Sum formula for scalar product of off-shell Bethe vectors

A. Hutsalyuk, **L**, S. Z. Pakuliak, É. Ragoucy, N. A. Slavnov

August 11, 2017

Table of contents

Generalized model

RTT-relation

$$R(u,v)\big(T(u)\otimes\mathbb{I}\big)\big(\mathbb{I}\otimes T(v)\big)=\big(\mathbb{I}\otimes T(v)\big)\big(T(u)\otimes\mathbb{I}\big)R(u,v),$$

where $T(u)$ is the monodromy matrix.
Bethe vectors belong to the space $\mathcal{H}$ in which the monodromy matrix entries act. We do not specify this space, however, we assume that it contains a *pseudovacuum vector* Ω, such that

$$\begin{aligned} T_{i,i}(u)\Omega &= \lambda_i(u)\Omega, \\ T_{i,j}(u)\Omega &= 0, \qquad\qquad i>j\,, \end{aligned}$$

where $\lambda_i(u)$ are some scalar functions.

Bethe vector $\mathfrak{gl}(2)$ case

In $\mathfrak{gl}(2)$ case the monodromy matrix is

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Then, the Bethe vector is

$$\mathbb{B}_n(\bar{u}) = B(u_1) \cdot \ldots \cdot B(u_n)\Omega = T_{12}(\bar{u})\Omega.$$

Bethe vector $\mathfrak{gl}(3)$ case

Bethe vector in the $\mathfrak{gl}(3)$ symmetry case

$$\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \\ = \sum \frac{\mathsf{K}_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{v}_{\rm II})\lambda_2(\bar{u})} \frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})f(\bar{u}_{\rm II},\bar{u}_{\rm I})}{f(\bar{v}_{\rm II},\bar{u})f(\bar{v}_{\rm I},\bar{u}_{\rm I})} T_{12}(\bar{u}_{\rm II})T_{13}(\bar{u}_{\rm I})T_{23}(\bar{v}_{\rm II})\Omega$$

The sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_{\rm I},\bar{v}_{\rm II}\}$ and $\bar{u} \Rightarrow \{\bar{u}_{\rm I},\bar{u}_{\rm II}\}$ with the restriction $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I} = n$, where $n = 0,1,\dots,\min(a,b)$.

Composite model

Consider a composite model, in which the monodromy matrix $T(u)$ is presented as a product of two partial monodromy matrices

$$T(u) = T^{(2)}(u)T^{(1)}(u).$$

Here every $T^{(l)}(u)$ satisfies the RTT-relation and has its own pseudovacuum vector $\Omega^{(l)}$, such that $\Omega = \Omega^{(1)} \otimes \Omega^{(2)}$. The operators $T^{(2)}_{i,j}(u)$ and $T^{(1)}_{k,l}(v)$ act in different spaces, and hence, they commute with each other. We assume that

$$T^{(l)}_{i,i}(u)\Omega = \lambda^{(l)}_i(u)\Omega, \;\; \lambda_i(u) = \lambda^{(1)}_i(u)\lambda^{(2)}_i(u).$$

Co-product formulas for $\mathfrak{gl}(N+1)$ Bethe vectors

Bethe vector for $\mathfrak{gl}(N+1)$ has N types of Bethe roots.
Co-product formula for Bethe vector is

$$\mathbb{B}(\bar{t}) = \sum \frac{\prod_{\nu=1}^{N} \alpha_{\nu}^{(2)}(\bar{t}_{\rm i}^{\nu}) f(\bar{t}_{\rm ii}^{\nu}, \bar{t}_{\rm i}^{\nu})}{\prod_{\nu=1}^{N-1} f(\bar{t}_{\rm ii}^{\nu+1}, \bar{t}_{\rm i}^{\nu})} \mathbb{B}^{(1)}(\bar{t}_{\rm i}) \mathbb{B}^{(2)}(\bar{t}_{\rm ii}).$$

And for dual Bethe vector

$$\mathbb{C}(\bar{s}) = \sum \frac{\prod_{\nu=1}^{N} \alpha_{\nu}^{(1)}(\bar{s}_{\rm ii}^{\nu}) f(\bar{s}_{\rm i}^{\nu}, \bar{s}_{\rm ii}^{\nu})}{\prod_{\nu=1}^{N-1} f(\bar{s}_{\rm i}^{\nu+1}, \bar{s}_{\rm ii}^{\nu})} \mathbb{C}^{(2)}(\bar{s}_{\rm ii}) \mathbb{C}^{(1)}(\bar{s}_{\rm i}).$$

Scalar product

The sum formula for scalar product is

$$S(\bar{s}|\bar{t}) = \mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \sum W(\bar{s}_{\mathrm{I}}, \bar{s}_{\mathrm{II}}|\bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}) \prod_{k=1}^{N} \alpha_k(\bar{s}_{\mathrm{I}}^k)\alpha_k(\bar{t}_{\mathrm{II}}^k)$$

with

$$W(\bar{s}_{\mathrm{I}}, \bar{s}_{\mathrm{II}}|\bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}) = \sum Z(\bar{s}_{\mathrm{I}}|\bar{t}_{\mathrm{I}}) \;\; Z(\bar{t}_{\mathrm{II}}|\bar{s}_{\mathrm{II}}) \frac{\prod_{k=1}^{N} f(\bar{s}_{\mathrm{II}}^k, \bar{s}_{\mathrm{I}}^k) f(\bar{t}_{\mathrm{I}}^k, \bar{t}_{\mathrm{II}}^k)}{\prod_{j=1}^{N-1} f(\bar{s}_{\mathrm{II}}^{j+1}, \bar{s}_{\mathrm{I}}^j) f(\bar{t}_{\mathrm{I}}^{j+1}, \bar{t}_{\mathrm{II}}^j)},$$

where we introduced the highest coefficient

$$Z(\bar{s}|\bar{t}) = W(\bar{s}, \emptyset|\bar{t}, \emptyset).$$

The highest coefficients in $\mathfrak{gl}(2)$ and $\mathfrak{gl}(3)$ cases

- In $\mathfrak{gl}(2)$ case $Z(\bar{x}|\bar{y})$ is Izergin-Korepin determinant.

$$K_n(\bar{x}|\bar{y}) = h(\bar{x},\bar{y}) \prod_{\ell<m}^{n} g(x_\ell,x_m)g(y_m,y_\ell) \ \det_n \left[t(x_i,y_j)\right].$$

- In $\mathfrak{gl}(3)$ case

$$Z_{a,b}(\bar{t};\bar{s}|\bar{x};\bar{y}) = (-1)^b \sum K_b(\bar{s}-c|\bar{w}_{\mathrm{I}})K_a(\bar{w}_{\mathrm{II}}|\bar{t})K_b(\bar{y}|\bar{w}_{\mathrm{I}})f(\bar{w}_{\mathrm{I}},\bar{w}_{\mathrm{II}}).$$

Here $\bar{w} = \{\bar{s},\ \bar{x}\}$. The sum is taken with respect to partitions of the set $\bar{w}$ into subsets $\bar{w}_{\mathrm{I}}$ and $\bar{w}_{\mathrm{II}}$.
There is no determinant formula in $\mathfrak{gl}(3)$ case.

Supercases

- $\mathfrak{gl}(1|1)$ case

$$Z_n(\bar{t}|\bar{x}) = g(\bar{t}, \bar{x}).$$

- $\mathfrak{gl}(2|1)$ case

$$Z_{a,b}(\bar{t};\bar{s}|\bar{x};\bar{y}) = h(\bar{\omega},\bar{t})\Delta'_a(\bar{t})\Delta'_b(\bar{y})\Delta_{a+b}(\bar{\omega}) \det_{a+b} \mathcal{N},$$

where $\bar{\omega} = \{\bar{x}, \bar{s}\}$ and the matrix $\mathcal{N}$ can be written as follows

$$\mathcal{N} = \begin{pmatrix} t(x_j, t_k) & | & g(x_j, y_k)\frac{h(x_j,\bar{x})}{h(x_j,\bar{t})} \\ - - - - & | & - - - - \\ t(s_j, t_k) & | & g(s_j, y_k)\frac{h(s_j,\bar{x})}{h(s_j,\bar{t})} \end{pmatrix}.$$

Off-shell Bethe vectors

Co-product formula and scalar product

The highest coefficient

Norm

Gaudin formula

The square of the norm of the on-shell Bethe vector reads

$$\mathbb{C}(\bar{t})\mathbb{B}(\bar{t}) = \prod_{\nu=1}^{N} \prod_{\substack{p,q=1 \\ p\neq q}}^{r_\nu} f(t_p^\nu, t_q^\nu) \left(\prod_{\nu=1}^{N-1} f(\bar{t}^{\nu+1}, \bar{t}^{\nu}) \right)^{-1} \det G,$$

where the matrix G is

$$G_{jk}^{(\mu,\nu)} = -c \frac{\partial \log \Phi_j^{(\mu)}}{\partial t_k^\nu},$$

where $\Phi_j^{(\mu)} = 1$ are Bethe equations.