Limiting curves for a class of self-similar adic transformations

Aleksei Minabutdinov

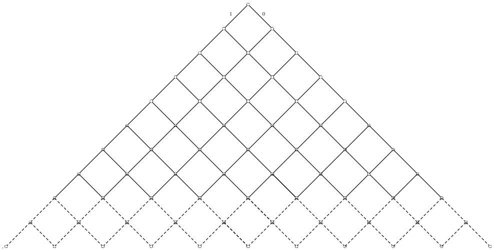
(based on a joint work with Andrei Lodkin)

National Research University
Higher School of Economics

10 August 2017

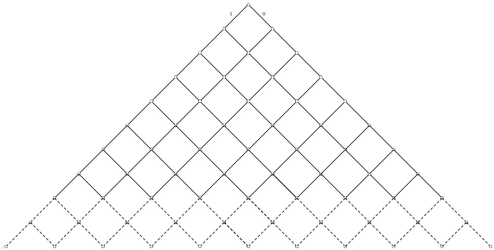
In the process of studying properties of the Pascal adic transformation, introduced to the Ergodic theory by A. M. Vershik in 1982, X. Mela, É. Janvresse, T. de la Rue and Y. Velenik (2005) found a new phenomenon which they called the *limiting curve*. Limiting curve provides explicit description of the deviations of ergodic sums along trajectories of dynamical systems.
Our work develops their results for a wider classes of functions and other dynamical systems.

The Pascal graph



Let I be $\{0,1\}^\infty$ and μ_q be the dyadic Bernoulli measure $\prod_1^\infty (q, 1-q)$, $q \in (0,1)$.

The Pascal graph



Let I be $\{0,1\}^\infty$ and μ_q be the dyadic Bernoulli measure $\prod_1^\infty (q, 1-q)$, $q \in (0,1)$.
A vertex of the Pascal graph has coordinates $(n,k), 0 \le k \le n$. We consider $\omega = (\omega_n)_{n=1}^\infty \in I$ as a path in the graph passing through the vertices $(n, k_n(\omega)), n \ge 1$.

Concatenations procedure

Let $N \in \mathbb{N}$. For each vertex $(n, k), n \geq N, 0 \leq k \leq n$, we consider a pair $(F_{n,k}, \varphi_{n,k})$ of piecewise linear continuous functions

$$F_{n,k} : [0, \binom{n}{k}] \to \mathbb{R}, \quad \varphi_{n,k} : [0, 1] \to [-1, 1].$$

Concatenations procedure

Let $N \in \mathbb{N}$. For each vertex $(n, k), n \geq N, 0 \leq k \leq n$, we consider a pair $(F_{n,k}, \varphi_{n,k})$ of piecewise linear continuous functions

$$F_{n,k} : [0, \binom{n}{k}] \to \mathbb{R}, \quad \varphi_{n,k} : [0, 1] \to [-1, 1].$$

Functions $F_{n,k}, n > N$, are defined recursively building up from some "initial functions" $F_{N,l}$ such that $F_{N,l}(0) = 0$, $0 \leq l \leq N$, using the following concatenation procedure:

Concatenations procedure

Let $N \in \mathbb{N}$. For each vertex $(n, k), n \geq N, 0 \leq k \leq n$, we consider a pair $(F_{n,k}, \varphi_{n,k})$ of piecewise linear continuous functions

$$F_{n,k} : [0, \tbinom{n}{k}] \to \mathbb{R}, \quad \varphi_{n,k} : [0,1] \to [-1,1].$$

Functions $F_{n,k}, n > N$, are defined recursively building up from some "initial functions" $F_{N,l}$ such that $F_{N,l}(0) = 0$, $0 \leq l \leq N$, using the following concatenation procedure:
For $n > N$ and $1 \leq k \leq n-1$ let

$$F_{n,k}(j) = \begin{cases} F_{n-1,k-1}(j), & 0 \leq j \leq \binom{n-1}{k-1}, \\ F_{n-1,k-1}(\binom{n-1}{k-1}) + F_{n-1,k}(j - \binom{n-1}{k-1}), & \binom{n-1}{k-1} < j \leq \binom{n}{k}, \end{cases}$$

where $0 \leq j \leq \binom{n}{k}$; and let $F_{n,0} = F_{n-1,0}$, $F_{n,n} = F_{n-1,n-1}$.

Concatenations procedure

Let $N \in \mathbb{N}$. For each vertex $(n,k), n \geq N, 0 \leq k \leq n$, we consider a pair $(F_{n,k}, \varphi_{n,k})$ of piecewise linear continuous functions

$$F_{n,k} : [0, \tbinom{n}{k}] \to \mathbb{R}, \quad \varphi_{n,k} : [0,1] \to [-1,1].$$

Functions $F_{n,k}, n > N$, are defined recursively building up from some "initial functions" $F_{N,l}$ such that $F_{N,l}(0) = 0$, $0 \leq l \leq N$, using the following concatenation procedure:
For $n > N$ and $1 \leq k \leq n-1$ let

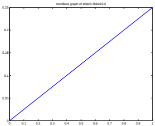
$$F_{n,k}(j) = \begin{cases} F_{n-1,k-1}(j), & 0 \leq j \leq \binom{n-1}{k-1}, \\ F_{n-1,k-1}(\binom{n-1}{k-1}) + F_{n-1,k}(j - \binom{n-1}{k-1}), & \binom{n-1}{k-1} < j \leq \binom{n}{k}, \end{cases}$$

where $0 \leq j \leq \binom{n}{k}$; and let $F_{n,0} = F_{n-1,0}$, $F_{n,n} = F_{n-1,n-1}$. We define

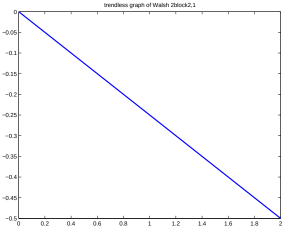
$$\varphi_{n,k}(t) = \frac{F_{n,k}(t \cdot \binom{n}{k}) - t \cdot F_{n,k}(\binom{n}{k})}{R_{n,k}},$$

where $R_{n,k} = \max_{t \in [0,1]} |F_{n,k}(t \cdot \binom{n}{k}) - t \cdot F_{n,k}(\binom{n}{k})|$, provided $|F_{n,k}(t \cdot \binom{n}{k}) - t \cdot F_{n,k}(\binom{n}{k})| \not\equiv 0$, and $R_{n,k} = 1$, otherwise).

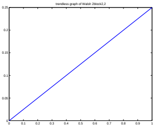
Functions $F_{n,k}$, $n = 2, 3$, for the given initial condition



$F_{2,0}$

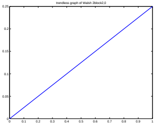


$F_{2,1}$

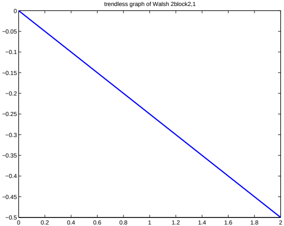


$F_{2,2}$

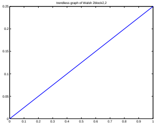
Functions $F_{n,k}$, $n = 2, 3$, for the given initial condition



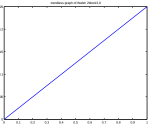
$F_{2,0}$



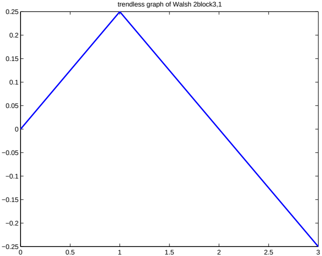
$F_{2,1}$



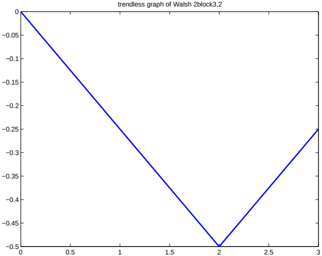
$F_{2,2}$



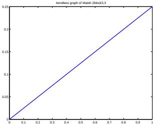
$F_{3,0}$



$F_{3,1}$

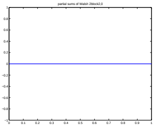


$F_{3,2}$

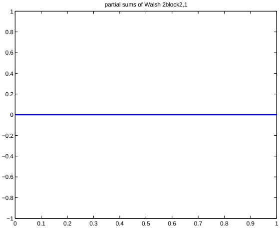


$F_{3,3}$

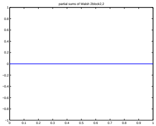
Functions $\varphi_{n,k}$, $n = 2, 3$, for the given initial condition



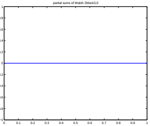
$\varphi_{2,0}$



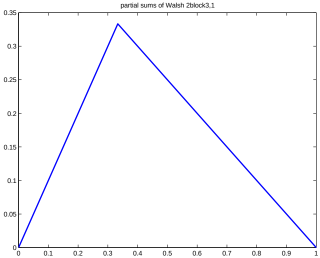
$\varphi_{2,1}$



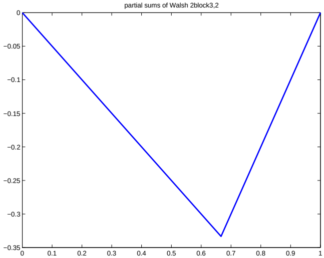
$\varphi_{2,2}$



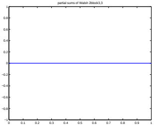
$\varphi_{3,0}$



$\varphi_{3,1}$

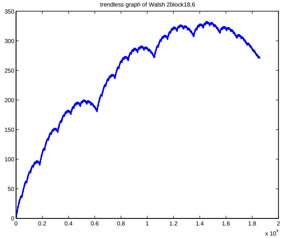


$\varphi_{3,2}$

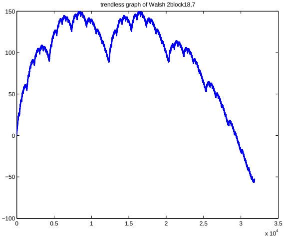


$\varphi_{3,3}$

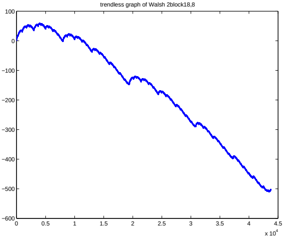
Functions $F_{18,k}$, $k = 6, 7, 8, 9$ for the given initial condition



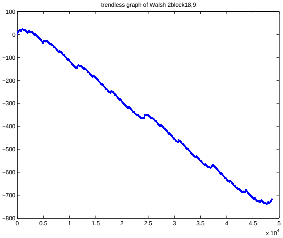
$F_{18,6}$



$F_{18,7}$

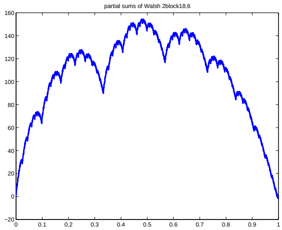


$F_{18,8}$

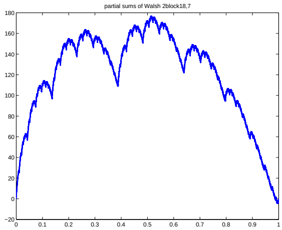


$F_{18,9}$

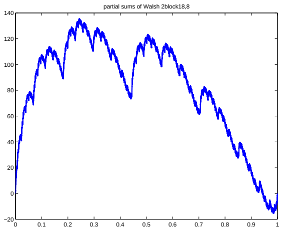
Functions $\varphi_{18,k}$, $k = 6, 7, 8, 9$ for the given initial condition



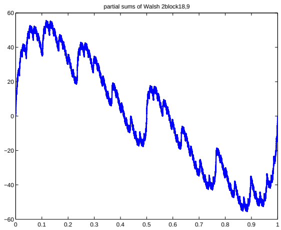
$\varphi_{18,6}$



$\varphi_{18,7}$



$\varphi_{18,8}$



$\varphi_{18,9}$

Limiting curves: existence

Measure space $(I, \mathcal{B}, \mu_q)$.
Let $\omega = (\omega)_n \in I$ be a path in the Pascal graph.
Assume that the vector $(F_{N,l}(\binom{N}{l}))_{l=0}^{N}$ containing endpoint values of initial functions is *not proportional* to the vector $(\binom{N}{l})_{l=0}^{N}$ of binomial coefficients.
Functions $\varphi_{n,k}$ are defined as above.

Theorem (É. Janvresse, T. de la Rue and Y. Velenik, '05)

For any sequence $(k_n(\omega))$ such that $\lim_n \frac{k_n(\omega)}{n} = q \in (0,1)$, one can extract a subsequence (n_j) such that $\varphi_{n_j, k_{n_j}(\omega)}$ converges in $C[0,1]$ to a continuous *limiting function*.

The graph of this limiting function is called a *limiting curve*.
In general, limiting curve depends on initial values, path ω and chosen subsequence n_j.

The Takagi function

The Takagi function $\tau(t)$ is a continuous nowhere differentiable function introduced by Teiji Takagi in 1903. It has appeared in a surprising number of different mathematical contexts, including mathematical analysis, probability theory and number theory.
It can be defined on the unit interval $x \in [0, 1]$ by

$$\tau(x) = \lim_{n} \tau_n(x),\ \tau_n(x) = \sum_{k=0}^{n} \frac{d(2^k x)}{2^k}, \quad d(x) = \min_{n \in \mathbb{Z}} |x - n|$$

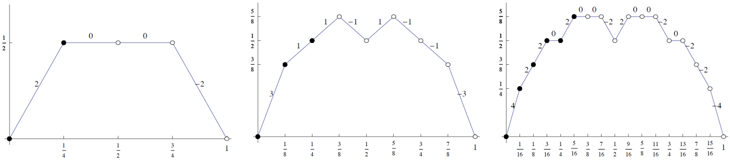


Figure: Approximants to Takagi function: (left to right) $\tau_2; \tau_3; \tau_4$ (picture by J. Lagarias).

The distribution function of the measure μ_q

The Bernoulli measure $\mu_q = \prod(q, 1-q)$ (if carried to $[0,1]$) can be defined by the distribution function $L_q(x) : [0,1] \to [0,1]$.

$$L_q : x = \sum_{k=1}^{\infty} \omega_k \frac{1}{2^k} \quad \mapsto \quad \sum_{k=1}^{\infty} \omega_k q^{k-s_{k-1}} (1-q)^{s_{k-1}},$$

where $s_k = \sum_{j=1}^{k} \omega_j, \ \omega_j \in \{0,1\}$.

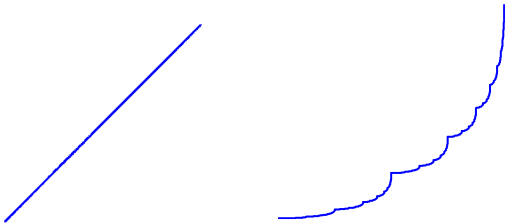


Figure: The graphs of $L_{0.5}$ (left) and $L_{0.3}$ (right)

A class of self-affine functions

Let q_1 and q_2 be distinct parameters from $(0,1)$. We consider the function $S_{q_1,q_2} : [0,1] \to [0,1]$ defined by $S_{q_1,q_2} = L_{q_2} \circ L_{q_1}^{-1}$. For $k \in \mathbb{N}$ we define the function $\mathcal{T}_q^k$ by the identity:

$$\mathcal{T}_q^k := \frac{\partial^k S_{q,a}}{\partial a^k}\bigg|_{a=q}, \; k \in \mathbb{N}.$$

The function $\frac{1}{2}\mathcal{T}_{1/2}^1$ is the famous Takagi function (M. Hata and M. Yamaguti).

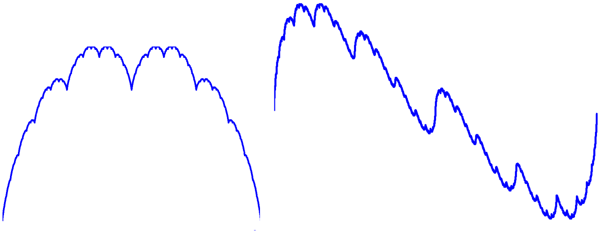


Figure: The graphs of $\mathcal{T}_{1/2}^1$ (left) and $\mathcal{T}_{1/2}^2$ (right)

A class of self-affine functions.

$$\mathcal{T}_q^k(q^i) = \frac{\partial^k}{\partial q^k} q^i = i(i-1)\ldots(i-k)q^{i-k-1},$$

$i \in \mathbb{N}$, in particular,

$$\mathcal{T}_q^1(q^i) = iq^{i-1}, \qquad \mathcal{T}_q^2(q^i) = i(i-1)q^{i-2}.$$

Taking into account certain self-affinity relations, the functions $\mathcal{T}_q^k$ are uniquely defined by these values.

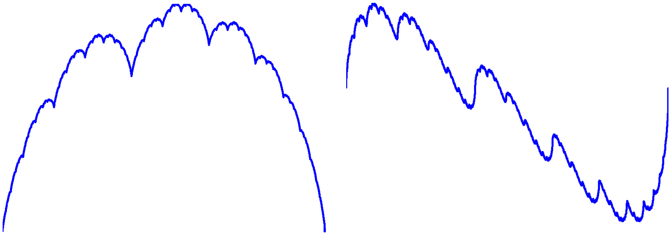


Figure: The graphs of $\mathcal{T}_{0.4}^1$ (left) and $\mathcal{T}_{0.4}^2$ (right)

Let's return to the Pascal adic

Theorem (É. Janvresse, T. de la Rue and Y. Velenik, '05)

Let $N = 1$ and $F_{1,0}(1) = 1, F_{1,1}(1) = -1$ be the initial condition[1]. For any sequence $(k_n(\omega))_{n=1}^{\infty}$ such that $k_n(\omega)/n \to q \in (0,1)$, one can extract a subsequence (n_j) such that $\varphi_{n_j, k_{n_j}(\omega)}$ converges in $C[0,1]$ to the function $\pm\mathcal{T}_q^1$.

[1] In fact, it was shown that the statement holds for a slightly more general class of initial conditions.

Let's return to the Pascal adic

Theorem (É. Janvresse, T. de la Rue and Y. Velenik, '05)

Let $N = 1$ and $F_{1,0}(1) = 1, F_{1,1}(1) = -1$ be the initial condition[1]. For any sequence $(k_n(\omega))_{n=1}^{\infty}$ such that $k_n(\omega)/n \to q \in (0,1)$, one can extract a subsequence (n_j) such that $\varphi_{n_j, k_{n_j}(\omega)}$ converges in $C[0,1]$ to the function $\pm\mathcal{T}_q^1$.

Question: What curves can arise for initial conditions other then $F_{1,0}(1) = 1$ *and* $F_{1,1}(1) = -1$*?*

[1] In fact, it was shown that the statement holds for a slightly more general class of initial conditions.

Let's return to the Pascal adic

Theorem (É. Janvresse, T. de la Rue and Y. Velenik, '05)

Let $N = 1$ and $F_{1,0}(1) = 1, F_{1,1}(1) = -1$ be the initial condition[1]. For any sequence $(k_n(\omega))_{n=1}^{\infty}$ such that $k_n(\omega)/n \to q \in (0,1)$, one can extract a subsequence (n_j) such that $\varphi_{n_j,k_{n_j}(\omega)}$ converges in $C[0,1]$ to the function $\pm\mathcal{T}_q^1$.

Question: What curves can arise for initial conditions other then $F_{1,0}(1) = 1$ *and* $F_{1,1}(1) = -1$*?*
The following theorem answers this question.

Theorem (A. Lodkin, A.M., 2016)

For μ_q almost any ω if a subsequence (n_j) is such that $\varphi_{n_j,k_{n_j}(\omega)}$ converges in $C[0,1]$ to a continues function then this function $\pm\mathcal{T}_q^1$.

[1] In fact, it was shown that the statement holds for a slightly more general class of initial conditions.

How to find the limiting curves explicitly?

Let $h_{n,k} = F_{n,k}(\binom{n}{k}), n \geq N$.

How to find the limiting curves explicitly?

Let $h_{n,k} = F_{n,k}(\binom{n}{k}), n \geq N$.
Function $h(k) = h_{n,k}, 0 \leq k \leq n$, is a function defined on the discrete space $\{0, 1, 2, 3 \ldots, n\}$ with binomial measure defined by the weighting function $w(k) = \binom{n}{k} q^k (1-q)^{n-k}$.

How to find the limiting curves explicitly?

Let $h_{n,k} = F_{n,k}(\binom{n}{k}), n \geq N$.
Function $h(k) = h_{n,k}, 0 \leq k \leq n$, is a function defined on the discrete space $\{0, 1, 2, 3 \ldots, n\}$ with binomial measure defined by the weighting function $w(k) = \binom{n}{k} q^k (1-q)^{n-k}$.
For a given $k \in \mathbb{N}$, any positive integer $x \in \mathbb{N}$ can be uniquely written in the $k-$cascade $x = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_{k-s}}{k-s}$, where $a_k > a_{k-1} > \cdots > a_{k-s} \geqslant 0$.

How to find the limiting curves explicitly?

Let $h_{n,k} = F_{n,k}(\binom{n}{k}), n \geq N$.
Function $h(k) = h_{n,k}, 0 \leq k \leq n$, is a function defined on the discrete space $\{0, 1, 2, 3 \ldots, n\}$ with binomial measure defined by the weighting function $w(k) = \binom{n}{k} q^k (1-q)^{n-k}$.
For a given $k \in \mathbb{N}$, any positive integer $x \in \mathbb{N}$ can be uniquely written in the $k-$cascade $x = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_{k-s}}{k-s}$, where $a_k > a_{k-1} > \cdots > a_{k-s} \geqslant 0$.
The following expression (generalizing Vandermonde's convolution formula) holds

$$F_{n,k}(x) = \sum_{j=0}^{N} h_{N,j} \partial_{n-k}^{N,j}(x),$$

where $\partial_k^{N,j}(x) = \binom{a_k - N}{k-j} + \binom{a_{k-1} - N}{k-1-j} + \cdots + \binom{a_{k-s} - N}{k-s-j}$ and $a_{k-s}(x) \geq N$.

How to find the limiting curves explicitly?

Let $h_{n,k} = F_{n,k}(\binom{n}{k}), n \geq N$.
Function $h(k) = h_{n,k}, 0 \leq k \leq n$, is a function defined on the discrete space $\{0, 1, 2, 3 \ldots, n\}$ with binomial measure defined by the weighting function $w(k) = \binom{n}{k} q^k (1-q)^{n-k}$.
For a given $k \in \mathbb{N}$, any positive integer $x \in \mathbb{N}$ can be uniquely written in the $k-$cascade $x = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_{k-s}}{k-s}$, where $a_k > a_{k-1} > \cdots > a_{k-s} \geqslant 0$.
The following expression (generalizing Vandermonde's convolution formula) holds

$$F_{n,k}(x) = \sum_{j=0}^{N} h_{N,j} \partial_{n-k}^{N,j}(x),$$

where $\partial_k^{N,j}(x) = \binom{a_k - N}{k-j} + \binom{a_{k-1} - N}{k-1-j} + \cdots + \binom{a_{k-s} - N}{k-s-j}$ and $a_{k-s}(x) \geq N$.
Thus initial condition is in fact given by $N+1$ numbers $h_{N,l}, 0 \leq l \leq N$.

How to find the limiting curves explicitly?

In order to find the limiting curve φ along the sequence $(n_j, k(n_j))_j$ (we write simply (n, k)) for the given initial conditions $h(l) = h_{N,l}, 0 \leq l \leq N$, we decompose the function $h(l)$ in the convenient basis $\{h^m(l)\}_{m=0}^N$.

How to find the limiting curves explicitly?

In order to find the limiting curve φ along the sequence $(n_j, k(n_j))_j$ (we write simply (n,k)) for the given initial conditions $h(l) = h_{N,l}, 0 \le l \le N$, we decompose the function $h(l)$ in the convenient basis $\{h^m(l)\}_{m=0}^N$.

Krawtchouk polynomials $K(k,q,n)$ are discrete orthogonal polynomials of a discrete variable $k \in \{0,1,\ldots,n\}$ associated with the binomial distribution $w(k) = \binom{n}{k} q^k (1-q)^{n-k}$, introduced by M. Krawtchouk in 1929.

They can be defined by the identity

$$K_m(k,q,n) = {}_2F_1\left[\begin{matrix} -k, & -m \\ & -n \end{matrix}; \frac{1}{q}\right] \tag{1}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Setting $h^m_{N,k} = (-2q)^m K_m(k,q,N), 0 \le k \le N, 0 \le m \le N$, by induction one can show that

$$F^m_{n,k}\left(\binom{n-i}{k-i}\right) = (-2q)^m K_m(k-i,q,n-i)\binom{n-i}{k-i}$$

We study the asymptotic behavior of $F_{n,k}$ for $n \to \infty, \frac{k}{n} \to q$.

Let $n \to \infty$, and $\xi_n = \frac{k-nq}{\sqrt{nq(1-q)}} = O(1)$.

- There is a classical asymptotic expansion of the Krawtchouk polynomial

$$K_m(k,q,n) = b_0 H_m(\xi_n) + O(n^{-(m+1)/2}),$$

where $H_m(x) \equiv (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2}$ is the Hermite polynomial and $b_0 = (-1)^m \left(\frac{pq}{2n}\right)^{m/2}$.

- There are also expansions by N. Temme and J. Lopez (with asymptotic property)

$$K_m(k,q,n) = \sum_{j=0}^{m} b_j(\xi_n) H_{m-j}(\xi_n)$$

For instance, we have

$$K_m(k,q,n) = b_0 H_m(\xi_n) + b_3(\xi_n) H_{m-3}(\xi_n) \frac{1}{n} + o(n^{-(m+2)/2})$$

Recall that

$$\varphi_{n,k}(t) = \frac{F_{n,k}(t \cdot \binom{n}{k}) - t \cdot F_{n,k}(\binom{n}{k})}{R_{n,k}}.$$

Note that $\lim_{n\to\infty} \frac{\binom{n-i}{k-i}}{\binom{n}{k}} = q^i$, provided $\frac{k}{n} \to q$.

Recall that

$$\varphi_{n,k}(t) = \frac{F_{n,k}(t \cdot \binom{n}{k}) - t \cdot F_{n,k}(\binom{n}{k})}{R_{n,k}}.$$

Note that $\lim_{n\to\infty} \frac{\binom{n-i}{k-i}}{\binom{n}{k}} = q^i$, provided $\frac{k}{n} \to q$.
Technically, for almost every sequence $(n, k_n(x))$ (in the sense of the μ_q measure) using the classical expansion of the Krawtchouk polynomials, we show that for $x_{i,k,n} = \binom{n-i}{k-i}$, $i = O(1)$,

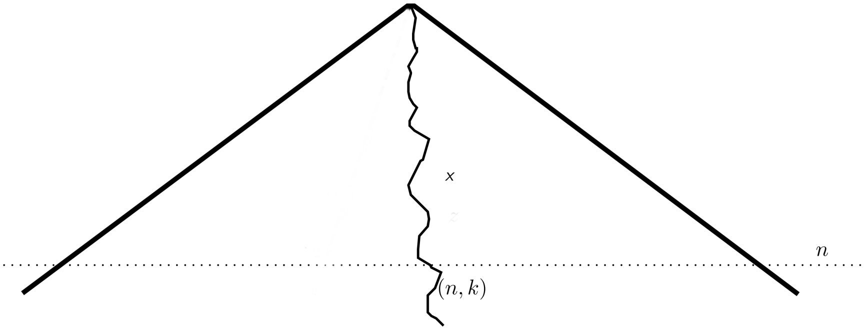
$$F^m_{n,k}(x_{i,k,n}) - \frac{x_{i,k,n}}{x_{0,k,n}} \cdot F^m_{n,k}(x_{0,k,n}) = iq^{i-1}R^m_{n,k} + o(R^m_{n,k})$$

$$R^m_{n,k} = \beta H_{m-1}(\xi_n),$$

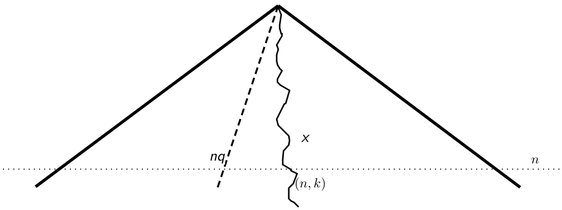
where $\beta = \beta(m, n, k, q)$, provided $\xi = \lim_n \xi_n$ is not the root of H_{m-1} this corresponds to $\mathcal{T}^1_q$ function.

Graphical interpretation

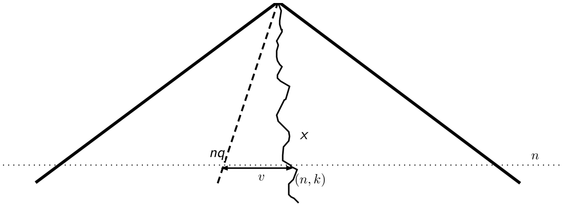
Consider an infinite path passing through the vertices $(n, k_n(x))$ of the Pascal graph.



Limiting curves

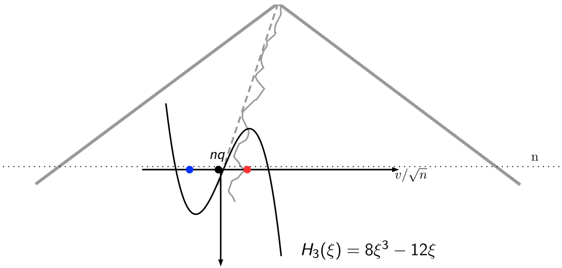


Limiting curves



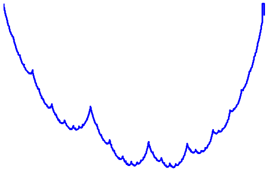
$v = k - nq,$

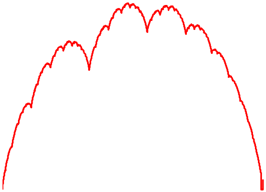
Limiting curves



$$v = k - nq,\ \xi = \frac{k-nq}{\sqrt{2q(1-q)n}} = \frac{v}{\sqrt{2q(1-q)n}},\ \ m = 4.$$

Limiting curves



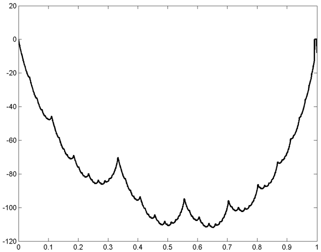


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

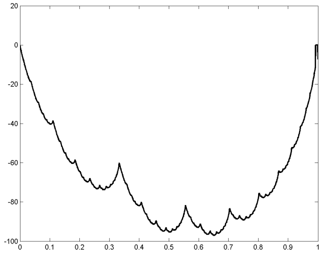


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

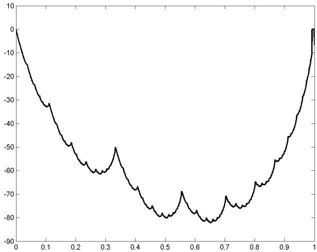


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

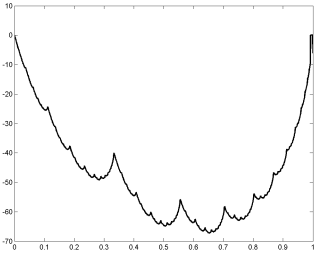


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

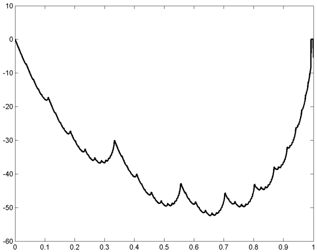


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

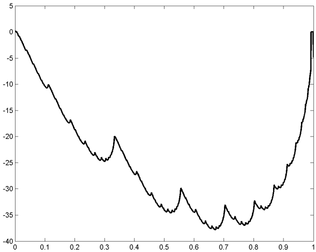


Limiting curves

Transition regimes

What if $v = O(1)$?

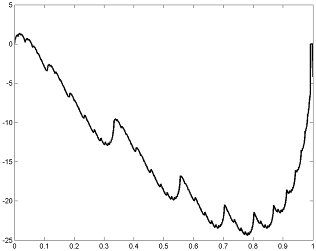
$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$



Limiting curves
Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

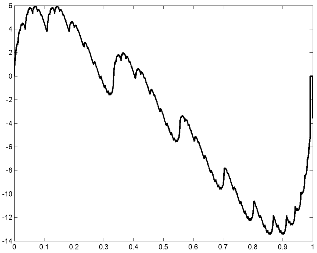


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

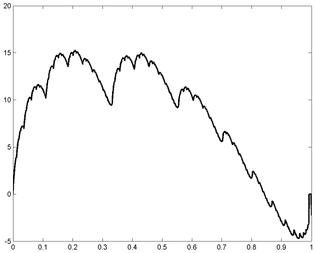


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

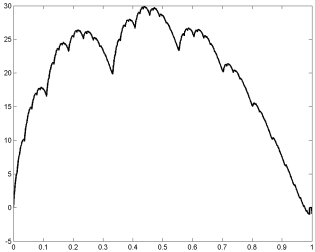


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

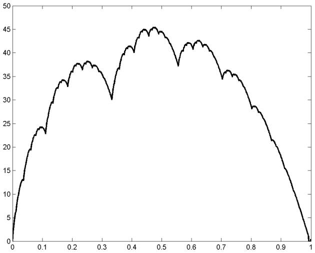


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

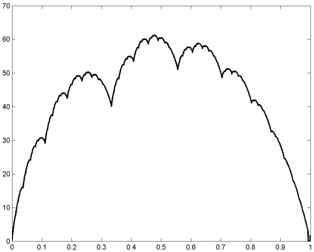


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

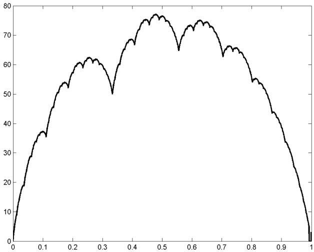


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

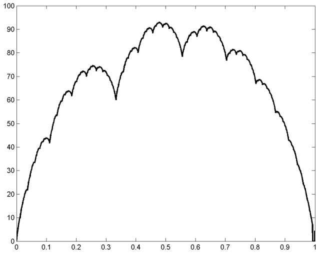


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$

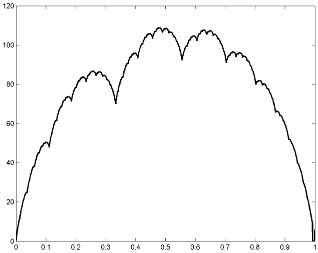


Limiting curves

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$



Limiting curves: dynamical systems interpretation

Let $(f_0, f_1, \ldots, f_n, \ldots)$ be a sequence with $f_i \in \mathbb{R}, i \in \mathbb{N}_0$.
Let F denote the partial sum, defined by

$$F(n) = \sum_{i=0}^{n-1} f_i, n \geq 0,$$

The function F is assumed to be linearly interpolated between consecutive integers.
Let the function $\varphi_n : [0, 1] \to [-1, 1]$ be defined by

$$\varphi_n(t) = \frac{F(t \cdot n) - t \cdot F(n)}{R_n},$$

(The normalizing coefficient $R_n = \max_{t \in [0,1]} |F(t \cdot n) - t \cdot F(n)|$, provided $|F(t \cdot n) - t \cdot F(n)| \not\equiv 0$, and $R_n = 1$, otherwise)

Definition (É. Janvresse, T. de la Rue and Y. Velenik)

Let (X, T) be a dynamical system, a function $f : X \to \mathbb{R}$ and a point $x \in X$. Define $f_i := f(T^i x), i \geq 0$, and considered cluster points in $C[0, 1]$ of the set $\{\varphi_n\}_{n\geq 1}$. Any cluster point $\varphi = \varphi_{x,f}$ is called a *limiting function*.

Definition (É. Janvresse, T. de la Rue and Y. Velenik)

Let (X, T) be a dynamical system, a function $f : X \to \mathbb{R}$ and a point $x \in X$. Define $f_i := f(T^i x), i \geq 0$, and considered cluster points in $C[0, 1]$ of the set $\{\varphi_n\}_{n \geq 1}$. Any cluster point $\varphi = \varphi_{x,f}$ is called a *limiting function*.

É. Janvresse, T. de la Rue and Y. Velenik showed that for the Pascal adic transformation the famous graph of the Takagi-Blancmange function arises as a limiting curve for certain functions f.

Adic dynamical systems

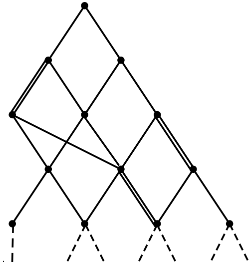


Figure: A graded graph (Bratteli diagram).

- The space X of infinite edge paths of some graded graph with some linear order on the incoming edges of each vertex. It is equipped with a partial order $\preceq$, which is lexicographical on the set of edge paths in X that belong to the same class of the tail partition.
- The adic transformation T is defined on $X \setminus \left(X_{\max} \cup X_{\min}\right)$ by sending $x \in X$ to its successor Tx, that is, the smallest y that satisfies $y \succ x$.

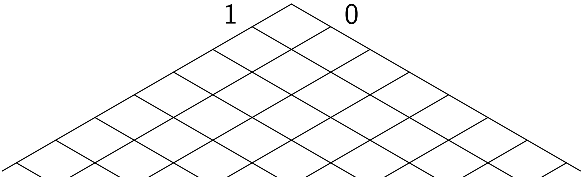
Let (X, T) be an adic transformation. This assumption is not restrictive due to the following theorem by A. M. Vershik:

Theorem

Any ergodic measure preserving transformation on a Lebesgue space is isomorphic to some adic transformation.

Let $\mathscr{F}_N$ denote the space of cylindric functions of rank N (i.e., functions that depend only on the first N coordinates of $x = (x_n)_0^\infty$).

The Pascal adic transformation



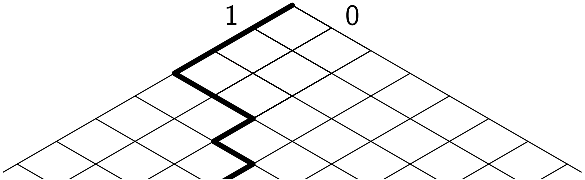
Let I be $\{0,1\}^\infty$ and μ_q be the dyadic Bernoulli measures $\prod_1^\infty(q, 1-q)$, $q \in (0,1)$.
We denote by P the Pascal adic transformation can be explicitly defined by:

$$x \mapsto Px; \quad P(0^{m-l}1^l\mathbf{10}\dots) = 1^l0^{m-l}\mathbf{01}\dots$$

(that is only the initial $m+2$ coordinates of x are being changed).

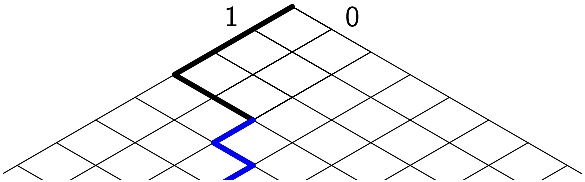
$$x \mapsto Px; \quad P(0^{m-s}1^{s}\mathbf{10}\ldots) = 1^{s}0^{m-s}\mathbf{01}\ldots \tag{2}$$

$$x = 11\mathbf{10}0101\ldots$$



$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \tag{2}$$

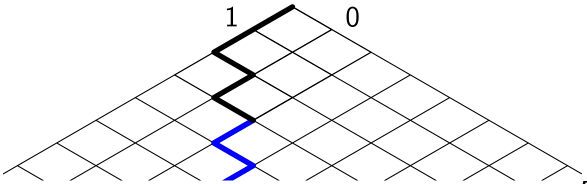
$$x = 11100\color{blue}{101}\dots$$



$$x \quad = 11100\color{blue}{\dots}$$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \qquad (2)$$

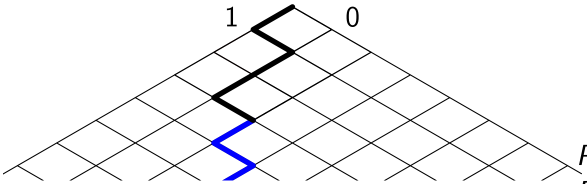
$$x = 11100101\dots$$



$$P(x) = 11010\dots$$
$$x = 11100\dots$$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \tag{2}$$

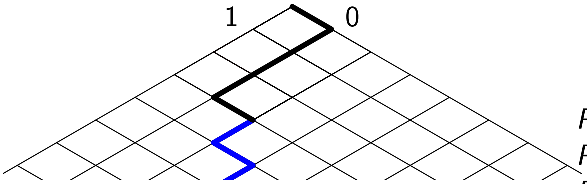
$$x = 11100101\dots$$



$P^2(x) = 10110\dots$
$P(x) = 11010\dots$
$x = 11100\dots$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\ldots) = 1^s0^{m-s}\mathbf{01}\ldots \qquad (2)$$

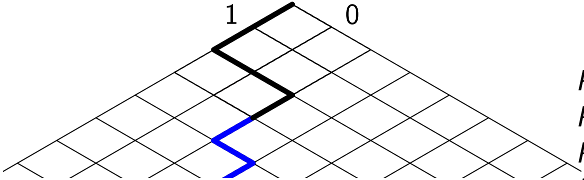
$$x = 11100\textcolor{blue}{101}\ldots$$



$$P^3(x) = 01110\ldots$$
$$P^2(x) = 10110\ldots$$
$$P(x) = 11010\ldots$$
$$x = 11100\ldots$$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \qquad (2)$$

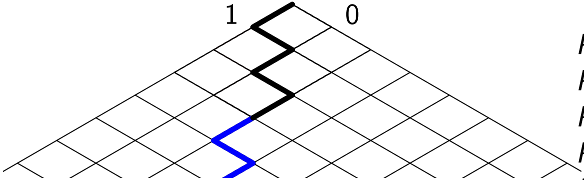
$$x = 11100101\dots$$



$$P^4(x) = 11001\dots$$
$$P^3(x) = 01110\dots$$
$$P^2(x) = 10110\dots$$
$$P(x) = 11010\dots$$
$$x = 11100\dots$$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \tag{2}$$

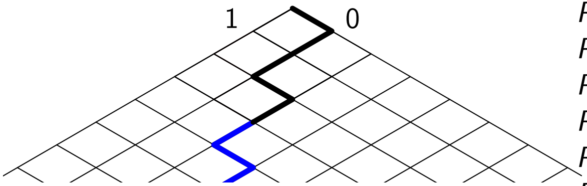
$$x = 11100101\dots$$



$$P^5(x) = 10101\dots$$
$$P^4(x) = 11001\dots$$
$$P^3(x) = 01110\dots$$
$$P^2(x) = 10110\dots$$
$$P(x) = 11010\dots$$
$$x = 11100\dots$$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \tag{2}$$

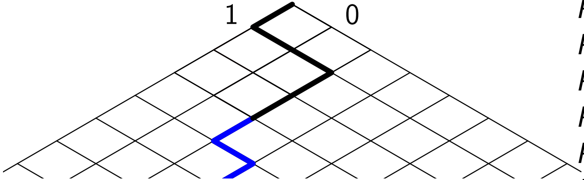
$$x = 11100101\dots$$



$P^6(x) = 01101\dots$
$P^5(x) = 10101\dots$
$P^4(x) = 11001\dots$
$P^3(x) = 01110\dots$
$P^2(x) = 10110\dots$
$P(x) = 11010\dots$
$x = 11100\dots$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s 0^{m-s}\mathbf{01}\dots \tag{2}$$

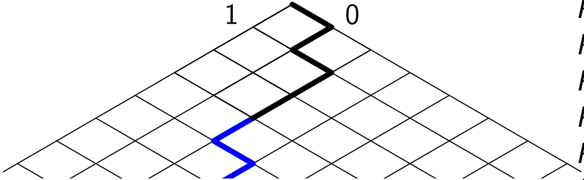
$$x = 11100101\dots$$



$$P^7(x) = 10011\dots$$
$$P^6(x) = 01101\dots$$
$$P^5(x) = 10101\dots$$
$$P^4(x) = 11001\dots$$
$$P^3(x) = 01110\dots$$
$$P^2(x) = 10110\dots$$
$$P(x) = 11010\dots$$
$$x = 11100\dots$$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \tag{2}$$

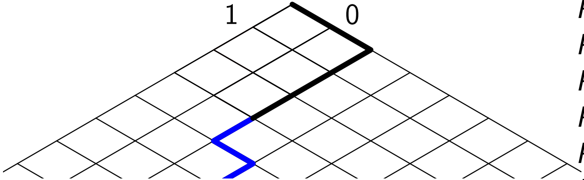
$$x = 11100\textcolor{blue}{101}\dots$$



$$\begin{aligned}
P^8(x) &= 01011\dots \\
P^7(x) &= 10011\dots \\
P^6(x) &= 01101\dots \\
P^5(x) &= 10101\dots \\
P^4(x) &= 11001\dots \\
P^3(x) &= 01110\dots \\
P^2(x) &= 10110\dots \\
P(x) &= 11010\dots \\
x &= 11100\dots
\end{aligned}$$

$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\ldots) = 1^s0^{m-s}\mathbf{01}\ldots \tag{2}$$

$$x = 11100101\ldots$$



$$
\begin{aligned}
P^9(x) &= 00111\ldots\\
P^8(x) &= 01011\ldots\\
P^7(x) &= 10011\ldots\\
P^6(x) &= 01101\ldots\\
P^5(x) &= 10101\ldots\\
P^4(x) &= 11001\ldots\\
P^3(x) &= 01110\ldots\\
P^2(x) &= 10110\ldots\\
P(x) &= 11010\ldots\\
x &= 11100\ldots
\end{aligned}
$$

Theorem

Let P be the Pascal adic transformation of the Lebesgue probability space $(I, \mathcal{B}, \mu_q)$, $N \in \mathbb{N}$, and $g \in \mathscr{F}_N$ be a function that is not cohomologous to a constant. Then for μ_q-a.e. x there exists a stabilizing sequence $l_n(x)$ such that the limiting function is $\alpha_{g,x}\mathcal{T}_q^1$, where $\alpha_{g,x} \in \{-1, 1\}$.

Wider class of dynamical systems

Polynomial adic systems

Let $p(x)$ be a positive integer polynomial, for example,
$p(x) = 1 + x + 3x^2$

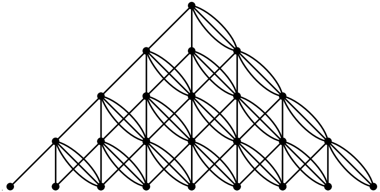


Figure: The graded graph associated to $p(x) = 1 + x + 3x^2$.

Wider class of dynamical systems

Polynomial adic systems

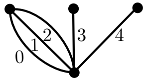


Figure: Canonical ordering for $p(x) = 1 + x + 3x^2$.

As for the Pascal adic, the set of all invariant ergodic measures for a polynomial system is a certain one-parameter family $\mu_{\boldsymbol{q}}$, $q \in (0, \frac{1}{a_0})$, of Bernoulli measures.

Denote by t_q the unique solution in $(0, 1)$ of the equation

$$a_0 q^d + a_1 q^{d-1} t + \cdots + a_d t^d - q^{d-1} = 0.$$

X. Mela and S. Bailey showed that these measures are as follows:

$$\mu_{\boldsymbol{q}} = \prod_0^\infty \Big(\underbrace{q, \ldots, q}_{a_0}, \underbrace{t_q, \ldots, t_q}_{a_1}, \underbrace{\frac{t_q^2}{q}, \ldots, \frac{t_q^2}{q}}_{a_2}, \ldots, \underbrace{\frac{t_q^d}{q^{d-1}}, \ldots, \frac{t_q^d}{q^{d-1}}}_{a_d} \Big).$$

Wider class of dynamical systems

Theorem

Let (X, T, μ_q) *be a polynomial system and* g *be a cylindric function from* $\mathscr{F}_N$*. Then for* μ_q*-a.e.* x *a limiting curve* $\varphi_x^g \in C[0,1]$ *exists if and only if the function* g *is not cohomologous to a constant.*

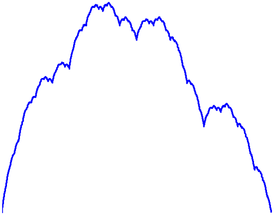


Figure: An example of a limiting curve

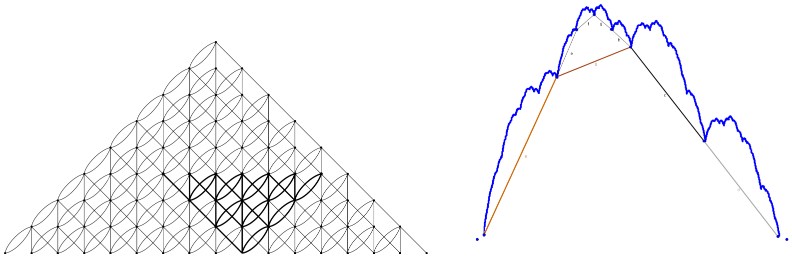


Figure: The Bratteli diagram and the polygonal approximation $p(x) = 2 + x + x^2$.

A class of self-affine functions

Everything can be generalized for $p(x) = a_0 + a_1 x + \cdots + a_d x^d$:
As above, for $q_1,\ q_2 \in (0, 1/a_0)$, functions $S^p_{q_1,q_2} : [0,1] \to [0,1]$ can be defined.
Similarly,

$$\mathcal{T}^k_{p,q_1} := \left.\frac{\partial^k S^p_{q_1,q_2}}{\partial q_2^k}\right|_{q_2=q_1},\ \ k \in \mathbb{N}.$$

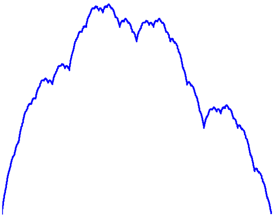


Figure: The graph of $\mathcal{T}^k_{p,q_1}$,$p(x) = 2 + x + x^2$.

Smooth limits of limiting curves

We answer the question by É. Janvresse et.al: is there a smooth curve in the limit?

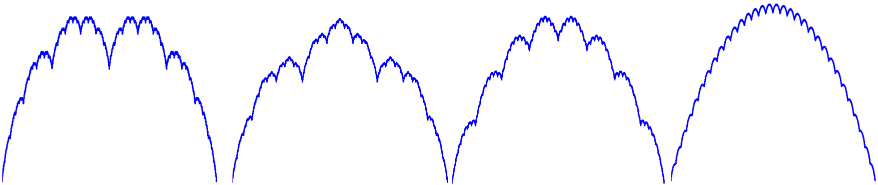


Figure: Limiting curves observed for the polynomial adic transformations associated with polynomial $p(x) = 1 + x + x^2 + \cdots + x^d$ for (from left to right): $d + 1 = 2, 3, 8, 32$ and symmetric measure.

Smooth limits of limiting curves

Statement:

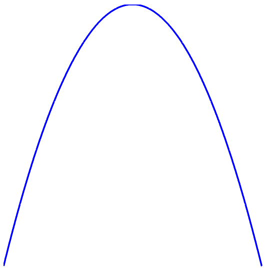


Figure: Limiting curve is parabola.