Bäcklund transformation of Painleve τ function from representation theory

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based on joint paper with M. Bershtein
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Painlevé VI equation

- Painleve VI equation (note that it has 4 parameters $\theta_0, \theta_z, \theta_1, \theta_\infty$)

$$\frac{d^2w}{dz^2} = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz} +$$
$$+ \frac{2w(w-1)(w-z)}{z^2(z-1)^2}\left(\left(\theta_\infty - \frac{1}{2}\right)^2 - \frac{\theta_0^2 z}{w^2} + \frac{\theta_1^2(z-1)}{(w-1)^2} - \frac{\left(\theta_z^2 - \frac{1}{4}\right)z(z-1)}{(w-z)^2}\right)$$

- PVI equation admits non-autonomous Hamiltonian form and τ form of order 4. This form is equivalent to the original form up to Jimbo asymptotics (see below).
- The natural framework for Painleve equation is the isomonodromic problem for the $sl(2)$ connections on the sphere with 4 punctures.

Bäcklund transformations of Painlevé VI

- Group of Backlund transformations (group of symmetries) for PVI is extended by outer automorphisms affine Weyl group of $D_4^{(1)}$. It is generated by simple reflections $s_0, s_1, s_t, s_\infty, s_\delta$ acting on the Cartan space $(\theta_0, \theta_z, \theta_1, \theta_\infty)$. Nontrivial actions are

$$s_i : \theta_i \mapsto -\theta_i,\ i = 0, t, 1, \quad s_\infty : \theta_\infty \mapsto 1 - \theta_\infty, \quad s_\delta : \theta_i \mapsto \theta_i - \delta,\ i = 0, 1, t, \infty,$$

  where $\delta = \frac{\theta_0 + \theta_z + \theta_1 + \theta_\infty}{2}$.

- We are interested in infinite order transformation $\pi_{z\infty} : (\theta_0, \theta_z, \theta_1, \theta_\infty) \mapsto (\theta_0, \theta_z + 1/2, \theta_1, \theta_\infty + 1/2)$.

Painlevé III(D_8) equation

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$$\frac{d^2 w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{2w^2}{z^2} - \frac{2}{z}.$$

- This equation is equivalent to the radial sine-Gordon equation.

$$v_{rr} + \frac{v_r}{r} = 1/2 \sin 2v$$

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- It is useful to consider such degeneration of the most general PVI equation, because it is simplier then others from the Painleve hierarchy.
- PIII(D_8) also admits Hamiltonian and τ form.
- Group of Backlund transformations is $\mathbb{Z}_2$ generated by $w \mapsto z/w$. In terms of radial sine-Gordon this is just $v \mapsto -v$.

Gamayun-Iorgov-Lisovyy formula 1

Formula for Painleve τ function [Gamayun-Iorgov-Lisovyy 12-13]

$$\tau(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^n z^{(\sigma+n)^2} C(\sigma+n)\mathcal{F}((\sigma+n)^2|z) \tag{1}$$

- Similar formula holds for all degeneracy chain from PVI to PIII(D_8): there are only different in structure constants C and conformal blocks $\mathcal{F}$. For PVI such expansion exists also in 0 and ∞.
- s, σ — integration constants. Periodicity $\tau(\sigma, s|z) = s^{-1}\tau(\sigma+1, s|z)$
- In case of PVI τ function depends on $\theta_0, \theta_z, \theta_1, \theta_\infty$ and so as C and $\mathcal{F}$
- $C(\sigma)$ is expressed in terms of Barnes G function, $\mathrm{G}(z+1) = \mathrm{G}(z)\Gamma(z)$.

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- $C(\sigma)$ is expressed in terms of Barnes G function, $\mathrm{G}(z+1) = \mathrm{G}(z)\Gamma(z)$.
- $\mathcal{F}(\Delta|z)$ for PVI case is 4-point $c=1$ conformal block of Virasoro Verma module with 4 external weights $\Delta_i = \theta_i^2$ and intermediate weight Δ.
- In case of PIII(D_8) this function is a Whittaker limit of Virasoro conformal block in a representation of the highest weight Δ.

Gamayun-Iorgov-Lisovyy formula 2

- AGT relation: for PIII(D_8) $\mathcal{F}(\Delta|z)$ is Nekrasov partition function for pure $SU(2)$ theory, for PVI case there are 4 additional matter fields, mass of which are expressed in terms of parameters θ_i.

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- This formula is a generalization of Jimbo asymptotics [Jimbo 82] for τ function. In case of PIII(D_8) this asymptotic reads

$$\tau(\sigma, \widetilde{s}|z) \propto z^{\sigma^2}\left(1+\frac{z}{2\sigma^2}-\frac{\widetilde{s}^{-1}}{(1-2\sigma)^2(2\sigma)^2}z^{1-2\sigma}+o(|z|)\right), \qquad (2)$$

  where $\widetilde{s}$ differs from s by rational function on σ.
- In PIII(D_8) Backlund transformation $\pi : \tau(\sigma, s|z) \mapsto \tau(\sigma + 1/2, s|z)$.

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- In PIII(D_8) Backlund transformation $\pi : \tau(\sigma, s|z) \mapsto \tau(\sigma+1/2, s|z)$.
- Formula (1) proven [Iorgov, Lisovyy, Teschner 14], [Bershtein, AS 14], [Gavrylenko, Lisovyy 16]. Our method is based on representation-theoretic calculations and such approach will be presented below for different goal.

Backlund transformations and τ forms)

- For PIII(D_8) there is a Toda-like equation on functions τ and $\tau_1 = \pi(\tau)$

$$\begin{cases} 1/2 D^2_{[\log z]}(\tau(z), \tau(z)) = z^{1/2}\tau_1(z)\tau_1(z), \\ 1/2 D^2_{[\log z]}(\tau_1(z), \tau_1(z)) = z^{1/2}\tau(z)\tau(z), \end{cases} \tag{3}$$

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- There is also alternative equivalent system of equations

$$D^2_{[\log z]}(\tau, \tau_1) - \frac{1}{2} z \frac{d}{dz}(\tau\tau_1) + \frac{1}{16}\tau\tau_1 = 0,$$
$$D^3_{[\log z]}(\tau, \tau_1) + \frac{1}{16} D^1_{[\log z]}(\tau, \tau_1) - \frac{1}{2} z \frac{d}{dz} D^1_{[\log z]}(\tau, \tau_1) = 0$$

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- For PVI equation for transformation $\pi_{z\infty}$ we have bilinear [Okamoto 85] Toda-like equation

$$\delta^2 \log \tau = \pi_{z\infty}(\tau)\pi^{-1}_{z\infty}(\tau)/\tau^2, \qquad \delta = z(z-1)\frac{d}{dz} \tag{4}$$

  It is believed to be equivalent to the original Painleve VI equation.

F $\oplus$ NSR algebra

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$$\begin{aligned}
&\{f_r, f_s\} = \delta_{r+s,0}, \quad \{f_r, G_s\} = 0 \\
&[L_n, L_m] = (n-m)L_{n+m} + 1/8(n^3 - n)c_{\mathrm{NSR}}\delta_{n+m,0} \\
&\{G_r, G_s\} = 2L_{r+s} + 1/2c_{\mathrm{NSR}}(r^2 - 1/4)\delta_{r+s,0} \\
&[L_n, G_r] = (n/2 - r)G_{n+r}.
\end{aligned} \tag{5}$$

- We consider NS and R sectors of this algebra — with $\mathbb{Z} + 1/2$ and $\mathbb{Z}$ indexes of odd generators correspondingly.

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- We consider NS and R sectors of this algebra — with $\mathbb{Z}+1/2$ and $\mathbb{Z}$ indexes of odd generators correspondingly.
- There exists embedding of Vir $\oplus$ Vir algebra in $\overline{\mathfrak{U}(\mathrm{F}\oplus\mathrm{NSR})}$ [Crncovic, Paunov, Sotkov, Stanishkov 90, Lashkevich 92]

$$
L^{(\eta)}(z) = \frac{1}{2}L(z) + \frac{1}{2}T_f(z) - (-1)^\eta \frac{i}{2} f(z)G(z), \quad \eta = 1,2 \tag{6}
$$

where $T_f(z) = 1/2 : f'(z)f(z) :$ – fermionic energy-momentum tensor.
$f(z), L(z), G(z)$ — currents which are builded from modes of F $\oplus$ NSR. This a specialization where $c_{\text{NSR}} = 1 \Rightarrow c_{\text{Vir}^{(\eta)}} = 1$.

$\mathsf{F} \oplus \mathsf{NSR}$ Verma module decomposition: NS sector

- Consider $\mathsf{F} \oplus \mathsf{NSR}$ and Vir Verma modules defined as usually.
- We have decomposition in NS sector

$$\pi_{\mathsf{F}\oplus\mathsf{NSR}}^{\Delta^{\mathsf{NS}}} \cong \bigoplus_{2n\in\mathbb{Z}} \pi_{\mathsf{Vir}\oplus\mathsf{Vir}}^{n}, \tag{7}$$

where h.w. of $\pi_{\mathsf{Vir}\oplus\mathsf{Vir}}^{n}$ is $(\sigma+n)^2$ and $(\sigma-n)^2$ wrt to Vir algebras.

- Roughly speaking it follows from character identity

$$\mathrm{ch}(\pi_{\mathsf{F}\oplus\mathsf{NSR}}^{\Delta^{\mathsf{NS}}}) = z^{\Delta^{\mathsf{NS}}} \prod_{k=1}^{\infty} \frac{(1+z^{k-\frac{1}{2}})^2}{1-z^k} =$$

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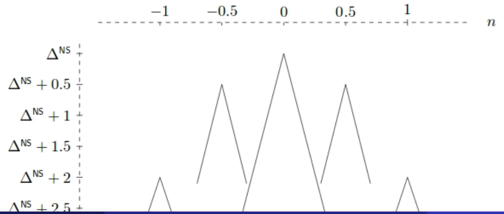
which follows from Jacobi triple product.



$F \oplus$ NSR module decomposition: R sector

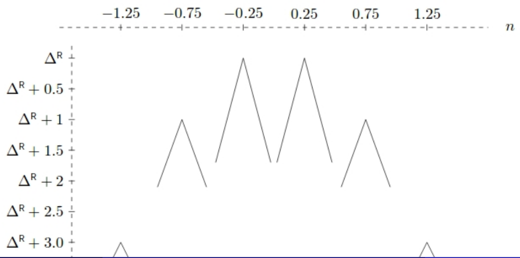
- In R sector we have

$$\mathrm{ch}(\pi^{\Delta^{\mathrm{R}}}_{\mathrm{F}\oplus\mathrm{NSR}}) = z^{\Delta^{\mathrm{R}}+\frac{1}{16}} \frac{\prod_{k=0}^{\infty}(1+z^k)^2}{\prod_{k=1}^{\infty}(1-z^k)} = 2\sum_{2n\in\mathbb{Z}+\frac{1}{2}} \frac{z^{\Delta^{\mathrm{R}}+2n^2-\frac{1}{16}}}{\prod_{k=1}^{\infty}(1-z^k)^2} = \sum_{\substack{2n\in\mathbb{Z}+1/2\\ \epsilon=0,1}} \mathrm{ch}(\pi^{\epsilon,n}_{\mathrm{Vir}\oplus\mathrm{Vir}}).$$

  Here ϵ denote two copies of $\mathrm{Vir}\oplus\mathrm{Vir}$ module with the same h.w. This implies

-

$$\pi^{\Delta^{\mathrm{R}}}_{\mathrm{F}\oplus\mathrm{NSR}} \cong \pi^{\Delta^{\mathrm{R}},0}_{\mathrm{F}\oplus\mathrm{NSR}} \oplus \pi^{\Delta^{\mathrm{R}},1}_{\mathrm{F}\oplus\mathrm{NSR}} \cong \bigoplus_{2n+1/2\in\mathbb{Z}} \pi^{n,0}_{\mathrm{Vir}\oplus\mathrm{Vir}} \oplus \bigoplus_{2n+1/2\in\mathbb{Z}} \pi^{n,1}_{\mathrm{Vir}\oplus\mathrm{Vir}}. \qquad (9)$$



Utilization of the decomposition: general scheme 1

- Introduce Whittaker vector — gener. function of vectors of Verma module.
- For Vir Verma module it is defined by

$$|W(z)\rangle = z^{\Delta}\sum_{N=0}^{\infty} z^N|N\rangle, \qquad |N\rangle \in \pi^{\Delta}_{\mathrm{Vir}}, \qquad L_0|N\rangle = (\Delta+N)|N\rangle, \quad (10)$$

where

$$L_1|W(z)\rangle = z|W(z)\rangle, \quad L_k|W(z)\rangle = 0, k>1. \quad (11)$$

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- For $\mathsf{F}\oplus\mathsf{NSR}$ Verma module in NS sector defining propoperties are

$$G_{1/2}|W_{\mathrm{NS}}(z)\rangle = z^{1/2}|W_{\mathrm{NS}}(z)\rangle, \quad G_r|W_{\mathrm{NS}}(z)\rangle = 0, r\geq 3/2 \tag{12}$$

- Then we have

Proposition

The decomposition of the $F\oplus NSR$ Whittaker vector of NS sector in terms of the subalgebra $Vir\oplus Vir$ has the form

$$|1\otimes W_{NS}(z)\rangle = \sum_{2n\in\mathbb{Z}} l_n(\sigma)\left(|W^{(1)}(z/4)\rangle_n\otimes|W^{(2)}(z/4)\rangle_n\right). \tag{13}$$

Utilization of the decomposition: general scheme 2

- We want to obtain bilinear relations on conformal blocks which are equivalent to the bilinear equations on τ functions.
- Whitt. limits of conformal blocks are squares of appropriate Whittaker vectors

$$\mathcal{F}(\Delta|z) = \langle W(1)|W(z)\rangle, \qquad \mathcal{F}(\Delta^{\text{NS}}|z) = \langle W_{\text{NS}}(1)|W_{\text{NS}}(z)\rangle \tag{14}$$

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- Introduce $H = L_0^{(1)} - L_0^{(2)}$. We want to calculate matrix elements of its powers in two ways.
- First way is to calculate z^H in terms of $\mathrm{Vir} \oplus \mathrm{Vir}$ generators which gave us generating function of Hirota differentials.
- Other way is to rewrite $H = -i \sum\limits_{r \in \mathbb{Z}+1/2} f_{-r} G_r$ and use properties of NSR Whittaker vectors.
- Comparing these two calculations we obtain required bilinear relations on Vir conformal blocks.

Utilization of the decomposition: Okamoto equation on PVI τ function 1

- We want to obtain Okamoto equation by using above scheme but with other Whittaker vector than in case of proof of Gamayun-Iorgov-Lisovyy formula.
- For Painleve VI case Vir Whittaker vector is defined in other way (it is often called chain vector)

$$|W(z)\rangle_{21} = z^{\Delta_1+\Delta_2} V^{\Delta_2}_{\Delta,\Delta_1}(z)|\Delta_1\rangle, \tag{15}$$

where $V^{\Delta_2}_{\Delta,\Delta_1} : \pi^{\Delta_1}_{\mathrm{Vir}} \mapsto \pi^{\Delta}_{\mathrm{Vir}}$ satisfy

$$[L_k, V_\Delta(z)] = \left(z^{k+1}\partial_z + (k+1)\Delta z^k\right) V_\Delta(z), \tag{16}$$

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- For NSR algebra there exist two types of vertex operators – even Φ^{NS} and odd Ψ^{NS}. Vertex Φ^{NS} correspond to the h.w.v. of $\mathrm{F} \oplus \mathrm{NSR}$ module.

Proposition

[Belavin-Bershtein-Feigin-Litvinov-Tarnopolsky 11]

$$\Phi^{NS}(z) \simeq V^{(1)}(z)V^{(2)}(z) \tag{17}$$

which generalize chain vector decomposition.

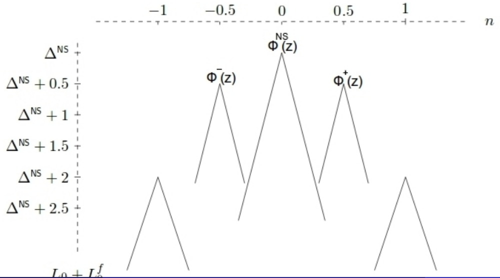
Utilization of the decomposition: Okamoto equation on PVI τ function 2

- We will use another vertex operator [Belavin-Bershtein-Feigin-Litvinov-Tarnopolsky 11] which correspond to highest weight vector of $\pi^{\pm 1/2}_{\mathsf{Vir}\oplus\mathsf{Vir}}$

$$\Phi^{\pm}_{2\sigma}(z) = \pm 2i\sigma f(z)\Phi^{\mathsf{NS}}_{2\sigma}(z) + \Psi^{\mathsf{NS}}_{2\sigma} \tag{18}$$

- Analogously to Φ^{NS} case we prove that

$$\Phi^{\pm}_{2\sigma}(z) = V^{(1)}_{\sigma\pm1/2}(z)V^{(2)}_{\sigma\mp1/2}(z) \tag{19}$$



Utilization of the decomposition: Okamoto equation on PVI τ function 3

- Note that from now we specify that we use NS sector. NS sector should be used in purpose to obtain Toda-like equations and R sector — to obtain Okamoto-like.
- Then we have

$$z^{2\Delta_0+2\Delta_t}\langle\Delta_\infty|\Phi^{\pm}_{2\theta_1}\Phi^{\pm}_{2\theta_z}|\Delta_0\rangle = \frac{4\theta_1\theta_z}{1-z}\mathcal{F}^{\text{NS}} + z^{-1/2}\widetilde{\mathcal{F}^{\text{NS}}} \tag{20}$$

$$z^{2\Delta_0+2\Delta_t}\langle\Delta_\infty|\Phi^{\pm}_{2\theta_1}\Phi^{\mp}_{2\theta_z}|\Delta_0\rangle = -\frac{4\theta_1\theta_z}{1-z}\mathcal{F}^{\text{NS}} + z^{-1/2}\widetilde{\mathcal{F}^{\text{NS}}} \tag{21}$$

  where $\widetilde{\mathcal{F}^{\text{NS}}}$ — conformal block builded from chains with one odd operator in each chain.
- For PVI τ function introduce the notation
  $\tau_{\mu,\nu}(\theta_0,\theta_z,\theta_1,\theta_\infty,\sigma|z) = \tau(\theta_0,\theta_z+\mu 1/2,\theta_1+\nu 1/2,\theta_\infty,\sigma|z), \quad \mu,\nu=\pm.$

Utilization of the decomposition: Okamoto equation on PVI τ function 4

- Subtracting above two relations we obtain bilinear relations on Vir conformal blocks without derivatives. It implies algebraic relation on τ functions

$$\frac{8\theta_1\theta_z}{1-z}\tau^2 = z^{1/2}(\tau_{+-}(\sigma+1/2)\tau_{-+}(\sigma-1/2) - \tau_{++}(\sigma+1/2)\tau_{--}(\sigma-1/2)). \tag{22}$$

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- In other hand from calculations with Φ^{NS} we know that

$$_{34}\langle W_{\text{NS}}(1)|H^2|W_{\text{NS}}(z)\rangle_{21} = -\frac{z^{1/2}}{1-z}\widetilde{\mathcal{F}^{\text{NS}}} \tag{23}$$

- Substituting this we obtain differential relation on conformal block which implies differential relation on τ function

$$(1-z)^2 D^2(\tau,\tau) = -2z^{1/2}(1-z)\tau_{++}(\sigma+1/2)\tau_{--}(\sigma-1/2) - 4\theta_1\theta_z z\tau^2. \tag{24}$$

- Up to certain Backlund transformation it is equiv. to the Okamoto equation.

Conclusion

- The aim of the talk is to present powerful method of obtaining various relations on conformal blocks useful for study of Painlevé equations.
- Nevertheless, there are no understanding what is the meaning of bilinear equations on τ functions in terms natural to Painlevé equation science, for instance, in terms of isomonodromic deformation problem.

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Thank you for your attention!