i}{2}) + iP_{12}$$

Spectrum is captured by nested Bethe ansatz equations

Sutherland 68; Kulish, Reshetikhin 83; …

The monodromy matrix

$$T(u) = R_{a1}(u-\theta_1)\dots R_{aJ}(u-\theta_J)g$$



We take generic inhomogeneities θ_n

and diagonal twist $g = \mathrm{diag}(\lambda_1, \dots, \lambda_{\mathrm{N}})$

Transfer matrix $\mathrm{Tr}_a\, T(u) = \sum_{n=0}^{J} T_n u^n$ gives commuting integrals of motion

We want to build their common eigenstates

Construction of eigenstates

$SU(2)$ use a creation operator evaluated on the Bethe roots

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_M)|0\rangle \qquad T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$SU(N)$ no simple analog despite
many efforts over 30 years

Sutherland, Kulish, Reshetikhin, Slavnov, Ragoucy, Pakouliak, Belliard, Mukhin, Tarasov, Varchenko, …

We conjecture for any $SU(N)$

$$|\Psi\rangle = B^{\mathrm{good}}(u_1)B^{\mathrm{good}}(u_2)\dots B^{\mathrm{good}}(u_M)|0\rangle$$

Surprisingly simple !

$B^{\mathrm{good}}(u)$ is an explicit polynomial in the monodromy matrix entries

The same operator also provides separated variables

SU(2) case

Separated variables for SU(2)

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_M)|0\rangle$$

$$B(u) = c\prod_{n=1}^{J}(u - x_n) \qquad [B(u), B(v)] = 0 \Longrightarrow [x_m, x_n] = 0$$

x_m are the separated coordinates

In their eigenbasis the wavefunction factorizes

$$\langle \mathbf{x}_1 \dots \mathbf{x}_J|\Psi\rangle = \prod(u_j - \mathbf{x}_k) = \prod_{k=1}^{J} Q_1(\mathbf{x}_k)$$

we normalize
$\langle \mathbf{x}_1 \dots \mathbf{x}_J|0\rangle = 1$

Removing degeneracy

$$B(u) = C\prod_{n=1}^{J}(u - x_n)$$ nilpotent, cannot be diagonalized

$$T(u) \longrightarrow T^{\mathrm{good}}(u) = K^{-1}T(u)K$$ K is a constant 2 x 2 matrix

$$T^{\mathrm{good}}(u) = \begin{pmatrix} A^{\mathrm{good}}(u) & B^{\mathrm{good}}(u) \\ C^{\mathrm{good}}(u) & D^{\mathrm{good}}(u) \end{pmatrix}$$

Gromov, Levkovich-Maslyuk, Sizov 16

All comm. rels are preserved, trace of T is unchanged

Now we can diagonalize B^{good}

$$B^{\mathrm{good}}(u) = C\prod_{n=1}^{J}(u - x_n) \Longrightarrow$$ separated variables x_m

Eigenstates from B^{good}

$$T^{\rm good}(u) = K^{-1} T(u) K$$

$$T^{\rm good}(u) = \begin{pmatrix} A^{\rm good}(u) & B^{\rm good}(u) \\ C^{\rm good}(u) & D^{\rm good}(u) \end{pmatrix}$$

Trace of T is unchanged

We can also build the states with $B^{\rm good}(u)$

$$|\Psi\rangle = B^{\rm good}(u_1) B^{\rm good}(u_2) \dots B^{\rm good}(u_M) |0\rangle$$

Surprisingly it's true even for generic K

Gromov, Levkovich-Maslyuk, Sizov 16

On-shell off-shell scalar product

$$\langle\Phi|\Psi\rangle = \langle 0|C(v_1)\dots C(v_M)B^{\text{good}}(u_1)B^{\text{good}}(u_2)\dots B^{\text{good}}(u_M)|0\rangle$$

For $K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ We have $B^{\text{good}} = A + B - C - D$

Since the on-shell state has exactly M spin-ups we can replace C with -Bgood

$$\langle\Phi|\Psi\rangle = \langle 0|B^{\text{good}}(v_1)\dots B^{\text{good}}(v_M)B^{\text{good}}(u_1)B^{\text{good}}(u_2)\dots B^{\text{good}}(u_M)|0\rangle$$

Next we insert a complete set of eigenstates

$$\langle\Phi|\Psi\rangle = \sum_x \mu(x)\langle 0|x\rangle\langle x|B^{\text{good}}(v_1)\dots B^{\text{good}}(v_M)B^{\text{good}}(u_1)B^{\text{good}}(u_2)\dots B^{\text{good}}(u_M)|0\rangle$$

Since the measure is essentially a Vandermond determinant:

$$\mu(\mathrm{x}) = \left(\frac{\alpha\lambda_2}{\lambda_1-\lambda_2}\right)^L \prod_{n=1}^{L}\left(\frac{\mathrm{x}_n-\theta_n}{i/2}\right) \frac{\prod\limits_{m<n}^{L}(\bar{\mathrm{x}}_m-\bar{\mathrm{x}}_n)}{\prod\limits_{m<n}^{L}(\theta_m-\theta_n)}$$

We arrive to Slavnov’s determinant formula almost for free

SU(3)

Eigenstates for SU(3)

$T(u)$ is a 3 x 3 matrix

The operator which should provide separated variables is [Sklyanin 92]

$$B(u) = T_{23}(u)T_{12}(u-i)T_{23}(u) - T_{23}(u)T_{13}(u-i)T_{22}(u) + T_{13}(u)T_{11}(u-i)T_{23}(u) - T_{13}(u)T_{13}(u-i)T_{21}(u)\ .$$

$$B(u) = C\prod_{n=1}^{3J}(u - x_n)$$

Again cannot be diagonalized

Replacing $T_{ij} \to T_{ij}^{\text{good}}$ we get B^{good}

$$T^{\text{good}} = K^{-1}TK$$

e.g. $K = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

It generates the eigenstates !

$$|\Psi\rangle = B^{\text{good}}(u_1)\dots B^{\text{good}}(u_M)|0\rangle$$

Gromov, FLM, Sizov 16

Conjecture supported by many tests

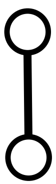
Eigenstates for SU(3)

$$|\Psi\rangle = B^{\text{good}}(u_1)\dots B^{\text{good}}(u_M)|0\rangle$$

Exactly like in SU(2)

$$[B^{\text{good}}(z), B^{\text{good}}(w)] = 0$$

u_j are the momentum-carrying Bethe roots fixed by usual nested Bethe equations



$$\prod_{n=1}^{L} \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} = \frac{\lambda_2}{\lambda_1} \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{R} \frac{u_j - v_k - i/2}{u_j - v_k + i/2}\,,$$

$$\prod_{n=1}^{M} \frac{v_j - u_n + i/2}{v_j - u_n - i/2} = \frac{\lambda_3}{\lambda_2} \prod_{k\neq j}^{R} \frac{v_j - v_k + i}{v_j - v_k - i}\,.$$

Separated variables are found from $B^{\text{good}}(u) = C \prod_{n=1}^{3J} (u - x_n)$

Factorization of wavefunction follows at once

Comparison with known constructions

Usual nested algebraic Bethe Ansatz gives

$$|\Psi\rangle = \sum_{a_i=2,3} F^{a_1 a_2 \ldots a_M} T_{1a_1}(u_1) T_{1a_2}(u_2) \ldots T_{1a_M}(u_M)|0\rangle$$

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

wavefunction of
auxiliary SU(2) chain

Sutherland,
Kulish, Reshetikhin 83

Our conjecture

$$|\Psi\rangle = B^{\text{good}}(u_1) \ldots B^{\text{good}}(u_M)|0\rangle$$

- Only a single operator
- No recursion
- Complexity in # of roots is linear, not exponential

Comparison with known constructions

Large literature on the construction of eigenstates

- representation as sum over partitions of roots

  Belliard, Pakouliak, Ragoucy, Slavnov 12, 14

- trace formulas

  Tarasov, Varchenko 94
  Mukhin, Tarasov, Varchenko 06
  Mukhin, Tarasov, Varchenko 07 …

- Drinfeld current construction

  Khoroshkin, Pakouliak 06
  Khoroshkin, Pakouliak, Tarasov 06
  Frappat, Khoroshkin, Pakouliak, Ragoucy 08
  Belliard, Pakouliak, Ragoucy 10
  Pakouliak, Ragoucy, Slavnov 14 …

- …

  Albert, Boos, Frume, Ruhling 00 …

Our proposal seems much more compact

We focussed on spin chains with
fundamental representation at each site

Many of the results should apply more generally
(using fusion procedure one can get any representation from the fundamental)

Extension to SU(N)

Classical B and quantization

In the classical limit
for SU(N)

$$B=\sum_{i_1,\ldots,i_{N-1}=1}^{N-1}\epsilon_{i_1\ldots i_{N-1}}T_{i_1N}(T^2)_{i_2N}\ldots(T^{N-1})_{i_{N-1}N}$$

Scott 94
Gekhtman 95

How to quantize this expression?

SU(3) result: $B(u)=T_{13}(u)T_{12|13}(u-i)+T_{23}(u)T_{12|23}(u-i)$

$$T_{j_1j_2|k_1k_2}(u)=\begin{vmatrix} T_{j_1k_1}(u) & T_{j_1k_2}(u+i)\\ T_{j_2k_1}(u) & T_{j_2k_2}(u+i)\end{vmatrix}$$

quantum minors

SU(4): make an ansatz, all coefficients fixed to 0 or 1 !

$$B(u)=\sum_{j,k}T_{j|4}(u)T_{k|j4}(u-i)T_{123|k4}(u-2i)$$

Results for SU(N)

We propose for any SU(N)

$$B(u) = \sum_{j,\dots,p} T_{j|N}(u) T_{k|jN}(u-i) \dots T_{12\dots|pN}(u-(N-2)i)$$

Gromov, FLM, Sizov 16

- Matches classical limit $N \leq 6$
- Commutativity $N \leq 5, J \leq 4$

Highly nontrivial checks !

We conjecture that $B^{\rm good}$ generates the states and separated variables

Tests

$$|\Psi\rangle = B^{\text{good}}(u_1)\dots B^{\text{good}}(u_M)|0\rangle$$

Numerically: many states with $J \leq 4$,

all states with $J \leq 2$ for $SU(3), SU(4), SU(5)$

Huge matrices, e.g. $4^4 \times 4^4 = 65536$ entries

No need for recursion $SU(5) \to SU(4) \to SU(3) \to SU(2)$!

$SU(3)$: analytic proof for up to 2 magnons, any J

For $SU(N)$ quantum B which provides SoV was also proposed without construction of eigenstates in:

- [F. Smirnov 2001], would be interesting to compare
- [Chervov, Falqui, Talalaev 2006-7], fails for $SU(4),\ J=2$
  Based on Manin matrices, perhaps can be improved

Supersymmetric spin chains

Our construction of eigenstates generalizes

to at least some $SU(M|N)$

[work in progress]

SOV is not known even for $SU(1|1)$

Future

- Rigorous proof, algebraic origins
- Can one extend Slavnov's determinant (on-shell/off-shell scalar product) beyond SU(2) ?
- Higher-loop guesses in AdS/CFT