The infinite-dimensional q-Beta distribution

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Large-N limit transition:

1. In point processes — ensembles of (randomly distributed) particles.

Example: Random Matrix Theory; finite particle configurations $\leftrightarrow$ spectra of $N \times N$ random matrices.

2. In representation theory.

$G(\infty) = \varinjlim G(N)$: finite particles configurations $\leftrightarrow$ labels of irreducible representations of groups $G(N)$.

Preliminaries: Schur-type polynomials

$$\varphi_n(x) = x^n + \text{lower degree terms}, \qquad n = 0, 1, 2, \dots$$

$$\text{Fix} \quad N = 2, 3, \dots$$

$$\nu = (\nu_1 \ge \nu_2 \ge \dots \ge 0), \quad \ell(\nu) \le N, \quad \text{a partition}$$

$$\varphi_{\nu|N}(x_1, \dots, x_N) := \frac{\det[x_j^{\nu_i+N-i}]_{i,j=1}^N}{\prod_{1 \le i < j \le N}(x_i - x_j)}$$

If $\varphi_n(x) = x^n$, then $\varphi_{\nu|N}$ is $S_{\nu|N}$, the Schur polynomial. In general,

$$\varphi_{\nu|N} = S_{\nu|N} + \text{lower degree terms}$$

It follows: $\{\varphi_{\nu|N} : \ell(\nu) \le N\}$ is a (non-homogeneous) basis in

$$\mathrm{Sym}(N) := \mathbb{C}[x_1, \dots, x_N]^{\mathrm{permutations}},$$

the algebra of N-variate symmetric polynomials.

Assume now that the φ_n are a system of **orthogonal** polynomials with a weight function $W(x)$ on $\mathbb{R}$.

Then, for each fixed $N = 2, 3, \ldots$, the polynomials $\varphi_{\nu|N}$ are a system of symmetric N-variate polynomials, orthogonal with respect to the weight measure

$$M_N := \frac{1}{Z_N} \cdot \prod_{i=1}^{N} W(x_i) \cdot \prod_{1 \le i < j \le N} (x_i - x_j)^2 \cdot dx_1 \ldots dx_N$$

Under mild assumptions on $W(x)$, the $\varphi_{\nu|N}$ form an orthogonal basis in the Hilbert space

$$L^2_{\text{symmetric}}(\mathbb{R}^N, M_N)$$

Ensembles of random particle configurations

A slightly different viewpoint (necessary for letting $N \to \infty$):

We replace $\mathbb{R}^N$ by

$$\Omega_N := N\text{-point configurations } X = (x_1 > \cdots > x_N)$$

and regard M_N as a probability measure on Ω_N (Z_N being changed).

Symmetric polynomials can be viewed as functions on Ω_N, and then the polynomials the $\varphi_{\nu|N}$ form an orthogonal basis in

$$H_N := L^2(\Omega_N, M_N).$$

The measure M_N on Ω_N generates an **ensemble of random particle configurations**. The particles are "interacting" due to repulsive logarithmic pair potential in

$$\prod_{1 \le i < j \le N} (x_i - x_j)^2 = \exp\left\{ -2 \sum_{i<j} \log \frac{1}{|x_i - x_j|} \right\}$$

Hence the term **log-gas system** with parameter $\beta = 2$.

Log-gas systems with parameter $\beta = 2\theta > 0$

A more general model of log-gas system:

$$M_N := \frac{1}{Z_N} \cdot \prod_{i=1}^{N} W(x_i) \cdot \prod_{1 \le i < j \le N} (x_i - x_j)^{2\theta} \cdot dx_1 \ldots dx_N$$

$$= \frac{1}{Z_N} \cdot \prod_{i=1}^{N} W(x_i) \cdot \exp\left\{ -2\theta \sum_{i<j} \log \frac{1}{|x_i - x_j|} \right\}$$

For **certain** weight functions $W(x)$ it is still possible to construct a system of symmetric polynomials, which constitute an orthogonal basis in the Hilbert space $L^2_{\mathrm{sym}}(\Omega_N, M_N)$ and have the form

$$\varphi_{\nu|N}(x_1, \ldots, x_N; \theta) = P_{\nu|N}(x_1, \ldots, x_N; \theta) + \text{lower degree terms},$$

where the $P_{\nu|N}(x_1, \ldots, x_N; \theta)$ are the **Jack symmetric polynomials**, a 1-parameter deformation of the Schur polynomials.

For instance, this holds for N-**dimensional Beta distribution**

$$M_N^{A,B,\theta} = \frac{1}{Z_N} \cdot \prod_{i=1}^{N} (1-x_i)^A (1+x_i)^B \times \prod_{1 \le i < j \le N} (x_i - x_j)^{2\theta} \cdot dx_1 \dots dx_N,$$

where $-1 \le x_N < \dots < x_1 \le 1$, $A, B > -1$, $\theta > 0$.

• A close formula for Z_N exists: the famous **Selberg's integral**.

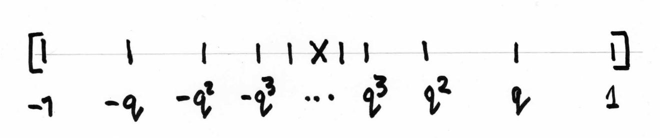
• The corresponding symmetric orthogonal polynomials $\varphi_{\nu|N}$ were studied by James & Constantine, Vretare, Debiard, Lassalle, Macdonald, ... They are a particular case of the **Heckman-Opdam's** Jacobi polynomials corresponding to root system BC_N.

• In the special case $\theta = 1$, the $\varphi_{\nu|N}$ are given by determinantal formula with $\varphi_n =$ classical Jacobi polynomials. But there is no such elementary formula for $\theta \neq 1$!

"Quantization" of $[-1, 1]$

Fix $q \in (0, 1)$. The q-version of $[-1, 1]$ is

$$\boxed{[-1,1]_q := \{-1, -q, -q^2, \dots\} \cup \{\dots, q^2, q, 1\}}$$



We will be dealing with particle configurations on $[-1, 1]_q$.

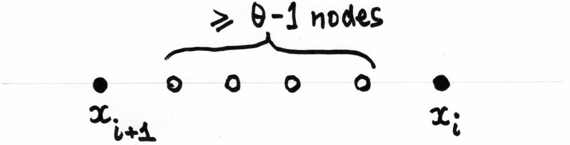
A (q,t)-analogue of N-particle Beta distribution

Set $t = q^\theta$, where (for simplicity only!) $\theta \in \mathbb{Z}_{\geq 1}$ (actually, $\theta > 0$ is allowed). Denote $(z;q) := \prod_{n \geq 0}(1 - zq^n)$, the infinite q-Pochhammer.

The N-**particle** (q,t)-**Beta distribution** with parameters (c,d):

$$\frac{1}{\mathcal{Z}_N} \cdot \prod_{i=1}^{N} |x_i| \frac{(x_i q; q)_\infty (-x_i q; q)_\infty}{(c x_i; q)_\infty (d x_i; q)_\infty} \cdot \left| \prod_{1 \leq i \neq j \leq N} \prod_{r=0}^{\theta-1} (x_i - x_j q^r) \right|$$

Here $X = (x_1 > \cdots > x_N) \subset [-1,1]_q$ and (c,d) are such that the weights are ≥ 0. **Important:** the weight of X is nonzero if and only if X is 'θ-**sparse**' meaning that any two points of X are separated by at least $\theta - 1$ unoccupied nodes of $[-1,1]_q$.



Constraints on (c,d):

• either (principal series) $d=\bar{c}$, $c \in \mathbb{C}\setminus\mathbb{R}$

• or (complementary series) $c,d \in \mathbb{R}$ are such that $[c,d]$ lies inside an interval between two neighboring points from

$$\dots \quad -q^{-2}, \qquad -q^{-1}, \quad -1,\ 1, \quad q^{-1}, \qquad q^{-2} \quad \dots$$

Degeneration to ordinary Beta distribution

Set $c=q^{A+1}$, $d=-q^{B+1}$ and let $q\to 1^-$.

• $[-1,1]_q$ approximates $[-1,1]$

• The q-Beta distribution $M_N^{q,t;c,d}$ degenerates into the ordinary Beta distribution $M_N^{A,B,\theta}$.

Stokman-Koornwinder's big q-Jacobi polynomials

Let $q \in (0,1)$, $t = q^\theta$ with $\theta \in \mathbb{Z}_{\geq 1}$, and (c,d) satisfy the constraints above.

Let $N \in \mathbb{Z}_{\geq 1}$ and $M_N^{q,t;\,c,d}$ denote the N-particle (q,t) Beta distribution living on the space of N-particle θ-sparse configurations on $[-1,1]_q$.

Let $P_{\nu|N}(x_1,\dots,x_N;q,t)$ be the Macdonald polynomials (ν ranges over partitions with $\ell(\nu) \leq N$).

Theorem [Stokman 1997; Stokman & Koornwinder 1997].

There exists a unique (non-homogeneous) basis $\{\varphi_{\nu|N}\}$ in the algebra of N-variate symmetric polynomials such that:

- $\varphi_{\nu|N}(x_1,\dots,x_N;q,t;\,c,d) = P_{\nu|N}(x_1,\dots,x_N;q,t) +$ lower degree terms;

- the corresponding functions $\varphi_{\nu|N}(X;q,t;\,c,d)$ are orthogonal with respect to $M_N^{q,t;\,c,d}$

Main result

Theorem. The whole picture, i.e. N-particle (q, t) Beta probability distribution + related system of N-variate symmetric orthogonal polynomials, survives in a large-N limit transition.

Detalization: convergence of polynomials

In our limit regime, parameters (c, d) **must vary** together with N:

$$c = \gamma t^{1-N}, \quad d = \delta t^{1-N}, \qquad \text{where } (\gamma, \delta) \text{ are fixed.}$$

Claim 1. For every fixed partition ν, there exists a limit

$$\lim_{N\to\infty} \varphi_{\nu|N}(\,\cdot\,; q, t;\, \gamma t^{1-N}, \delta t^{1-N}) = \Phi_\nu(\,\cdot\,; q, t; \gamma, \delta).$$

The result is a symmetric function whose top degree homogeneous component is $P_\nu(\,\cdot\,; q, t)$, the Macdonald symmetric function with index ν.

Detalization: convergence of measures

Let Ω denote the set of all θ-sparse configurations in $[-1,1]_q$. It is compact space, being a closed subspace of $2^{[-1,1]_q}$. Its stratification:

$$\Omega = \Omega_\infty \cup \Omega_{\text{fin}} = \Omega_\infty \cup \bigcup_{N \geq 0} \Omega_N$$

(infinite and N-point configurations).

Assume (γ, δ) are in the principal or complementary series and in the latter case additionally require that γ, δ have the **same sign**.

Claim 2. Under these assumptions, as $N \to \infty$, the N-particle (q,t) Beta distributions with varying parameters $(c,d) = (\gamma t^{1-N}, \delta t^{1-N})$ weakly converge to a probability distribution $M_\infty^{q,r;\gamma,\delta}$ concentrated on Ω_∞.

We call $M_\infty^{q,r;\gamma,\delta}$ the **infinite-dimensional** (q,t) Beta distribution.

Detalization: orthogonal system

Let Sym denote the algebra of symmetric functions.

There is a natural embedding $\mathrm{Sym} \to C(\Omega)$, because symmetric functions $F \in \mathrm{Sym}$ can be evaluated at any $X \in \Omega$.

Claim 3. Under this embedding, the limit symmetric functions $\Phi_\nu(\,\cdot\,; q, t; \gamma, \delta)$ give rise to an orthogonal basis in the Hilbert space $L^2(\Omega_\infty, M_\infty^{q,r;\,\gamma,\delta})$.

Corollary. The limit measure $M_\infty^{q,t;\,\gamma,\delta}$ admits the following characterization: it is a unique probability distribution on Ω, orthogonal to all basis functions $\Phi_\nu(X; q, t; \gamma, \delta)$ with $\nu \neq 0$.

In the special case $\theta = 1$, the measure $M_\infty^{q,t;\,\gamma,\delta}$ is **determinantal**. Its correlation kernel is computed in **[Gorin-O., 2016]**, it is expressed through ${}_2\phi_1$.

Degeneration at $q = 1$ destroys large-N limit transition

Recall that the degeneration, as $q \to 1^-$, assumes

$$(c, d) = (q^{A+1}, -q^{B+1}), \quad \text{where } A, B > -1.$$

Therefore, (c, d) must be in the complementary series and, moreover, c and d must have **opposite sign**.

On the other hand, the limit regime assumes

$$(c, d) = (\gamma t^{1-N}, \delta t^{1-N}).$$

This in turn requires to take $\gamma > 0$, $\delta < 0$. But then $(c, d) \to (+\infty, -\infty)$ and hence, for N large enough, (c, d) **cannot** be in the complementary series as $[c, d]$ will not be contained in one of the intervals between the points

$$\dots \quad -q^{-2}, \qquad -q^{-1}, \quad -1,\ 1, \quad q^{-1}, \qquad q^{-2} \quad \dots$$

Conclusion. Our large-N limit regime is a **specific "q" phenomenon**.

Comments on proof

Claim 1 (convergence of Stokman-Koornwinder's N-variate big q-Jacobi polynomials $\varphi_{\nu|N}$ as $N \to \infty$).

Proof is based on expansion of the $\varphi_{\nu|N}$'s on multivariate **interpolation Macdonald polynomials** of Knop-Okounkov-Sahi.

Claim 2 (convergence of N-particle (q,t) Beta distributions.)

Existence of weak limit follows from Claim 1. A fine point is that the resulting limit measure on Ω is concentrated on $\Omega_\infty \subset \Omega$. This is proved using the **method of intertwiners**.

Claim 3 (orthogonality).

Not evident that each of the limit functions $X \to \Phi_\nu(X;q,t;\gamma,\delta)$ gives a nonzero vector in the Hilbert space $L^2(\Omega_\infty, M_\infty^{q,t;\gamma,\delta})$. This is proved by computation of the norms.

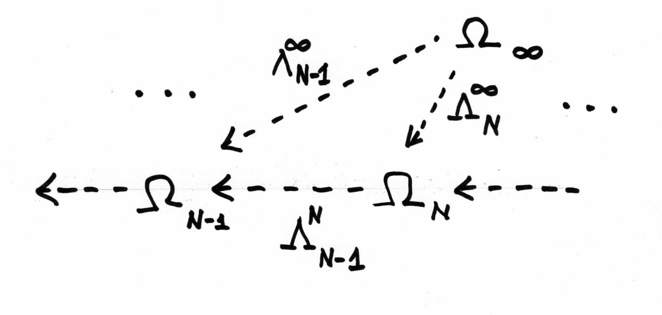
The special case $\theta = 1$ is investigated in **[O., 2017]**.

Method of intertwiners: the boundary

Theorem 1. The space Ω_∞ arises as the **entrance boundary** of a (highly inhomogeneous) Markov chain

$$\Omega_1 \overset{\Lambda_1^2}{\dashleftarrow} \Omega_2 \overset{\Lambda_2^3}{\dashleftarrow} \Omega_3 \overset{\Lambda_3^4}{\dashleftarrow} \dots$$

Here Ω_N is the space of all N-particle θ-sparse configurations on $[-1,1]_q$ and Λ_{N-1}^N are certain **stochastic matrices**. Their construction is based on [Okounkov, 1998].



Method of intertwiners: coherency relation

Theorem 2. The N-particle (q,t) Beta distributions with parameters $(\gamma t^{1-N}, \delta t^{1-N})$ (denote them by M_N) are **consistent** with the stochastic matrices Λ^N_{N-1} in the sense that

$$M_N \Lambda^N_{N-1} = M_{N-1}, \quad \text{row vector} \times \text{matrix} = \text{row vector}$$

In more detail, this **coherency relation** means

$$\sum_{X \in \Omega_N} M_N(X) \Lambda^N_{N-1}(X,Y) = M_{N-1}(Y), \quad \text{for every fixed } Y \in \Omega_{N-1}.$$

Theorem 1 & Theorem 2 **automatically imply** that the limit measure $\lim_{N\to\infty} M_N$ (the infinite-particle (q,t) Beta distribution) lives on $\Omega_\infty \subset \Omega$.

Comparison with RMT

• Continuum $\mathbb{R}$ is replaced by a q-**lattice**.

• The "Jack-type" two-point factor $|x_i - x_j|^{2\theta}$, responsible for pair interaction, is replaced by "Macdonald-type" factor

$$\left| \prod_{1 \le i \ne j \le N} \prod_{r=0}^{\theta-1} (x_i - x_j q^r) \right|$$

• No space scaling in the large-N limit transition

Summary: infinite-variate orthogonal polynomials

• A system $\{\Phi_\nu\}$ of "symmetric orthogonal polynomials with **infinitely many variables**" is constructed. They are indexed by arbitrary partitions ν and form an orthogonal basis in $L^2(\Omega_\infty, M_\infty)$.

• The functions Φ_ν actually live in the algebra Sym of symmetric functions and have the form

$$\Phi_\nu(x_1, x_2, \dots) = P_\nu(x_1, x_2, \dots; q, t) + \text{lower degree terms},$$

where $P_\nu(\,\cdot\,; q, t)$ is **Macdonald symmetric function**.

• Each Φ_ν can be explicitly written as (a kind of) (terminating) **basic hypergeometric series**.