Scaling limits of critical Ising correlations: convergence, fusion rules, applications to SLE.

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Joint work with

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- Clément Hongler (EPF Lausanne)

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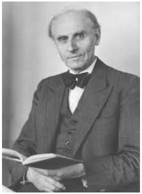

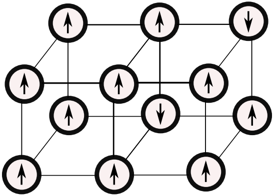

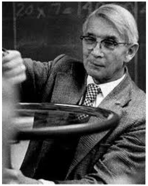

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- The thermal motion produces randomness

$$\mathbb{P}(\sigma) = e^{-\beta\mathcal{H}(\sigma)}/Z, \quad \beta = \frac{k}{T}, \quad Z = \sum_{\sigma} e^{-\beta\mathcal{H}(\sigma)}$$

(Gibbs-Boltzmann disctribution)

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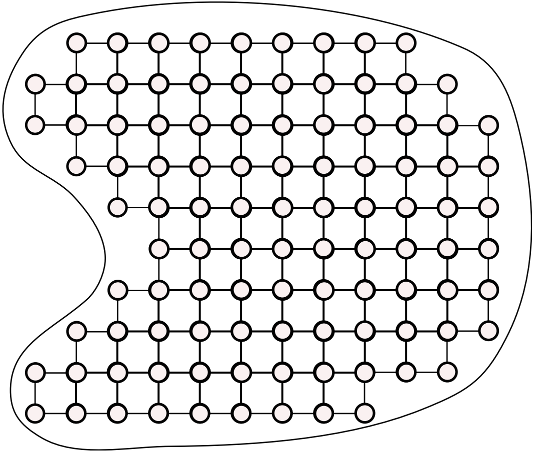

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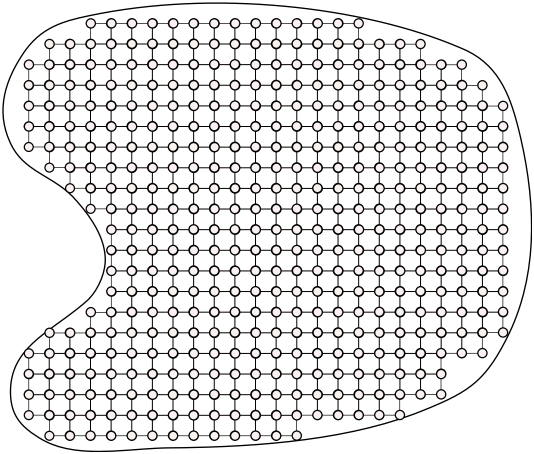

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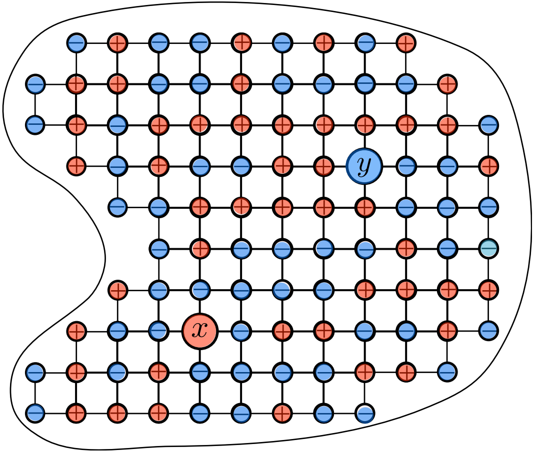

The mathematical formulation

A mathematical manifestation of the phase transition:

- take the scaling limit: $\Omega^\delta \subset \delta\mathbb{Z}^2$ approximates $\Omega \subset \mathbb{C}$ as $\delta \to 0$.
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- from now on: $\beta = \beta_c = \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

A convergence result (Chelkak–Hongler–K. I., 2015)

As $\delta \to 0$, one has

$$\mathbb{E}_{\Omega^\delta}\left(\sigma_{x_1}\dots\sigma_{x_n}\right) \sim C^n \cdot \delta^{\frac{n}{8}} \cdot \langle \sigma_{x_1}\dots\sigma_{x_n}\rangle_\Omega.$$

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(Similar results for other boundary conditions.)

More general correlations

Ideally, one would like generalize the result for more general random variables ("lattice fields")

$$\mathbb{E}_{\Omega^\delta}\left(\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\right) \sim \delta^{\Delta_1+\cdots+\Delta_n}\langle\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_\Omega$$

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In general, we get much *more complicated covariance rules*.
Example:

$$\mathcal{O}_1(x_1) = \sigma_{x_1+\delta} - \sigma_{x_1}.$$

Then, it is natural to expect

$$\mathbb{E}_{\Omega^\delta}\left(\mathcal{O}_1(x_1)\cdot\sigma_{x_2}\ldots\sigma_{x_n}\right) \sim \delta^{1+\frac{n}{8}}\partial_{\mathfrak{Re}\,x_1}\langle\sigma_{x_1}\ldots\sigma_{x_n}\rangle.$$

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(see Hongler-Kytölä-Viklund for more general fields)

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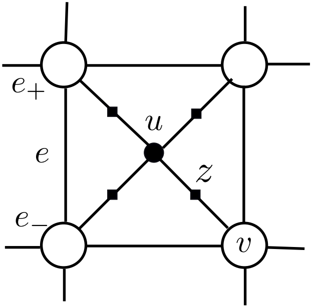

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- For boundary conditions, one may partition the boundary into free, plus, or minus parts.

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- Moreover, there is a natural way to choose the signs of $\mathbb{E}(\mu_\gamma \sigma_{v_1} \dots \sigma_{v_n})$ as $v_1, \dots, v_n$ and $u_1, \dots, u_m$ move around in the lattice.
- With this choice, $\mathbb{E}(\mu_\gamma \sigma_{v_1} \dots \sigma_{v_n}) \prod_{i,j}(u_i - v_j)^{\frac{1}{2}}$ *is a well-defined function of* $u_1, \dots, u_m, v_1, \dots, v_n$. *We write* $\mathbb{E}(\mu_{u_1} \dots \mu_{u_m} \sigma_{v_1} \dots \sigma_{v_n})$ *instead of* $\mathbb{E}(\mu_\gamma \sigma_{v_1} \dots \sigma_{v_n})$.

What is μ_u – continued

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\begin{aligned}
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&\quad = \frac{1}{Z} \sum_{\sigma:\Omega^\delta \to \{\pm 1\}} e^{-2\beta \sum_{(xy)\cap\gamma\neq\emptyset} \sigma_x\sigma_y} \sigma_{v_1} \dots \sigma_{v_n} e^{\beta \sum_{x\sim y} \sigma_x \sigma_y} \\
&\quad = \frac{1}{Z} \sum_{\sigma:\Omega^\delta \to \{\pm 1\}} \sigma_{v_1} \dots \sigma_{v_n} e^{\beta \sum_{(xy)\cap\gamma=\emptyset} \sigma_x\sigma_y - \beta \sum_{(xy)\cap\gamma\neq\emptyset} \sigma_x\sigma_y} \\
&\qquad\qquad = \frac{1}{Z} \sum_{\substack{\sigma:\Omega^\delta_{[u_1,u_2]} \to \{\pm 1\} \\ \sigma(v)=-\sigma(v^*)}} e^{\frac{\beta}{2} \sum_{x\sim y} \sigma_x\sigma_y} \sigma_{v_1} \dots \sigma_{v_n}.
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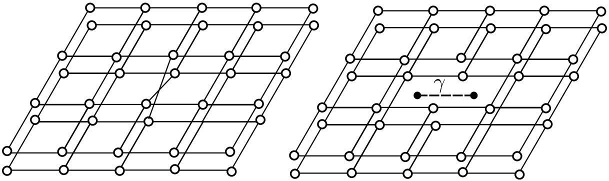

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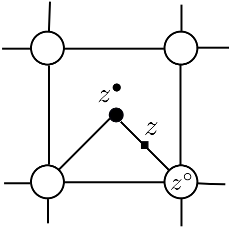

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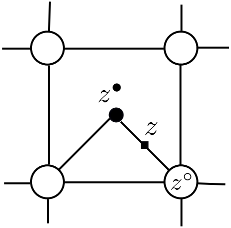

What we mean by this is that any expression of the form

$$\mathbb{E}(\psi_{z_1} \dots \psi_{z_k} \mu_{u_1} \dots \mu_{u_m} \sigma_{v_1} \dots \sigma_{v_n})$$

is well defined

- up to sign at any particular point;
- as a (multi-valued) function of $z_1, \dots v_m$ living on the Riemann surface of

$$\prod (z_i - u_j)^{\frac{1}{2}} \prod (z_i - v_j)^{\frac{1}{2}} \prod (u_i - v_j)^{\frac{1}{2}}.$$

Properties of ψ_z

- Discrete holomorphicity *within correlations*, that is, any correlation of the form

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is discrete holomoprhic in z (away from other marked points implicit in $\mathcal{O}$).

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- Discrete holomorphicity *within correlations*, that is, any correlation of the form

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- The *anti-symmetry*

$$\mathbb{E}(\mathcal{O}_1 \psi_z \mathcal{O}_2 \psi_w \mathcal{O}_3) = -\mathbb{E}(\mathcal{O}_1 \psi_w \mathcal{O}_2 \psi_z \mathcal{O}_3)$$

and the *Pfaffian structure* of the correlations

$$\frac{\mathbb{E}(\psi_{z_1} \ldots \psi_{z_k} \sigma_{v_1} \ldots \sigma_{v_n})}{\mathbb{E}(\sigma_{v_1} \ldots \sigma_{v_n})} = \mathrm{Pf}\, \frac{\mathbb{E}(\psi_{z_i} \psi_{z_j} \sigma_{v_1} \ldots \sigma_{v_n})}{\mathbb{E}(\sigma_{v_1} \ldots \sigma_{v_n})}$$

Convergence theorem

- As $\delta \to 0$, one has

$$\mathbb{E}_{\Omega^\delta}\left(\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right) \sim \prod_{i=1}^{n} C_i \cdot \delta^{\Delta_1+\dots+\Delta_n} \cdot \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\Omega,$$

  where each $\mathcal{O}_i$ can be any of σ, ϵ, μ, ψ.
- The *correlation functions* $\langle \sigma_{x_1}\dots\sigma_{x_n}\rangle_\Omega$ is conformally covariant:

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\Omega = \prod_{i=1}^{n} \varphi'(x_i)^{\Delta_i'} \prod_{i=1}^{n} \overline{\varphi'(x_i)}^{\Delta_i''} \cdot \langle \mathcal{O}_1(\varphi(x_1))\dots\mathcal{O}_n(\varphi(x_n))\rangle_{\varphi(\Omega)}.$$

- In the upper half-plane or an annulus, the formulas are (in principle) explicit.
- For boundary conditions, admit free, plus, minus, or combinations thereof.

Overview of the proof

- It suffices to consider $\mathbb{E}(\psi_{z_1}\dots\psi_{z_k}\sigma_{v_1}\dots\sigma_{v_n})$. E. g., $\mu_u = (u-v)^{\frac{1}{2}}\psi_{(u+v)/2}\sigma_v$, where $v\sim u$.
- By Pfaffian formula, it suffices to consider the asymptotics of

$$\frac{\mathbb{E}(\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n})}{\mathbb{E}(\sigma_{v_1}\dots\sigma_{v_n})} \quad \text{and} \quad \mathbb{E}(\sigma_{v_1}\dots\sigma_{v_n}),$$

  where v_i are away from each other, and z, w may be either away from other marked points, or immediately adjacent to v_i or each other.
- Use that

$$\frac{\mathbb{E}(\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n})}{\mathbb{E}(\sigma_{v_1}\dots\sigma_{v_n})}$$

  is discrete holomoprhic, solves a well-posed discrete boundary value problem, and has a "discrete pole" at $z=w$ with a residue proportional to its value (which is, more or less, equal to one).

Convergence of discrete holomoprhic functions

- When points are far apart, we deduce F

$$\delta^{-1}\frac{\mathbb{E}_{\Omega^\delta}(\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1}\dots\sigma_{v_n})}\to C\cdot\frac{\langle\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n}\rangle_\Omega}{\langle\sigma_{v_1}\dots\sigma_{v_n}\rangle_\Omega},$$

  where the RHS is a holomoprhic spinor solving a well-posed, conformally covariant boundary-value problem, with singularities of the type

$$(z-w)^{-1}\quad\text{and}\quad\alpha_j(z-v_j)^{-\frac{1}{2}}.$$

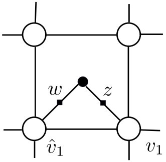
- When z is adjacent to v_1 (that is, at distance δ from v_1), we expect an additional factor of $\delta^{-\frac{1}{2}}$. This turns out to be is indeed true:

$$\delta^{-1}(z-v_1)^{\frac{1}{2}}\frac{\mathbb{E}_{\Omega^\delta}(\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1}\dots\sigma_{v_n})}\to C'\cdot\frac{\langle\psi_w\mu_{v_1}\dots\sigma_{v_n}\rangle_\Omega}{\langle\sigma_{v_1}\dots\sigma_{v_n}\rangle_\Omega},$$

  where the fraction in the RHS is equal to α_1. Similarly when $w\sim v_j$ and/or $w\sim z$.

Pure spin correlations

- Let $\hat{v}_1$ be adjacent to v_1, and take z, w as follows:



$$\frac{\mathbb{E}_{\Omega^\delta}(\psi_z \psi_w \sigma_{v_1} \dots \sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \dots \sigma_{v_n})} = \delta (z - v_1)^{-\frac{1}{2}} (w - v_1)^{-\frac{1}{2}} \frac{\mathbb{E}_{\Omega^\delta}(\sigma_{\hat{v}_1} \sigma_{v_2} \dots \sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \sigma_{v_2} \dots \sigma_{v_n})}$$

- This allows one to compute the limits of ratios

$$\frac{\mathbb{E}_{\Omega^\delta}(\sigma_{\hat{v}_1} \dots \sigma_{\hat{v}_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \dots \sigma_{v_n})} \to \frac{\langle \sigma_{\hat{v}_1} \dots \sigma_{\hat{v}_n} \rangle_\Omega}{\langle \sigma_{v_1} \dots \sigma_{v_n} \rangle_\Omega}.$$

- Finally, we use that

$$\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \dots \sigma_{v_{2n}}) \sim \mathbb{E}_{\mathbb{C}^\delta}(\sigma_{v_1} \sigma_{v_2}) \dots \mathbb{E}_{\mathbb{C}^\delta}(\sigma_{v_{2n-1}} \sigma_{v_{2n}})$$

as $v_1 \to v_2, \dots, v_{2n-1} \to v_{2n}$.

Fusion rules (or Operator Product Expansions)

Fusion rules are a collection of *asymptotic expansions* of correlation functions as marked point collide together.

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Example:

$$\sigma_v \sigma_{\hat{v}} = |v - \hat{v}|^{-\frac{1}{4}} \left(1 + \frac{1}{2} \epsilon_w |v - \hat{v}| + o(v - \hat{v}) \right), \quad v \to \hat{v}.$$

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This is understood as follows:

$$\langle \sigma_v \sigma_{\hat{v}} \mathcal{O} \rangle_\Omega = |v - \hat{v}|^{-\frac{1}{4}} \langle \sigma_v \sigma_{\hat{v}} \mathcal{O} \rangle_\Omega + \frac{1}{2} |v - \hat{v}|^{\frac{3}{4}} \langle \epsilon_v \mathcal{O} \rangle + o(v - \hat{v})^{\frac{3}{4}}, \quad v \to \hat{v},$$

where $\mathcal{O}$ is anything (containing spins, energies, disorders and fermions) away from v.

The rules

$$\psi_{\hat{w}}\psi_w = 2(\hat{w}-w)^{-1} + O(\hat{w}-w),$$

$$\psi_{\hat{w}}\psi^\star_w = -2i\epsilon_w + O(\hat{w}-w),$$

$$\psi_{\hat{w}}\epsilon_w = i(\hat{w}-w)^{-1}\psi^\star_w + O(1),$$

$$\psi_{\hat{w}}\mu_w = e^{-\frac{i\pi}{4}}(\hat{w}-w)^{-\frac{1}{2}}\left(\sigma_w + O(\hat{w}-w)\right),$$

$$\sigma_{\hat{w}}\sigma_w = |\hat{w}-w|^{-\frac{1}{4}}\left(1 + \frac{1}{2}\epsilon_w|\hat{w}-w| + o(\hat{w}-w)\right),$$

$$\mu_{\hat{w}}\mu_w = |\hat{w}-w|^{-\frac{1}{4}}\left(1 - \frac{1}{2}\epsilon_w|\hat{w}-w| + o(\hat{w}-w)\right),$$

$$\mu_{\hat{w}}\sigma_w = |\hat{w}-w|^{\frac{1}{2}}(\psi^{\eta_{\hat{w}w}}_w + O(\hat{w}-w)),$$

$$\epsilon_{\hat{w}}\epsilon_w = |\hat{w}-w|^{-2} + O(1),$$

$$\epsilon_{\hat{w}}\sigma_w = \frac{1}{2}|\hat{w}-w|^{-1}\sigma_w + O(1),$$

$$\epsilon_{\hat{w}}\mu_w = -\frac{1}{2}|\hat{w}-w|^{-1}\mu_w + O(1).$$

Overview of the proof

$$\langle \mathcal{O}_1(w)\mathcal{O}_2(\hat{w})\mathcal{O}\rangle_\Omega \sim \alpha_1|w-\hat{w}|^{\gamma_1}\langle \mathcal{O}_3(w)\mathcal{O}\rangle_\Omega + \ldots, \quad w \to \hat{w}.$$

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- By Pfaffian formulae, reduce everything to the case

$$\mathcal{O} = \sigma_{v_1}\ldots\sigma_{v_n} \quad \text{or} \quad \mathcal{O} = \psi_{z_1}\sigma_{v_1}\ldots\sigma_{v_n} \quad \text{or} \quad \mathcal{O} = \psi_{z_1}\psi_{z_2}\sigma_{v_1}\ldots\sigma_{v_n}.$$

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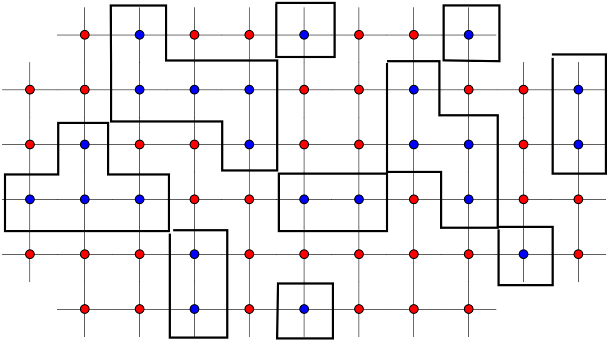
- Use properties of Riemann boundary value problem (uniqueness and compactness arguments are mostly enough)

Application to SLE_3 variants

Spins configurations can be put in correspondence to loop configurations:



Configurations := $\{S \subset \text{Edges}((\Omega^\delta)^\star) : \partial S = 0 \mod 2\}$.
$\mathbb{P}(S) = \frac{1}{Z} x^{|S|}$, where $x = e^{-2\beta} = \sqrt{2} - 1$.

Application to SLE_3 variants

It is natural to generalize this to:

$$\text{Configurations} := \{S \subset \mathsf{Edges}((\Omega^\delta)^\star) : \partial S = u_1, \ldots, u_m \bmod 2\}.$$

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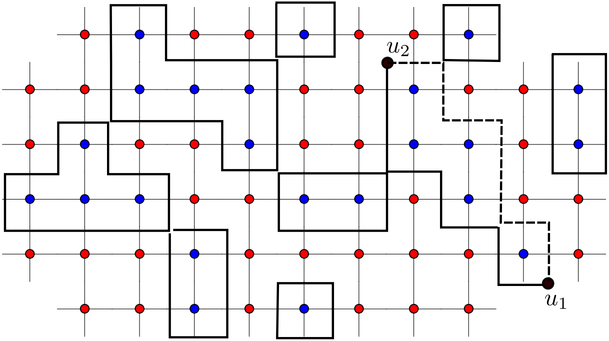
Apart from loops, there are now *interfaces* connecting $u_1, \ldots, u_m$ in some order.
On the lever of spins, this corresponds to *disorder insertions*, that is, tilting the probability measure by $\mu_\gamma = e^{-2\beta \sum_{(xy)\cap\gamma\neq\emptyset} \sigma_x \sigma_y}$ with γ such that $\partial\gamma = \{u_1, \ldots, u_m\} \bmod 2$.

Application to SLE_3 variants



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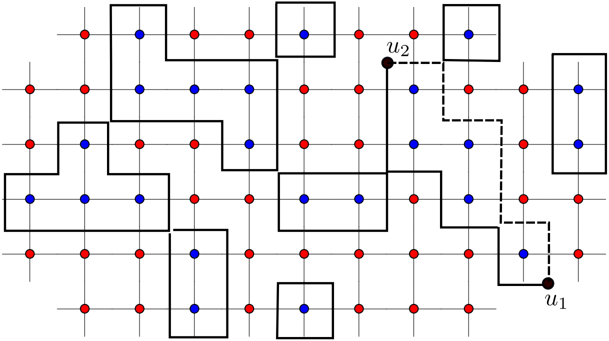

Bold: a random condiguration S with $\partial S = \{u_1, u_2\}$.
Dashed: a "disorder line" γ with $\partial\gamma = \{u_1, u_2\}$.

Martingale observables

Let $\beta_{[n]}$ be the initial segment of the interface starting from u_1.
Then,

$$\frac{\mathbb{E}_{\Omega^\delta \setminus \beta_{[n]}}(\mathcal{O}\mu_{\gamma \setminus \beta_{[n]}})}{\mathbb{E}_{\Omega^\delta \setminus \beta_{[n]}}(\mu_{\gamma \setminus \beta_{[n]}})}$$

is a martingale with respect to $\mathfrak{F}(\beta_{[n]})$.
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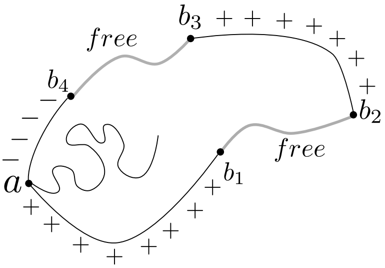
Usually, the most convenient choice is $\mathcal{O} = \psi_z \psi_w$ with $w \sim u_j$ for some j (as is the case for the original Smirnov's observable)

Martingale observables

This is enough to characterize the scaling limit of γ



$$da(t) = \sqrt{3}dB_t - \frac{3/2}{a(t)-b_1}dt - \frac{3/2}{a(t)-b_2}dt - \frac{3/2}{a(t)-b_3}dt$$
$$+ 3\left(a(t) - \frac{b_1\sqrt{b_3-b_2} + b_2\sqrt{b_3-b_1}}{\sqrt{b_3-b_2}+\sqrt{b_3-b_1}}\right)^{-1} dt.$$

Thank you!