Positive representations of quantum groups

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Based on a joint work with Gus Schrader (Columbia University).

Positive representations: generalities

Positive representations are certain modules of a quantum group with many nice properties, for example:

- they are bimodules for a quantum group and its modular dual;
- Chevalley generators act there by positive essentially self-adjoint operators.

Conjecture

Positive representations form a "continuous braided monoidal category".

In 1999, Ponsot and Teschner proved the conjecture for $U_q(\mathfrak{sl}_2)$, namely

$$\mathcal{P}_{s_1} \otimes \mathcal{P}_{s_2} = \int_{\mathbb{R}_{>0}}^{\oplus} \mathcal{P}_s d\mu(s),$$

where $d\mu(s) = 4 \sinh\left(2\pi\hbar s\right) \sinh\left(2\pi\hbar^{-1} s\right) ds$.

Goals and applications

Goal: prove the conjecture for other Dynkin types.

Applications:

1. Non-compact Chern-Simons theory $\xrightarrow{?}$ knot invariants.
2. 4d $\mathcal{N} = 2$ gauge theory $\xleftrightarrow{AGT}$ Liouville theory, 2d CFT
   Certain Virasoro modules $\simeq$ Positive rep's of $U_q(\mathfrak{sl}_2)$
   (as braided monoidal categories)
   To prove the equivalence one needs to calculate $3j$ and $6j$ symbols.
3. Classical limits of $6j$-symbols reproduce hyperbolic volumes of non-ideal tetrahedra.
4. Finite-dimensional representations of quantum groups arise as "analytic continuation" in $s \in \mathbb{R}$ of positive representations $\mathcal{P}_s$.

Quantum tori

The moduli space $\mathcal{X}_{G,\widehat{S}}$ of framed G local systems on $\widehat{S}$ is covered by toric charts $\mathcal{T}_{\mathbf{i}} \simeq (\mathbb{C}^\times)^d$ with log-canonical coordinates

$$\{x_j, x_k\} = \epsilon_{kj} x_j x_k.$$

On quantum level, toric charts $\mathcal{T}_{\mathbf{i}}$ become quantum tori:

$$\mathcal{T}_{\mathbf{i}}^q = \langle X_1, \dots, X_d \rangle / \{X_j X_k = q^{2\epsilon_{kj}} X_k X_j\}.$$

These quantum tori have the same field of fractions:

$$\mathcal{F} := \mathrm{Frac}\ \mathcal{T}_{\mathbf{i}}^q = \mathrm{Frac}\ \mathcal{T}_{\mathbf{i}'}^q,$$

and are "glued" by quantum cluster mutations that preserve $\mathcal{F}$. The quantized moduli space $\mathcal{X}_{G,\widehat{S}}^q$ consists of those elements of $\mathcal{F}$ that are Laurent polynomials in any chart $\mathcal{T}_{\mathbf{i}}^q$.

Cluster realization of $\mathrm{U}_q(\mathfrak{sl}_n)$

Theorem (Schrader-S. '16)

Let $\widehat{S}$ be a punctured disk with two marked points. Then there is an embedding of algebras

$$\mathrm{U}_q(\mathfrak{sl}_n) \hookrightarrow \mathcal{X}^q_{PGL_n, \widehat{S}},$$

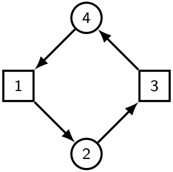
with the property that for each Chevalley generator of the quantum group, there is a cluster in which this generator is a cluster monomial.

Conjecture

$$\mathrm{U}_q(\mathfrak{sl}_n) \simeq \left(\mathcal{X}^q_{PGL_n, \widehat{S}} \right)^{S_n}.$$

Idea: Pull-back representations of the field of fractions $\mathcal{F}$ under the embedding $U_q(\mathfrak{sl}_n) \hookrightarrow \mathcal{F}$.

Example: $\mathrm{U_q}(\mathfrak{sl}_2)$



$$E \mapsto X_1(1+qX_2), \qquad K \mapsto q^2 X_1 X_2 X_3,$$
$$F \mapsto X_3(1+qX_4), \qquad K' \mapsto q^2 X_3 X_4 X_1.$$

KK' is central.

Recall that

$$E = \mu_2(X_1) \quad \text{and} \quad F = \mu_4(X_3).$$

Positive representations of quantum tori

Let

$$q = e^{\pi i \hbar^2}, \quad \hbar^2 \in \mathbb{R}_+ \setminus \mathbb{Q}_+.$$

We embed a quantum cluster chart $\mathcal{T}^q$ into a Heisenberg algebra $\mathcal{H}$ generated by $x_1, \dots, x_d$ with relations

$$[x_j, x_k] = \frac{i}{2\pi} \epsilon_{jk},$$

by the homomorphism

$$X_j \mapsto e^{2\pi \hbar x_j}.$$

The algebra $\mathcal{H}$ has a family of irreducible Hilbert space representations V_χ parameterized by central characters $\chi \in \mathrm{Hom}(\ker \epsilon, \mathbb{R})$, in which the generators X_j act by unbounded self-adjoint operators.

Modular duality

Now consider $q^\vee = e^{\pi i/\hbar^2}$, obtained from $q = e^{\pi i \hbar^2}$ by the transformation $\hbar \mapsto 1/\hbar$.
We also have an embedding of $\mathcal{T}^{q^\vee}$ into the Heisenberg algebra $\mathcal{H}$ given by

$$\tilde{X}_j = e^{2\pi\hbar^{-1}x_j},$$

so

$$\tilde{X}_j\tilde{X}_k = (q^\vee)^{2\epsilon_{kj}}\tilde{X}_k\tilde{X}_j.$$

Note that the generators $\tilde{X}_k$ commute with the original ones $X_j = e^{2\pi\hbar x_j}$:

$$X_j\tilde{X}_k = e^{2\pi i\epsilon_{kj}}\tilde{X}_kX_j = \tilde{X}_kX_j,$$

since $\epsilon_{kj} \in \mathbb{Z}$.

Mutations for $|q| = 1$

Cluster mutation in direction k is now realized by conjugation by *non-compact quantum dilogarithm*

$$\Phi^{\hbar}(z) = \frac{\Gamma_q(e^{2\pi\hbar z})}{\Gamma_{q^\vee}(e^{2\pi\hbar^{-1} z})}.$$

We have

$$\overline{\Phi^{\hbar}(\overline{z})} = \frac{1}{\Phi^{\hbar}(z)},$$

so since x_k is self-adjoint, the operator $\Phi^{\hbar}(x_k)$ is a unitary operator on V_χ.

This means the mutated operators $\mu_k(X_j)$ are also positive self-adjoint, and we get a unitary representation of the groupoid of cluster transformations.

Example: positive representations of $U_q(\mathfrak{sl}_2)$

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, and the self-adjoint, unbounded operators

$$\hat{p} = \frac{i}{2\pi}\frac{\partial}{\partial x}, \quad \hat{x} = x.$$

Then for all $s \in \mathbb{R}$, we have positive self-adjoint operators

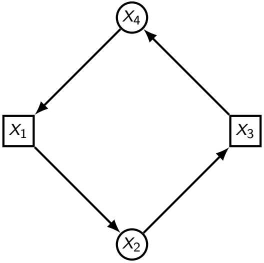
$$X_1 = e^{2\pi\hbar(\hat{p}-s)}, \qquad X_2 = e^{2\pi\hbar(\hat{x}+2s)}$$
$$X_3 = e^{2\pi\hbar(-\hat{p}-s)}, \qquad X_4 = e^{2\pi\hbar(-\hat{x}+2s)}$$

satisfying the cyclic quiver relations

$$q^2 X_k X_{k+1} = X_{k+1} X_k, \quad k \in \mathbb{Z}/4\mathbb{Z},$$

where $q = e^{\pi i \hbar^2}$.

Example: positive representations of $U_q(\mathfrak{sl}_2)$



Chevalley generators of $U_q(\mathfrak{sl}_2)$ act by positive, self-adjoint operators

$$E \mapsto e^{2\pi\hbar(\hat{p}-s)} + e^{2\pi\hbar(\hat{p}+\hat{x}+s)} = \mu_2\left(e^{2\pi\hbar(\hat{p}-s)}\right), \qquad K \mapsto e^{2\pi\hbar\hat{x}},$$

$$F \mapsto e^{-2\pi\hbar(\hat{p}+s)} + e^{-2\pi\hbar(\hat{p}+\hat{x}-s)} = \mu_4\left(e^{-2\pi\hbar(\hat{p}+s)}\right), \quad K' \mapsto e^{-2\pi\hbar\hat{x}}.$$

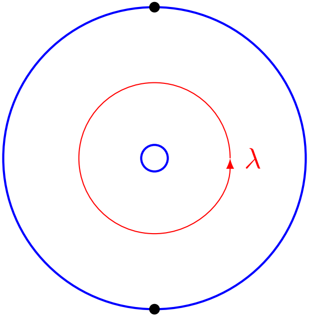
The $U_q(\mathfrak{sl}_2)$ Casimir element Ω acts by

$$\Omega \mapsto e^{4\pi\hbar s} + e^{-4\pi\hbar s}.$$

Geometric approach: cutting and gluing isomorphisms

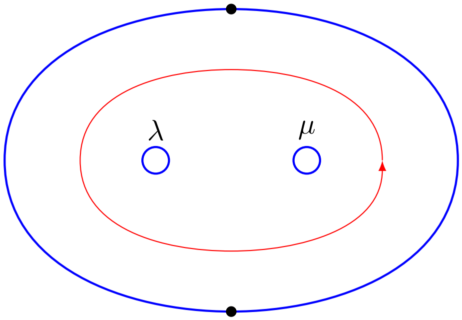
With Gus, we have a geometric approach to the conjecture using quantum higher Teichmüller theory.

First observation: central character of $\mathcal{P}_\lambda$ is determined by eigenvalues of the holonomy around the puncture.



Geometric approach: cutting and gluing isomorphisms

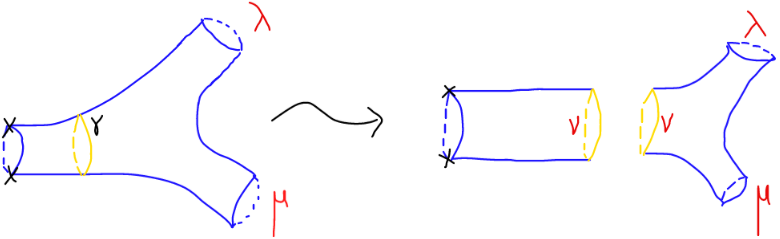
Picture for $P_\lambda \otimes P_\mu$:



Casimirs of $\Delta\left(U_q(\mathfrak{g})\right)$ are determined by the holonomy around the red loop.

Teichmüller theory interpretation of the decomposition of the tensor product of positive representations:

$$P_\lambda \otimes P_\mu = \int_\nu P_\nu \otimes M^\nu_{\lambda,\mu} d\nu.$$



Reduction to q-difference Toda spectral problem

Algebraically: the picture let us read off a natural sequence of cluster transformations, which identifies the traces of holonomies with the Hamiltonians of the q-difference *open Toda lattice.*

Fix a Coxeter element $c = s_1 s_2, \ldots, s_n$ of the symmetric group S_{n+1}. Let $H \subset SL_{n+1}(\mathbb{C})$ be a Cartan subgroup, $B_\pm$ a pair of opposite Borels, and consider the double Bruhat cell

$$SL_{n+1}^{c,c} = B_+ c B_+ \cap B_- c B_-.$$

Then $\dim SL_{n+1}^{c,c}/\mathrm{Ad}_H = 2n$, and the conjugation-invariant functions define an integrable system.

Reduction to q-difference Toda spectral problem

The eigenfunctions of the quantum Toda Hamiltonians have been determined by Kharchev-Lebedev-Semenov-Tian-Shansky: they are the *q-Whittaker functions*.

Theorem (Kharchev-Lebedev-Semenov-Tian-Shansky, Kashaev)

The q-Whittaker functions are orthogonal and complete for $\mathfrak{g} = \mathfrak{sl}_2$.

Conjecture

The q-Whittaker functions are orthogonal and complete for other types.

The intertwiner

At the end of the day, we construct an intertwining operator

$$\mathcal{I}\colon \mathcal{P}_\lambda \otimes \mathcal{P}_\mu \longrightarrow \int_{\mathcal{C}^+}^{\oplus} \mathcal{P}_\nu \otimes M_{\lambda,\mu}^{\nu} \, dm(\nu),$$

here $dm(\lambda)$ is the Sklyanin measure

$$dm(\lambda) = \prod_{j<k} 4 \sinh\left(\pi\hbar(\lambda_j - \lambda_k)\right) \sinh\left(\pi\hbar^{-1}(\lambda_j - \lambda_k)\right),$$

The intertwiner $\mathcal{I}$ is an integral operator, whose kernel is a product of a number of quantum dilogarithms.

Towards a modular functor

To a surface $\widehat{S}$, higher Teichmüller theory assigns an algebra $\mathcal{A}_{\widehat{S}}$ acting on a Hilbert space $\mathcal{H}_{\widehat{S}}$. Suppose we cut $\widehat{S}$ into

$$\widehat{S} = \widehat{S}_1 \cup \widehat{S}_2.$$

Question: How does a representation $\mathcal{H}_{\widehat{S}}$ decompose with respect to the subalgebra $\mathcal{A}_{\widehat{S}_1}$?

Our construction allows to answer this question, when we cut out a pair of pants. By similar methods, we can cut a handle by a non-separating cycle.

The missing ingredient is cutting/gluing of a hole with a disk.

Thank you for listening!