Path model for decomposition of the graded tensor powers into Weyl modules

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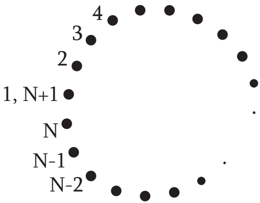
Outline

- Multiplicities for decomposition non-graded tensor powers
- Feigin Loktev fusion product and Weyl modules
- Path model for decomposition of the graded tensor powers into Weyl modules

Spin chains and decomposition of tensor powers



$\mathfrak{g}$ - simple Lie algebra

$$[S_m^\alpha, S_n^\beta] = c_{\alpha\beta\gamma} S_n^\gamma \delta_{mn}$$

$$S_{N+1}^\alpha = S_1^\alpha \qquad (j = 1, 2, 3)$$

$$\mathfrak{H}_N = \prod_{n=1}^{N} \bigotimes \mathfrak{h}_n,$$

$$S_n^\alpha = I \otimes \cdots \otimes \underbrace{S^\alpha}_{n} \otimes \cdots \otimes I$$

Bethe ansatz gives a complete set of solutions $\Longrightarrow$

$\Longrightarrow$ Bethe states correspond to $\mathfrak{g}$ - highest weight vectors

Spin chains and decomposition of tensor powers

Bethe(1931)

$$Z(l,N) = \frac{N-2l+1}{N-l+1}\binom{N}{l}$$

multiplicity of $V_{(N-2l+1)\omega}$ in tensor product $V_\omega^{\otimes N}$ for sl_2

Kirillov, Reshetikhin (1987)

$$M_{\lambda;\mathbf{n}} = \sum_{\substack{m_{a,j}\geq 0 \\ q_a=0,\ p_{a,j}\geq 0}} \prod_\alpha \binom{m_{\alpha,i}+p_{\alpha,i}}{m_{\alpha,i}}$$

multiplicity of V_λ in tensor product of $n_{\alpha,i}$ modules with highest weights $i\omega_\alpha$

$\lambda = \sum l_\alpha w_\alpha \quad q_\alpha = l_\alpha + \sum_{i,\beta} iC_{\alpha,\beta}m_{\beta,i} - \sum_i in_{\alpha,i} = 0$

$p_{\alpha,i} = \sum_{j\geq 1} n_{\alpha,j}min(i,j) -$
$\sum_\beta sgn(C_{\alpha,\beta})\sum_{j\geq 1} min(|C_{\alpha,\beta}|j, |C_{\beta,\alpha}|i)m_{\beta,j}$

Our approach

We need to obtain a generalization of Bethe formula with explicit dependence on N

- $m(\nu, N)$ was defined on P^+.
- $M(\nu, N)$ is defined on the whole weight lattice P.

Formal character:

$$chL^\mu = \sum_\nu (dimL_\nu)e^\nu$$

Singular element:

$$\Psi^\mu = \sum_{w \in W} \epsilon(w)\, e^{w \circ (\mu+\rho)-\rho}$$

Our approach

Old problem:

$$ch((L^{\mu})^{\otimes N}) = \sum_{\nu} m\,(\nu, N)\, chL^{\nu}$$

New problem:

$$\Psi((L^{\mu})^{\otimes N}) = \sum_{\xi \in P} M\,(\xi, N)\, e^{\xi},$$

$$M\,(w(\nu+\rho)-\rho, N)_{|w\in W} = \epsilon\,(w)\, m(\nu, N).$$

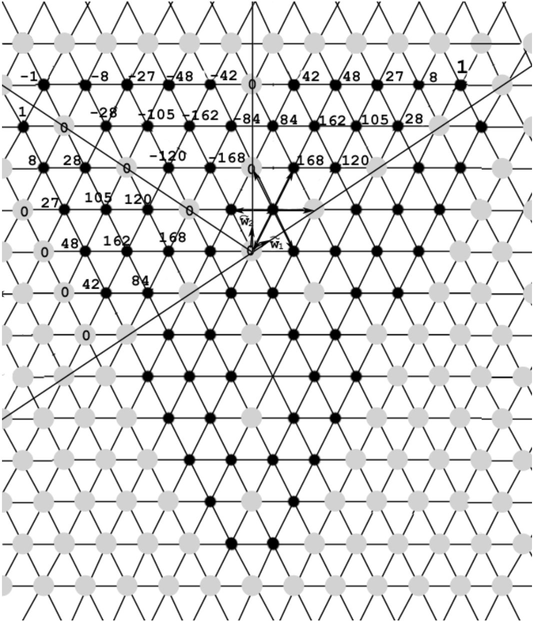
$$M\,(\xi, N)\,|_{\overline{C^{(0)}}} = m\,(\xi, N)$$

Tensor powers of fundamental modules for A_n, B_n

Multiplicity function $M(\xi, N)$

1. is equal to zero outside the orbit of the highest weight $\mu = N\omega$.
2. equal to zero on boundaries of Weyl chambers (shifted on $-\rho$)
3. Antiinvariant w.r.t. Weyl group transformations: $w \circ M^{\otimes^p \omega}(\xi, N) = \varepsilon(w)\, M^{\otimes^N \omega}(\xi, N)$; $w \in W$.
4. Satisfies boundary conditions $M^{\otimes^N \omega}(N\omega + \rho, N) = 1$

Tensor powers of fundamental modules for A_n, B_n



Tensor powers of fundamental modules for A_n

$$M^{\otimes^N \omega}_{(A_n)}(\{a_i\}, N) =$$

$$= N! \frac{\prod_{i=1}^{n} a_i \prod_{j=1}^{n-1} (a_j + a_{j+1}) \cdots \prod_{j=1}^{2} \left(\sum_{k=1}^{n-1} a_{j+k}\right) \left(\sum_{k=1}^{n} a_k\right)}{\prod_{l=0}^{n} \left(\frac{1}{n+1} \left(N + \frac{1}{2} n (n+1) + \sum_{s=1}^{l} (-s a_s) + \sum_{t=l}^{n-1} (n-t) a_{t+1}\right)\right.}$$

$\{a_i\}$ - coordinates of weight ν in $\tilde{\omega}_i$ (shifted on $-\rho$)

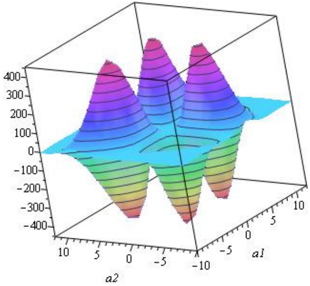
Tensor powers of fundamental modules for B_n

$$M^{\otimes^N \omega}_{(B_n)}(\{a_i\}, N) =$$

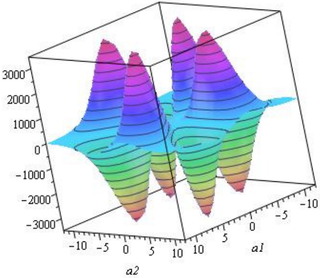
$$= \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k}\left(\frac{N+a_{k+1}+2n-1}{2}\right)!\left(\frac{p-a_{k+1}+2n-1}{2}\right)!} \prod_{l=1}^{n} a_l \prod_{i<j} \left(a_i^2 - a_j^2\right)$$

$\{a_i\}$ - coordinates in basis $X = \{x_i : \vec{x_i} \parallel \vec{e_i}, |x_i| = |\frac{e_i}{2}|\}$ (shifted)

Properties



A_2



B_2

Properties

- *with fixed coordinates* $\{a_i\}$ *at* $N \longrightarrow \infty$.

$$\mathfrak{g} = A_2 \quad M^{\otimes^N \omega}_{(A_2)}(\{1,1\}\,, N) \sim \left(\frac{1}{6} \frac{\sqrt{3} e^{5ln3} (\frac{1}{N})^4}{\pi} + O(\left(\frac{1}{N}\right)^5) \right) 3^N$$

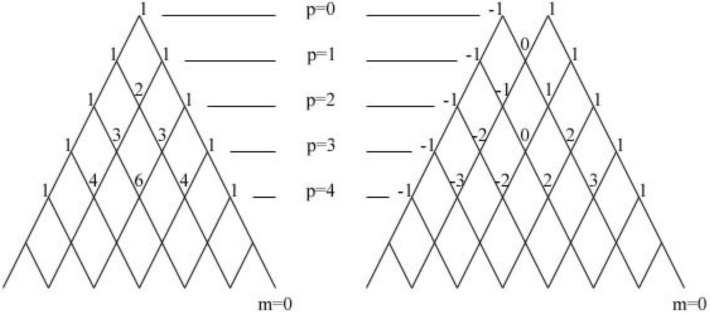
$$\mathfrak{g} = B_2 \quad M^{\otimes^N \omega}_{(B_2)}(\{2,3\}\,, N) \sim \left(\frac{15}{8} \frac{e^{9ln2} (\frac{1}{N})^5}{\pi} + O(\left(\frac{1}{N}\right)^6) \right) (2^N)^2$$

- *with fixed coordinates* $\{b_i\}$, *at* $N \longrightarrow \infty$

$$M_{A_2}(N; b_1, b_2)_{|N \to \infty} \quad \sim \quad \frac{b_2 + 1}{\left(\frac{1}{2}(b_1 + b_2) + 1\right)! \left(\frac{1}{2}(b_1 - b_2)\right)!} N^{b_1}.$$

Bethe formula and Pascal triangle

$$Z(p,l) = \binom{p}{m} - \binom{p}{m-1}$$



Graded tensor products

Current algebra $\mathfrak{g}[[t]]$ of $\mathfrak{g}$-valued complex polynomials with generators $X[n] = X \otimes t^n$ where $X \in \mathfrak{g}$, $n \in \mathbf{Z}_+$.

Feigin-Loktev fusion product

- $\mathfrak{g}[t]$-generators $X[n] \in \mathfrak{g}[t]$ act on $v \in L^\mu$ by $X[n] \cdot v = z^n X v$. and on $V = L^{\mu_1} \otimes \cdots \otimes L^{\mu_n}$ by the rule

$$X[m](v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} z_i^m v_1 \otimes \cdots \otimes X v_i \otimes \cdots \otimes v_n.$$

- define filtration on the tensor product of the irreducible modules $V = L^{\mu_1} \otimes \cdots \otimes L^{\mu_n}$.

$$F^0 V \subset F^1 V \subset \cdots \qquad \text{where} \quad F^i V = U^{\leq i}(\mathfrak{g}[[t]]) v_1 \otimes \cdots \otimes v_n,$$

$$\overline{V} = \oplus_{j \geq 0} F^j V / F^{j-1} V = L^{\mu_1} \star \cdots \star L^{\mu_n}$$

$$M_\lambda^{\mu_1, \ldots \mu_n}(q) = \sum_{k \geq 0} q^k \dim \mathrm{Hom}_{\mathfrak{g}}(F_k, L^\lambda)$$

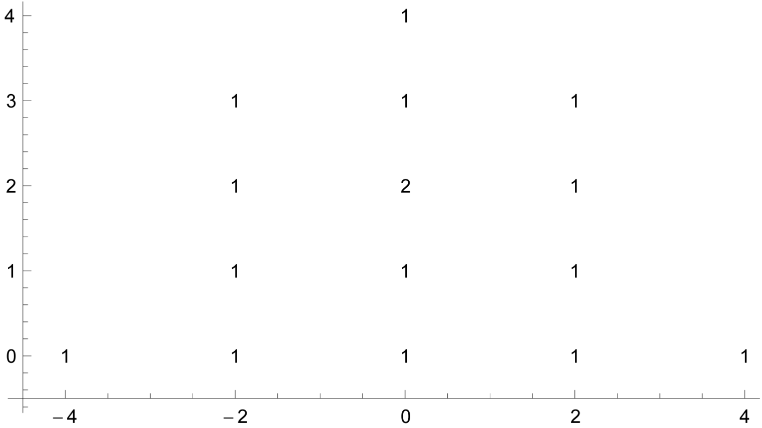
Local Weyl module

If all $L^{\mu_1}, \ldots, L^{\mu_n}$ are fundamental $\mathfrak{g}$-modules then $L^{\mu_1} \star \cdots \star L^{\mu_n}$ coincides with $W(\lambda), \lambda = \mu_1 + \cdots + \mu_n$

Local Weyl module is a cyclic $\mathfrak{g}[[t]]$-module generated by the action on the highest weight vector v with the following conditions:

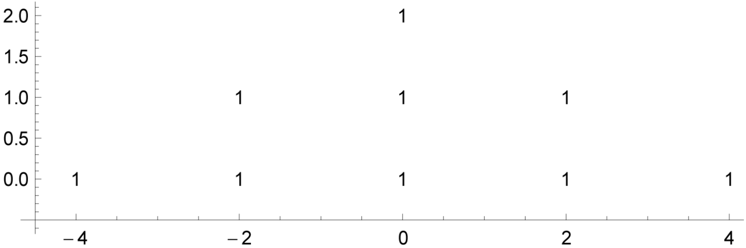
- $hv = \lambda(h)v, h \in \mathfrak{h}$
- $e_\alpha[n]v = 0, n \geq 0, e_\alpha \in \mathfrak{n}^+, \alpha \in \Delta^+$
- $h[n]v = 0, n > 0$
- $f_\alpha^{<\lambda,\alpha>+1} v = 0$

Graded weight diagrams



Weight diagram of the sl_2-Weyl module $W(4\omega)$

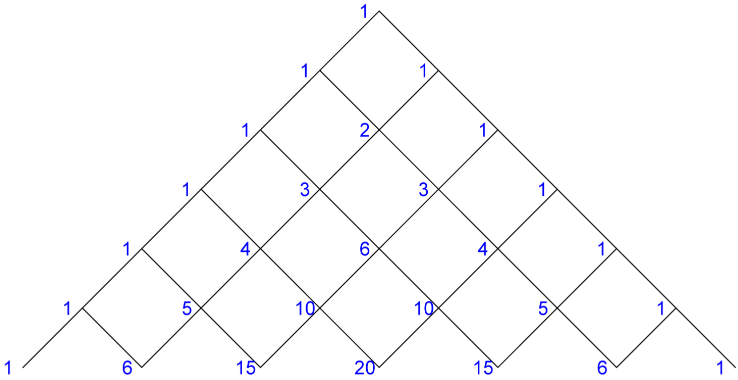
Graded weight diagrams



Weight diagram of the Feigin-Loktev sl_2-module $L^{2\omega} \star L^{2\omega}$

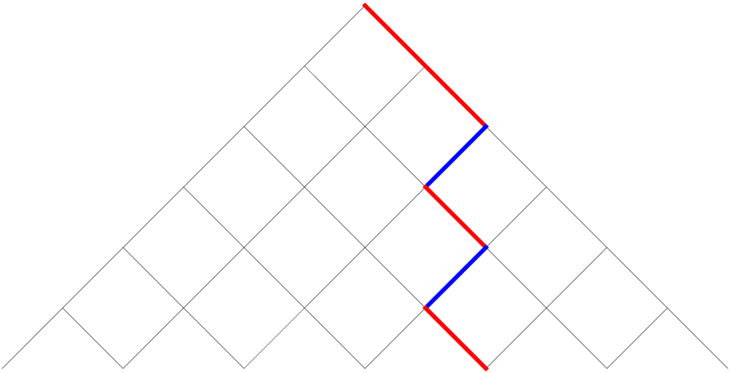
Pascal pyramids and weighted paths

Generalized Pascal pyramid $\Pi_{\mathfrak{g}} = (V, E)$
vertices $v(p, \lambda)$, $\quad p = 0, 1 \ldots \qquad \lambda \in P$
edges $e(p, \lambda_1, \lambda_2)$ connect $v(p, \lambda_1)$ and $v(p+1, \lambda_2) \quad \lambda_1 - \lambda_2 \in \mathcal{N}$



Pascal pyramids and weighted paths

Ordering of weights of L^{ω_1} compatible with the lexicographical ordering on $W_{\mathfrak{g}}$.
If $w_1 \prec w_2$ in $W_{\mathfrak{g}}$, then $w_1\omega_1 \prec w_2\omega_1$.
Take all pairs of consequential steps $(e_i, e_{i+1}) = (\lambda, \mu)$. If $\lambda \succ \mu$, the weight of path is multiplied by q^i.



Recurrent formulas for graded characters

$$ch\left(L^{2\omega}\right) = ch\left(L^{\omega}\right)^{\star(2)} - q\,ch\left(L^{0}\right)$$

$$ch\left(L^{\omega}\right)^{\star k} \star \left(L^{2\omega}\right)^{\star n} =$$

$$= ch\left(L^{\omega}\right)^{\star(k+2)} \star \left(L^{2\omega}\right)^{\star(n-1)} - q^{n+k} ch\left(L^{\omega}\right)^{\star(k)} \star \left(L^{2\omega}\right)^{\star n-1}$$

$$ch\left(L^{\omega}\right) \star \left(L^{3\omega}\right) =$$

$$= ch\left(L^{\omega}\right)^{\star(4)} - (q^3+q^2) ch\left(L^{\omega}\right)^{2} - (q^3-q^2) ch\left(L^{0}\right)$$