# Positive representations of quantum groups and higher Teichmüller theory

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Based on joint work with Alexander Shapiro (U. Toronto).

# Quantum group $\mathrm{U_q}(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a Hopf algebra; i.e. an associative algebra with a co-associative algebra map

$$\Delta : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}).$$

called the coproduct.

The Hopf algebra $\mathrm{U}(\mathfrak{g})$ is *co-commutative*;

$$\begin{aligned} \Delta^{op} &:= \mathrm{Flip} \circ \Delta \\ &= \Delta. \end{aligned}$$

So modules for $\mathrm{U}(\mathfrak{g})$ give rise to symmetric tensor categories: we have

$$\mathrm{Flip} : \;\; V \otimes W \simeq W \otimes V.$$

# Quantum group $\mathrm{U_q}(\mathfrak{g})$

The *quantum group* $\mathrm{U_q}(\mathfrak{g})$ is a Hopf algebra deformation of the enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Coproduct in $\mathrm{U_q}(\mathfrak{g})$ is no longer co-commutative; instead $\mathrm{U_q}(\mathfrak{g})$ is *quasi-triangular:* it has an $R$-matrix $R \in \mathrm{U_q}(\mathfrak{g}) \otimes \mathrm{U_q}(\mathfrak{g})$ satisfying

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{U_q}(\mathfrak{g})^{\otimes 3},$$
$$R\Delta R^{-1} = \Delta^{op},$$

where $\Delta^{op}$ is the opposite coproduct.
So modules for $U_q(\mathfrak{g})$ give rise to *braided tensor categories*.

# Positive representations of quantum groups

Let $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$, and set

$$q = e^{\pi i \hbar^2}.$$

The positive representations of $U_q(\mathfrak{g})$ are certain $*$-representations of the split real quantum group with many nice properties:

- the Chevalley generators of the quantum group act by positive essentially self-adjoint operators;
- they are bimodules for the quantum group $U_q(\mathfrak{g})$ and its Langlands dual $U_{q^\vee}({}^L\mathfrak{g})$, where $q, q^\vee$ are related by the modular $S$-duality:

$$q^\vee = e^{\pi i / \hbar^2}.$$

# Positive representations in rank 1

The positive representations of $U_q(\mathfrak{sl}_2)$ were first constructed by Faddeev, Ponsot and Teschner in 1999. These representations $\mathcal{P}_s \simeq L^2(\mathbb{R})$ are labelled by a continuous spin parameter $s \in \mathbb{R}_{\geq 0}$.

Ponsot and Teschner showed that the positive representations of $U_q(\mathfrak{sl}_2)$ form a "continuous tensor category": one has

$$\mathcal{P}_{s_1} \otimes \mathcal{P}_{s_2} = \int_{\mathbb{R}_{\geq 0}}^{\oplus} \mathcal{P}_s d\mu(s).$$

The measure is $d\mu(s) = 4\sinh\left(2\pi\hbar s\right)\sinh\left(2\pi\hbar^{-1}s\right) ds$.

# Positive representations in higher rank

In 2011, Frenkel and Ip constructed positive representations for $U_q(\mathfrak{sl}_n)$. Shortly after, Ip generalized the construction to the other finite Dynkin types.

The positivity of Chevalley generators and modular duality are manifest for higher rank positive representations.

But it remained an open problem to prove their closure under tensor product.

# Advertisement

In 2016 joint work with A. Shapiro, we showed how these representations can be constructed from quantized moduli spaces of local systems on surfaces, which I'll now explain.

In his talk at 16:00 on Thursday, Sasha will explain how to use our construction to establish the closure under tensor product for positive representations of $U_q(\mathfrak{sl}_n)$.

# Moduli spaces of framed local systems

A *marked surface* $\widehat{S}$ is a compact oriented surface $S$ with a finite set $\{x_1, \ldots, x_k\} \subset \partial S$ of marked points on the boundary.

Its punctured boundary is $\partial\widehat{S} := \partial S \setminus \{x_1, \ldots, x_k\}$.

Let $G$ be a complex semisimple Lie group of adjoint type, and $B \subset G$ a Borel subgroup. A *framed $G$-local system* on $\widehat{S}$ is:

1. a $G$-local system $\mathcal{L}$ on $S$, together with
2. a flat section of the restriction of the associated flag bundle $(\mathcal{L} \times_G G/B)|_{\partial\widehat{S}}$.

**Definition**

$$\mathcal{X}_{G,\widehat{S}} := \text{moduli of framed } G\text{-local systems on } \widehat{S}$$

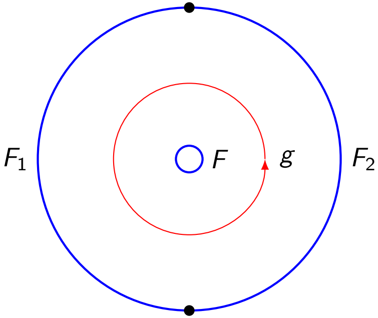
# Example

Let $\widehat{S}$ be a punctured disk with 2 marked points. Then

$$\mathcal{X}_{G,\widehat{S}} = \{(g, F_1, F_2, F) \,|\, gF = F\} \,/G$$

where

$$g \in G, \qquad F, F_1, F_2 \in G/B.$$



# Grothendieck-Springer resolution

The variety $\mathcal{X}_{G,\widehat{S}}$ from the previous example is a close cousin of the Grothendieck-Springer resolution

$$\widetilde{G} = \{(g,F) \,|\, gF = F\} \subset G \times (G/B).$$

The resolution map is

$$\widetilde{G} \longrightarrow G, \quad (g,F) \longmapsto g.$$

Poisson structure on the total space of the resolution was studied by Evens-Lu in 2007. Cluster structure on the total space was described by R. Brahami in 2010, and on the target by Gekhtman-Shapiro-Vainshtein in 2015.

# $\mathcal{X}_{G,\widehat{S}}$ is a cluster variety

**Fock-Goncharov:** For $G = PGL_n(\mathbb{C})$, $\mathcal{X}_{G,\widehat{S}}$ is a *cluster Poisson variety:* it is covered up to codimension 2 by an atlas of toric charts

$$\mathcal{T}_\Sigma\colon (\mathbb{C}^*)^d \longrightarrow \mathcal{X}_{G,\widehat{S}},$$

labelled by quivers $Q_\Sigma$. The Poisson brackets are determined by the adjacency matrix $\epsilon_{jk}$ of $Q_\Sigma$:

$$\{x_j, x_k\} = \epsilon_{kj} x_j x_k.$$

Different charts are related by subtraction-free birational transformations called *cluster mutations*.

# Higher Teichmüller spaces

Existence of subtraction-free gluing maps yields a well-defined notion of the totally positive subset $\mathcal{X}^+_{G,\widehat{S}} \subset \mathcal{X}_{G,\widehat{S}}$. It means that $\mathcal{X}^+_{G,\widehat{S}}$ consists of points where all toric coordinates are positive.

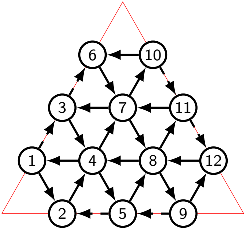
If $G = PGL_2(\mathbb{C})$, the variety $\mathcal{X}^+_{G,\widehat{S}}$ can be identified with a component in the moduli space $\mathcal{M}_{flat}(\widehat{S}, PSL_2(\mathbb{R}))$, isomorphic to a decorated Teichmüller space.

So $\mathcal{X}^+_{G,\widehat{S}}$ are higher rank analogs of Teichmüller spaces.

# Clusters from ideal triangulations

Shrink closed components of $\partial\widehat{S}$ to punctures. A triangulation of $\widehat{S}$ is *ideal* if its vertices are at punctures or marked points.

Each ideal triangulation provides a cluster chart. Fill each triangle with the following quiver (here $G = PGL_4$):



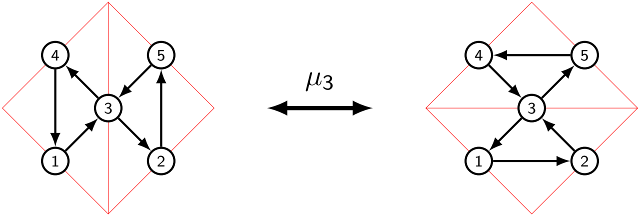

When two triangles share an edge, we "amalgamate" them, i.e. identify the quiver vertices lying on that edge, and replace the corresponding pairs of cluster variables by their products.

# Cluster mutations

Mutation at a vertex $X_i$ proceeds in 3 steps:

1. reverse all incident edges;
2. for each pair of edges $j \to i$ and $i \to k$ create an edge $k \to j$;
3. delete pairs of opposite edges;

For example, for $G = PGL_2$:




Mutations for $G = PGL_2$ correspond to flips of triangulation. For $G = PGL_n$, one flip is realized by a sequence of $\binom{n+1}{3}$ mutations.

# Canonical quantization of cluster varieties

Promote each cluster chart to a quantum torus algebra

$$\mathcal{T}_{\Sigma}^{q} = \langle X_1, \dots, X_d \rangle / \{X_j X_k = q^{2\epsilon_{kj}} X_k X_j\}.$$

They are "glued" by quantum mutations, which are the algebra automorphisms of conjugation by the *quantum dilogarithm* $\Gamma_q(X_k)$, where

$$\Gamma_q(X) = \prod_{n=1}^{\infty} \frac{1}{1 + q^{2n+1} X}.$$

Let $X_2 X_1 = q^2 X_1 X_2$, then

$$\begin{aligned}
\mu_2(X_1) &= \frac{1}{(1+qX_2)(1+q^3X_2)\dots} \cdot X_1 \cdot (1+qX_2)(1+q^3X_2)\dots \\
&= X_1 \frac{(1+qX_2)(1+q^3X_2)\dots}{(1+q^3X_2)(1+q^5X_2)\dots} = X_1(1+qX_2).
\end{aligned}$$

# Cluster realization of $\mathrm{U}_q(\mathfrak{sl}_n)$

**Theorem (S.-Shapiro '16)**

*Let $\widehat{S}$ be a punctured disk with two marked points, and let $\mathcal{X}^q_{PGL_n,\widehat{S}}$ be the corresponding quantum cluster algebra. Then there is an embedding of algebras*

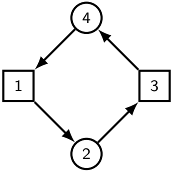
$$\mathrm{U}_q(\mathfrak{sl}_n) \hookrightarrow \mathcal{X}^q_{PGL_n,\widehat{S}},$$

*with the property that for each Chevalley generator of the quantum group, there is a cluster in which this generator is a cluster variable.*

**Theorem (Ip '16)**

*The same is true for other Dynkin types.*

# Example: $\mathrm{U_q}(\mathfrak{sl}_2)$



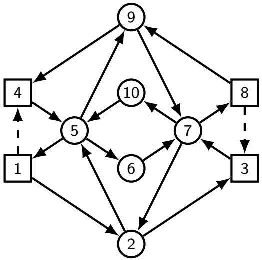

$$E \mapsto X_1(1+qX_2), \qquad K \mapsto q^2 X_1 X_2 X_3,$$
$$F \mapsto X_3(1+qX_4), \qquad K' \mapsto q^2 X_3 X_4 X_1.$$

$KK'$ is central.

Note that

$$E = \mu_2(X_1) \qquad \text{and} \qquad F = \mu_4(X_3).$$

# Example: $\mathrm{U_q}(\mathfrak{sl}_3)$




$$E_1 \mapsto X_1(1+qX_2), \qquad K_1 \mapsto q^2 X_1 X_2 X_3,$$
$$E_2 \mapsto X_4(1+qX_5(1+qX_6(1+qX_7))), \qquad K_2 \mapsto q^4 X_4 X_5 X_6 X_7 X_8,$$
$$F_1 \mapsto X_3(1+qX_7(1+qX_{10}(1+qX_5))), \qquad K_1' \mapsto q^4 X_3 X_7 X_{10} X_5 X_1,$$
$$F_2 \mapsto X_8(1+qX_9), \qquad K_2' \mapsto q^2 X_8 X_9 X_4.$$

$K_1 K_1'$ and $K_2 K_2'$ are central.

# Positive representations of quantum cluster varieties

**Idea:** The quantum cluster algebra $\mathcal{X}^q_{PGL_n,\widehat{S}}$ is a subalgebra in a quantum torus algebra $\mathcal{T}^q$.

So one can construct representations of $\mathrm{U_q}(\mathfrak{sl}_n) \subset \mathcal{X}^q_{PGL_n,\widehat{S}}$ by pulling back representations of the quantum torus algebra $\mathcal{T}^q$.

# Positive representations of quantum cluster varieties

We can embed a quantum cluster chart $\mathcal{T}^q$ into a Heisenberg algebra $\mathcal{H}$ generated by $x_1, \ldots, x_d$ with relations

$$[x_j, x_k] = \frac{i}{2\pi}\epsilon_{jk},$$

by the homomorphism

$$X_j \mapsto e^{2\pi\hbar x_j}.$$

The algebra $\mathcal{H}$ has a family of irreducible Hilbert space representations $V_\chi$ parameterized by central characters $\chi \in \mathrm{Hom}(\ker\epsilon, \mathbb{R})$, in which the generators $X_j$ act by unbounded self-adjoint operators.

# Example: positive representations of $U_q(\mathfrak{sl}_2)$

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, and the self-adjoint, unbounded operators

$$\hat{p} = \frac{i}{2\pi}\frac{\partial}{\partial x}, \quad \hat{x} = x.$$

Then for all $s \in \mathbb{R}$, we have positive self-adjoint operators

$$X_1 = e^{2\pi\hbar(\hat{p}-\frac{s}{2})}, \quad X_2 = e^{2\pi\hbar(\hat{x}+s)}$$

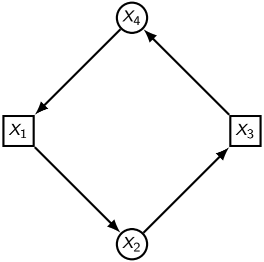
$$X_3 = e^{2\pi\hbar(-\hat{p}-\frac{s}{2})}, \quad X_4 = e^{2\pi\hbar(-\hat{x}+s)}$$

satisfying the cyclic quiver relations

$$q^2 X_k X_{k+1} = X_{k+1} X_k, \quad k \in \mathbb{Z}/4\mathbb{Z},$$

where $q = e^{\pi i \hbar^2}$.

# Example: positive representations of $U_q(\mathfrak{sl}_2)$




Chevalley generators $E, F$ of $U_q(\mathfrak{sl}_2)$ act by positive, self-adjoint operators

$$E \mapsto \mu_2(X_1), \qquad F \mapsto \mu_4(X_3).$$

The $U_q(\mathfrak{sl}_2)$ Casimir element $\Omega$ acts by

$$\Omega \mapsto e^{2\pi\hbar s} + e^{-2\pi\hbar s}.$$

# If you wait until Thursday...

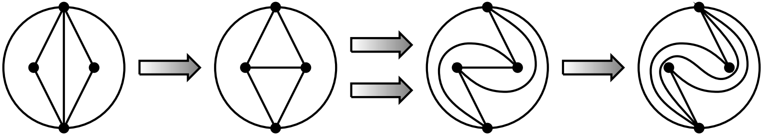
On Thursday, A. Shapiro will explain how the geometric construction of positive representations can be used to establish their closure under tensor product.

Thanks for listening!

# The $R$-matrix

## Theorem (Schrader-S. '16)

*Let $\widehat{S}$ be a twice punctured disk with two marked points on its boundary, and let $\mathcal{X}^q_{PGL_n,\widehat{S}}$ be the corresponding quantum cluster algebra. The previous theorem embeds $U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{sl}_n)$ into $\mathcal{X}^q_{PGL_n,\widehat{S}}$. The conjugation by the $R$-matrix can be identified with a half Dehn twist permuting the punctures.*



## Theorem (Ip '16)

*The same is true for the other finite Dynkin types.*