Heisenberg spin chains by separation of variables

recent advances

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The Heisenberg spin-1/2 chain: an archetype of quantum integrable models

The XXZ spin-1/2 Heisenberg chain

$$H_{XXZ} = \sum_{m=1}^{N} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\} - h \sum_{m=1}^{N} \sigma_m^z$$

- space of states: $\mathcal{H} = \otimes_{n=1}^{N} \mathcal{H}_n$ with $\mathcal{H}_n \simeq \mathbb{C}^2$
- $\sigma_m^{x,y,z} \in \mathrm{End}(\mathcal{H}_n)$: local spin-1/2 operators (Pauli matrices) at site m
- $\Delta = \cosh \eta$: anisotropy parameter $\to \Delta = 1$ for XXX (isotropic) chain
- h : magnetic field
- usually periodic boundary conditions are considered: $\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}$

⋆ First model solved via Bethe ansatz [Bethe, 1931]

⋆ More algebraic solution in the framework of the Quantum Inverse Scattering Method (QISM) [Faddeev, Sklyanin, Takhtajan, 1979]

$\rightsquigarrow$ solution based on the representation theory of the Yang-Baxter algebra

The (periodic) XXX/XXZ spin-1/2 chain is probably the most widely studied quantum integrable model:

- $\star$ It has a simple Yang-Baxter algebra:

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\varphi(\lambda)}{\varphi(\lambda+\eta)} & \frac{\varphi(\eta)}{\varphi(\lambda+\eta)} & 0 \\ 0 & \frac{\varphi(\eta)}{\varphi(\lambda+\eta)} & \frac{\varphi(\lambda)}{\varphi(\lambda+\eta)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } \varphi(\lambda) = \begin{cases} \lambda & \text{(XXX chain)} \\ \sinh(\lambda) & \text{(XXZ chain)} \end{cases}$$

  - monodromy matrix: $T(\lambda) = R_{0N}(\lambda) \dots R_{01}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

$$R(\lambda-\mu)\,(T(\lambda)\otimes 1)\,(1\otimes T(\mu)) = (1\otimes T(\mu))\,(T(\lambda)\otimes 1)\,R(\lambda-\mu)$$

  - transfer matrix: $t(\lambda) = \mathrm{tr}\, T(\lambda) = A(\lambda) + D(\lambda)$

$$[t(\lambda), t(\mu)] = [t(\lambda), H] = 0$$

- $\star$ there exists a reference state (the state $|\,0\,\rangle \equiv |\uparrow\uparrow\dots\uparrow\rangle$) such that

$$C(\lambda)\,|\,0\,\rangle = 0, \quad A(\lambda)\,|\,0\,\rangle = a(\lambda)|\,0\,\rangle, \quad D(\lambda)\,|\,0\,\rangle = d(\lambda)|\,0\,\rangle$$

$\rightsquigarrow$ algebraic Bethe ansatz (ABA) can be applied and eigenstates of $t(\lambda)$ can be constructed as Bethe states:

$$|\,\{\lambda\}\,\rangle = \prod_{k=1}^{n} B(\lambda_k)|\,0\,\rangle \in \mathcal{H}, \quad \langle\,\{\lambda\}\,| = \langle\,0\,|\prod_{k=1}^{n} C(\lambda_k) \in \mathcal{H}^*$$

$\rightarrow$ eigenstates ("on-shell" Bethe states) if $\{\lambda\}$ solution of Bethe eq.

$\rightarrow$ "off-shell" Bethe states otherwise

It is now possible to have access to correlation functions from the study of the periodic XXZ chain by algebraic Bethe Ansatz

- either numerically [Caux et al. 2005...]
- either analytically: large distance asymptotic behavior at the thermodynamic limit... [Kitanine, Kozlowski, Maillet, Slavnov, VT 2008, 2011...]

Both approaches are based

- on the form factor decomposition of the correlation functions:

$$\langle\psi_g|\sigma_n^\alpha\,\sigma_{n'}^\beta|\psi_g\rangle = \sum_{\substack{\text{eigenstates}\\ |\,\psi_i\,\rangle}} \langle\psi_g|\sigma_n^\alpha|\psi_i\rangle\cdot\langle\psi_i|\sigma_{n'}^\beta|\psi_g\rangle$$

- on the exact determinant representations for the form factors $\langle\psi_i|\sigma_n^\alpha|\psi_j\rangle$ in finite volume [Kitanine, Maillet, VT 1999] , obtained from
    - the action of local operators on Bethe states (using the solution of the quantum inverse problem $\sigma_n^- = t(0)^{n-1}\,B(0)\,t(0)^{-n}$)
    - the use of Slavnov's determinant representation for the scalar products of Bethe states [Slavnov 89]

$$\langle\{\mu\}_{\text{off-shell}}|\{\lambda\}_{\text{on-shell}}\rangle \propto \det_{1\le j,k\le n}\left[\frac{\partial\tau(\mu_j|\{\lambda\})}{\partial\lambda_k}\right]$$

Generalizations to more complicated integrable models ?

Limitations of the ABA approach:

- it requires the clear identification of a reference state $|\,0\,\rangle$
  $\rightsquigarrow$ there are some interesting models for which ABA cannot be applied

- even if ABA is a priori applicable, the completeness of the eigenstate construction is a delicate issue

- the ABA Bethe states have a complicated combinatorial structure
  $\rightsquigarrow$ the generalization of Slavnov's formula for the scalar products of Bethe states is a difficult problem (one does not know any model-independent procedure to compute these scalar products)

Integrable generalizations of the XXZ Heisenberg chain

It has several interesting generalizations which are still integrable (in the sense that one can still define a family of commuting transfer matrices):

- ⋆ XYZ model (related to 8-vertex model):

$$H_{\mathrm{XYZ}} = \sum_{m=1}^{N} \left\{ J_x\, \sigma_m^x \sigma_{m+1}^x + J_y\, \sigma_m^y \sigma_{m+1}^y + J_z\, \sigma_m^z \sigma_{m+1}^z \right\}$$

- ⋆ Open spin chains (with boundary magnetic fields):

$$H_{\mathrm{XXZ}}^{\mathrm{open}} = \sum_{m=1}^{N-1} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\} + h_-^x \sigma_1^x + h_-^y \sigma_1^y + h_-^z \sigma_1^z + h_+^x \sigma_N^x + h_+^y \sigma_N^y + h_+^z \sigma_N^z$$

$\rightsquigarrow$ relation with open Asymmetric Simple Exclusion Process (ASEP) [de Gier, Essler 05]

- ⋆ higher spins or higher ranks. . .

The reflection algebra for the XXZ open spin chain

The open spin chains are solvable in the framework of the representation theory of the reflection algebra (or boundary Yang-Baxter algebra) [Sklyanin 88]

- generators $\mathcal{U}_{ij}(\lambda)$, $1 \le i, j \le n$ $\leftarrow$ elements of the boundary monodromy matrix $\mathcal{U}(\lambda)$
- commutation relations given by the reflection equation:

$$R_{12}(\lambda-\mu)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda+\mu+\eta)\,\mathcal{U}_2(\mu) = \mathcal{U}_2(\mu)\,R_{12}(\lambda+\mu+\eta)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda-\mu)$$

$\hookrightarrow$ most general 2×2 solution of the refl. eq.
[de Vega, Gonzalez-Ruiz; Ghoshal, Zamolodchikov 93] :

$$K(\lambda;\zeta,\kappa,\tau) = \frac{1}{\sinh\zeta}\begin{pmatrix} \sinh(\lambda-\frac{\eta}{2}+\zeta) & \kappa e^{\tau}\sinh(2\lambda-\eta) \\ \kappa e^{-\tau}\sinh(2\lambda-\eta) & \sinh(\zeta-\lambda+\frac{\eta}{2}) \end{pmatrix}$$

$\rightsquigarrow$ 2 boundary matrices $K^{\pm}(\lambda)$ describing boundary fields in left/right boundaries

$\rightsquigarrow$ $\mathcal{U}(\lambda) = T(\lambda)\,K_-(\lambda)\,\sigma^y\,T^t(-\lambda)\,\sigma^y = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}$

$\rightsquigarrow$ transfer matrix: $\mathcal{T}(\lambda) = \mathrm{tr}\{K^+(\lambda)\,\mathcal{U}(\lambda)\}$ $\qquad [\mathcal{T}(\lambda),\mathcal{T}(\mu)] = 0$

$$H^{\text{open}}_{XXZ} \propto \frac{d}{d\lambda}\mathcal{T}(\lambda)$$

The open spin chains: limitations of the solution by ABA

- ⋆ In the diagonal case ($\kappa_\pm = 0$, boundary fields along σ_1^z and σ_N^z only):
  - the state $|\,0\,\rangle$ can still be used as a reference state to construct the eigenstates $\rightsquigarrow$ ABA can be applied [Sklyanin 88]
  - $\exists$ generalization of Slavnov's formula for the scalar products of Bethe states [Tsuchiya 98; Wang 02]
  - correlation functions can be computed (but no simple closed formula for the form factors) [Kitanine et al. 07]

- ⋆ it is possible to generalize Bethe ansatz equations to other cases with nevertheless some constraints on the boundary fields [Nepomechie 03] , but
  - problems in the ABA construction of a complete set of Bethe states [Cao et al 03; Yang, Zhang 07; Filali, Kitanine 11]
    $\rightsquigarrow$ scalar products and correlation functions cannot be computed

- ⋆ most general boundaries ? an ABA solution is missing. . .

A complementary approach to ABA: Sklyanin's quantum Separation of Variables (SOV) [Sklyanin 85,90]

Idea: Use the "operator roots" $\hat{b}_j$ of the operator $B(\lambda)$ from the monodromy matrix to construct a basis of the space of state which "separate the variables" for the transfer matrix spectral problem

Conditions on $B(\lambda)$:

- $[B(\lambda), B(\mu)] = 0$,
- $B(\lambda)$ is a (usual, trigonometric, elliptic...) polynomial of degree N
- $B(\lambda)$ is diagonalizable with simple spectrum

$\rightsquigarrow$ N commuting "operators roots" $\hat{b}_j$ (with $\mathrm{Spec}(\hat{b}_j) \cap \mathrm{Spec}(\hat{b}_k) = \emptyset$ if $j \neq k$) which can be used to define a basis of the space of states $\mathcal{H}$:

$$| \mathbf{b} \rangle \quad \text{with} \quad \mathbf{b} = (b_1, \dots, b_N) \in \mathrm{Spec}(\hat{b}_1) \times \cdots \times \mathrm{Spec}(\hat{b}_N)$$
$$\hat{b}_n | \mathbf{b} \rangle = b_n | \mathbf{b} \rangle$$

This basis is moreover such that

$$A(\hat{b}_n) | b_1, \dots, b_n, \dots, b_N \rangle = \Delta_+(b_n) | b_1, \dots, b_n + \eta, \dots, b_N \rangle$$
$$D(\hat{b}_n) | b_1, \dots, b_n, \dots, b_N \rangle = \Delta_-(b_n) | b_1, \dots, b_n - \eta, \dots, b_N \rangle$$

A complementary approach to ABA: Sklyanin's quantum Separation of Variables (SOV) [Sklyanin 85,90]

$\rightsquigarrow$ In this basis, the multi-dimensional spectral problem for the transfer matrix $t(\lambda) = A(\lambda) + D(\lambda)$ can be reduced to a set of N one-dimensional finite-difference spectral problems:

$$t(\lambda)\,|\,\Psi_\tau\,\rangle = \tau(\lambda)\,|\,\Psi_\tau\,\rangle,$$

$$\text{with} \quad |\,\Psi_\tau\,\rangle = \sum_{\mathbf{b}=(b_1,\dots,b_N)} \psi_\tau(b_1,\dots,b_N)\,|\,\mathbf{b}\,\rangle,$$

is solved by

$$\psi_\tau(b_1,\dots,b_N) = \prod_{n=1}^{N} Q_\tau(b_n)$$

where $Q_\tau(b_n)$ and $\tau(b_n)$ are solution of a discrete version of Baxter's T-Q equation, for $n \in \{1,\dots,N\}$, $b_n \in \mathrm{Spec}(\hat{b}_n)$:

$$\tau(b_n)\,Q_\tau(b_n) = \Delta_+(b_n)\,Q(b_n+\eta) + \Delta_-(b_n)\,Q_\tau(b_n-\eta)$$

Remark: the completeness is given by construction

SOV for the antiperiodic XXZ chain

One can here apply this process to the antiperiodic monodromy matrix (with inhomogeneity parameters $\xi_1, \ldots, \xi_N$):

$$\bar{T}(\lambda) = \sigma^x \, R_{0N}(\lambda - \xi_N) \ldots R_{02}(\lambda - \xi_2) \, R_{01}(\lambda - \xi_1)$$
$$= \begin{pmatrix} \bar{A}(\lambda) & \bar{B}(\lambda) \\ \bar{C}(\lambda) & \bar{D}(\lambda) \end{pmatrix} = \begin{pmatrix} C(\lambda) & D(\lambda) \\ A(\lambda) & B(\lambda) \end{pmatrix}$$

- $\bar{B}(\lambda) = D(\lambda)$ is a (trigonometric) polynomial of degree N with N operator roots $\hat{b}_n$
- $\mathrm{Spec}(\hat{b}_n) = \{\xi_n, \xi_n - \eta\}$
  $\rightarrow$ the simplicity condition is fulfilled if $\xi_j \neq \xi_k, \xi_k \pm \eta$ for $j \neq k$

$\rightsquigarrow$ basis $|\, \mathbf{b} \,\rangle$ of $\mathcal{H}$ and $\langle\, \mathbf{b} \,|$ of $\mathcal{H}^*$ which separate the variables for the spectral problem for the antiperiodic transfer matrix $\bar{t}(\lambda) = \bar{A}(\lambda) + \bar{D}(\lambda)$

Remark: Since $\mathbf{b}$ is of the form $(\xi_1 - h_1\eta, \ldots, \xi_N - h_N\eta)$ with $\mathbf{h} = (h_1, \ldots, h_N) \in \{0,1\}^N$, we shall use from now on the notation $|\, \mathbf{h} \,\rangle$ and $\langle\, \mathbf{h} \,|$ instead of $|\, \mathbf{b} \,\rangle$ and $\langle\, \mathbf{b} \,|$.

SOV for the antiperiodic XXZ chain: Spectrum and eigenstates of the antiperiodic transfer matrix

If the inhomogeneity parameters of the model are such that

$$\Lambda_i \cap \Lambda_j = \emptyset, \text{ if } i \neq j, \quad \text{where} \quad \Lambda_i = \mathrm{Spec}(\hat{b}_i) = \{\xi_i, \xi_i - \eta\}, \ 1 \leq i \leq N$$

the antiperiodic transfer matrix $\bar{t}(\lambda) = \bar{A}(\lambda) + \bar{D}(\lambda)$ has simple spectrum, and a function $\tau(\lambda)$ is an eigenvalue of $\bar{t}(\lambda)$ if and only if

1. it is a $\begin{cases} \text{polynomial (XXX case)} \\ \text{trigonometric polynomial (XXZ case)} \end{cases}$ of degree $N-1$.
2. it satisfies the discrete system of equations

$$\tau(\xi_j)\,\tau(\xi_j - \eta) = -a(\xi_j)\,d(\xi_j - \eta), \qquad \forall j \in \{1, \ldots, \mathrm{N}\}.$$

The $\bar{t}(\lambda)$-eigenstate associated with the eigenvalue $\tau(\lambda)$ is

$$|\,\Psi_\tau\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^{\mathrm{N}}} \prod_{a=1}^{\mathrm{N}} Q_\tau(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}+\boldsymbol{h}\boldsymbol{\eta}}\,|\,\mathbf{h}\,\rangle$$

where $V_{\boldsymbol{\xi}+\boldsymbol{h}\boldsymbol{\eta}} = \prod_{b<a} \varphi(\xi_a + h_a\eta - \xi_b - h_b\eta)$, and where $Q_\tau \in \mathrm{Fun}(\cup_{j=1}^{N}\Lambda_j)$ satisfies

$$\tau(b_n)\,Q_\tau(b_n) = -a(b_n)\,Q_\tau(b_n - \eta) + d(b_n)\,Q_\tau(b_n + \eta),$$

for $b_n \in \{\xi_n, \xi_n - \eta\}, \ 1 \leq n \leq N$.

Form factors of local operators in antiperiodic XXZ chain

- Scalar product of left/right eigenstates:

  ⋆ The transfer matrix eigenstates are particular cases of "separate states":

  $$\langle\,\alpha\,| = \sum_{\mathbf{h}\in\{0,1\}^{\mathsf{N}}} \prod_{a=1}^{\mathsf{N}} \alpha(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}-\boldsymbol{h}\eta}\,\langle\,\mathbf{h}\,|$$

  $$|\,\beta\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^{\mathsf{N}}} \prod_{a=1}^{\mathsf{N}} \beta(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}+\boldsymbol{h}\eta}\,|\,\mathbf{h}\,\rangle$$

  with $\alpha, \beta \in \mathrm{Fun}(\cup_{j=1}^{N}\Lambda_j)$

  ⋆ scalar product for SOV states: $\langle\mathbf{h}\,|\,\mathbf{k}\rangle = \dfrac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\boldsymbol{\xi}-\boldsymbol{h}\eta}}$

  where
  $V_{\boldsymbol{\xi}} = \prod_{k<j} \varphi(\xi_j - \xi_k) = \det_{1\le i,j\le N}[\tilde{\varphi}(\xi_i)^{j-1}]$

$\rightsquigarrow$ determinant representation for the scalar product of left/right separate states (for XXX):

$$\langle\,\alpha\,|\,\beta\,\rangle = \det_{1\le i,j\le N}\left[\sum_{h=0}^{1} \alpha(\xi_i - h\eta)\,\beta(\xi_i - h\eta)\,(\xi_i + h\eta)^{j-1}\right]$$

Form factors of local operators in antiperiodic XXZ chain

- Scalar product of left/right eigenstates:

  Eigenstates written as "separate states"

  $\rightsquigarrow$ determinant representation for the scalar product of left/right separate states (for XXX):

$$\langle \alpha \,|\, \beta \rangle = \det_{1\le i,j\le N}\left[\sum_{h=0}^{1} \alpha(\xi_i - h\eta)\,\beta(\xi_i - h\eta)\,(\xi_i + h\eta)^{j-1}\right]$$

- Solution of the quantum inverse problem:

$$\sigma_m^- = \prod_{k=1}^{m-1} \bar{t}(\xi_k)\cdot \bar{B}(\xi_m)\cdot \prod_{k=1}^{m}\left[\bar{t}(\xi_k)\right]^{-1}$$

  and similar expressions for σ_m^+, σ_m^z...

- SOV action of $\bar{B}(\xi_m)$ on $|\,\mathbf{h}\,\rangle$ $\rightarrow$ form factors reduce to scalar products of separate states

$\rightsquigarrow$ determinant representations for the finite-size form factors of the XXX (or XXZ model)

The XXZ open spin chain by SOV

- Similar construction can be performed for the XXZ open spin chain with (at least) one triangular boundary matrix [Niccoli 13]
- In the XXX case, the most general boundaries can be reduced to this case by means of the $SU(2)$ symmetry [cf. also Frahm et al. 08]
- In the XXZ case, the most general boundaries can be reduced to this case by means of a Vertex-IRF transformation (dynamical gauge transformation) [cf. Baxter 73; Cao et al, 03...]

$$R_{12}(\lambda-\mu)\, S_1(\lambda|\beta)\, S_2(\mu|\beta+\sigma_1^z) = S_2(\mu|\beta)\, S_1(\lambda|\beta+\sigma_2^z)\, R_{12}^{\mathrm{dyn}}(\lambda-\mu|\beta)$$

$$K_-^{\mathrm{dyn}}(\lambda|\beta) = S^{-1}(-\lambda+\eta/2|\beta)\, K_-(\lambda)\, S(\lambda-\eta/2|\beta)$$

with

$$S(\lambda|\beta) = \begin{pmatrix} e^{\lambda-\eta(\beta+\alpha)} & e^{\lambda+\eta(\beta-\alpha)} \\ 1 & 1 \end{pmatrix}$$

$\rightsquigarrow$ new boundary monodromy matrix $\mathcal{U}_-^{\mathrm{dyn}}(\lambda|\beta)$

$$R_{21}^{\mathrm{dyn}}(\lambda-\mu|\beta)\, \mathcal{U}_1^{\mathrm{dyn}}(\lambda|\beta+\sigma_2^z)\, R_{12}^{\mathrm{dyn}}(\lambda+\mu-\eta|\beta)\, \mathcal{U}_2^{\mathrm{dyn}}(\mu|\beta+\sigma_1^z)$$
$$= \mathcal{U}_2^{\mathrm{dyn}}(\mu|\beta+\sigma_1^z)\, R_{21}^{\mathrm{dyn}}(\lambda+\mu-\eta|\beta)\, \mathcal{U}_1^{\mathrm{dyn}}(\lambda|\beta+\sigma_2^z)\, R_{12}^{\mathrm{dyn}}(\lambda-\mu|\beta)$$

$\rightsquigarrow$ spectrum and eigenvectors of
$\mathcal{T}^{\mathrm{dyn}}(\lambda|\beta) = S_{1\dots N}(\{\xi\}|\beta)^{-1}\, \mathcal{T}(\lambda)\, S_{1\dots N}(\{\xi\}|\beta)$

- Similar formulas also hold for the scalar products of separate states

Problems...

All these results (characterization of the transfer matrix spectrum and eigenstates, expressions for the scalar products/form factors...) depend on a non-trivial way on the inhomogeneity parameters of the model

$\rightsquigarrow$ the study of the **homogeneous** ($\to$ physical model) or **thermodynamic** limits is not easy !

$\rightsquigarrow$ **2 main problems to be solved:**

1. reformulate the discrete characterization (in terms of discrete T-Q equations) of the spectrum in a more convenient way, i.e. in terms of continuous T-Q equations

   $\rightsquigarrow$ Bethe equations and Bethe-type representation for the eigenstates

2. transform the determinant representations for the scalar products/form factors into a more convenient form for the consideration of the homogeneous/thermodynamic limit

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From discrete to continuous T-Q equations

SOV characterization of the spectrum/eigenstates of the transfer matrix:

- ⋆ eigenvalue $\tau(\lambda)$ characterized by
  - its functional form (polynomial of a given degree...)
  - the fact that it satisfies a discrete system of equations at the (shifted) inhomogeneity parameters:

    there exists $Q_\tau \in \mathrm{Fun}(\cup_{j=1}^{N}\Lambda_j)$ $(\Lambda_j = \{\xi_j, \xi_j - \eta\})$ s.t.

    $\tau(b)\, Q_\tau(b) = -a(b)\, Q_\tau(b-\eta) + d(b)\, Q_\tau(b+\eta)$ on $\cup_{j=1}^{N} \Lambda_j$
- ⋆ The corresponding eigenvector $|\,\Psi_\tau\,\rangle$ is constructed in terms of $\mathbf{Q}_\tau$

Question: Can we identify a class of function Σ_Q on $\mathbb{C}$ such that, for each eigenvalue $\tau(\lambda)$, there exists a unique $Q(\lambda) \in \Sigma_Q$

- ⋆ which interpolates the $2N$ discrete values $Q_\tau(\xi_n)$ and $Q_\tau(\xi_n - \eta)$
- ⋆ s.t. $\tau(\lambda)\, Q(\lambda) = -a(\lambda)\, Q(\lambda-\eta) + d(\lambda)\, Q(\lambda+\eta)$?

$\rightsquigarrow$ complete description of the spectrum in terms of Bethe equations, and rewriting of the eigenstates as Bethe-type states

From discrete to continuous T-Q equations: the antiperiodic XXX case

Theorem

The following propositions are equivalent:

1. $\tau(\lambda)$ *is an eigenvalue of the antiperiodic transfer matrix*
2. $\tau(\lambda)$ *is an entire function of* λ *such that there exists a unique polynomial* $Q(\lambda)$ *of the form*

$$Q(\lambda) = \prod_{j=1}^{R} (\lambda - \lambda_j), \qquad R \leq N, \qquad \lambda_1, \dots \lambda_R \in \mathbb{C} \setminus \{\xi_1, \dots, \xi_N\},$$

such that $\tau(\lambda)$ *and* $Q(\lambda)$ *satisfy the functional T-Q equation*

$$\tau(\lambda)\, Q(\lambda) = -a(\lambda)\, Q(\lambda - \eta) + d(\lambda)\, Q(\lambda + \eta).$$

$\rightsquigarrow$ the complete characterization of the antiperiodic transfer matrix spectrum (and eigenstates) is given by the solutions of the Bethe equations for $R \leq N$:

$$a(\lambda_j) \prod_{k=1}^{R} (\lambda_j - \lambda_k - \eta) = d(\lambda_j) \prod_{k=1}^{R} (\lambda_j - \lambda_k + \eta), \quad 1 \leq j \leq R$$

with corresponding eigenstates which can be written as Bethe-type vectors:

$$\bar{B}(\lambda_1) \dots \bar{B}(\lambda_R) \,|\, \Omega \,\rangle \quad \text{with} \quad |\, \Omega \,\rangle = \otimes_{n=1}^{N} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

From discrete to continuous T-Q equations: the antiperiodic XXZ case

Construction of the Q-operator using the "pair-propagation through a vertex" property of the six-vertex model [Batchelor et al. 95]

$\rightsquigarrow$ eigenvalues of the form $Q(\lambda) = \prod_{j=1}^{N} \sinh\left(\frac{\lambda - \lambda_j}{2}\right), \quad \lambda_1, \dots, \lambda_N \in \mathbb{C}$ (1)

Starting from the SOV complete characterization of the spectrum in terms of discrete equations we have indeed proven that

Theorem [Niccoli, VT 2015]

The following propositions are equivalent:

1. $\tau(\lambda)$ is an eigenvalue of the antiperiodic transfer matrix
2. $\tau(\lambda)$ is an entire function of λ such that $\tau(\lambda + i\pi) = (-1)^{\mathsf{N}-1}\,\tau(\lambda)$, and there exists a unique function $Q(\lambda)$ of the form (1) such that
$$\tau(\lambda)\,Q(\lambda) = -a(\lambda)\,Q(\lambda - \eta) + d(\lambda)\,Q(\lambda + \eta).$$
This function $Q(\lambda)$ is s.t. $\big(Q(\xi_j), Q(\xi_j + i\pi)\big) \neq (0,0)\ \forall j \in \{1, \dots, \mathsf{N}\}$.

From discrete to continuous T-Q equations: the antiperiodic XXZ case

$$Q(\lambda) = \prod_{j=1}^{N} \sinh\left(\frac{\lambda - \lambda_j}{2}\right), \quad \lambda_1, \dots, \lambda_N \in \mathbb{C} \tag{1}$$

Theorem [Niccoli, VT 2015]

The following propositions are equivalent:

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This function $Q(\lambda)$ is s.t. $\big(Q(\xi_j), Q(\xi_j + i\pi)\big) \neq (0, 0)\ \forall j \in \{1, \dots, \mathrm{N}\}$.

$\rightsquigarrow$ the complete characterization of the antiperiodic transfer matrix spectrum (and eigenstates) is given by the solutions of the Bethe equations:

$$a(\lambda_j) \prod_{k=1}^{N} \sinh\left(\frac{\lambda_j - \lambda_k - \eta}{2}\right) = d(\lambda_j) \prod_{k=1}^{N} \sinh\left(\frac{\lambda_j - \lambda_k + \eta}{2}\right), \quad 1 \le j \le N$$

with corresponding eigenstates which can still be written as Bethe-type vectors:
$\hat{B}(\lambda_1) \dots \hat{B}(\lambda_N)\,|\,\Omega\,\rangle$ but here $\hat{B} \neq \bar{B}$!

From discrete to continuous T-Q equations: the XXX/XXZ open case

- If Nepomechie's constraint on the boundary parameters is satisfied, it is possible to reformulate the SOV discrete characterization of the spectrum in terms of polynomial (in λ^2 for XXX and in $\sinh^2\lambda$ for XXZ) Q-solutions of a functional T-Q equation of the form

$$\tau(\lambda)\, Q(\lambda) = \mathbf{A}(\lambda)\, Q(\lambda-\eta) + \mathbf{A}(-\lambda)\, Q(\lambda+\eta).$$

  where $\mathbf{A}(\lambda)$ depends on the boundary parameters

  $\rightsquigarrow$ the SOV construction also provides the corresponding Bethe states

- If Nepomechie's constraint is not satisfied, such a reformulation is presently not known.

  $\rightsquigarrow$ It was instead proposed in different contexts [Cao et al. 2013; Kitanine, Maillet, Niccoli 2013; Belliard, Crampé 2013] to consider instead polynomials solutions of a T-Q with an inhomogeneous term:

$$\tau(\lambda)\, Q(\lambda) = \mathbf{A}(\lambda)\, Q(\lambda-\eta) + \mathbf{A}(-\lambda)\, Q(\lambda+\eta) + F(\lambda),$$

  with $F(\xi_n) = F(\xi_n+\eta) = 0,\ n = 1, \ldots, N$.

Remark. It is possible to rewrite the separate states in a Bethe-type form, i.e. as multiple action of commuting operators $\bar{\mathcal{B}}(\lambda)$ on a reference state $|\,\Omega\,\rangle$

Determinant representations for the scalar products and form factors: antiperiodic XXX case [Kitanine, Maillet, Niccoli, VT 15]

For two separate states

$$\langle\,\alpha\,| = \sum_{\mathbf{h}\in\{0,1\}^{\mathsf{N}}} \prod_{a=1}^{\mathsf{N}} \alpha(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}-\boldsymbol{h}\eta}\,\langle\,\mathbf{h}\,|, \qquad |\,\beta\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^{\mathsf{N}}} \prod_{a=1}^{\mathsf{N}} \beta(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}+\boldsymbol{h}\eta}\,|\,\mathbf{h}\,\rangle$$

with $\alpha(\lambda) = \prod_{j=1}^{p}(\lambda - \alpha_j)$, $\beta(\lambda) = \prod_{j=1}^{q}(\lambda - \beta_j)$ and $V_{\boldsymbol{\xi}} = \det_{1\le i,j\le N}[\xi_i^{j-1}]$

$$\langle\,\alpha\,|\,\beta\,\rangle = \det_{1\le i,j\le N}\left[\sum_{h=0}^{1} \alpha(\xi_i - h\eta)\,\beta(\xi_i - h\eta)\,(\xi_i + h\eta)^{j-1}\right]$$

- This determinant can be transformed, through some algebraic identities, to a similar determinant in which the role of the set of variables $\{\xi_j\}$ and $\{\alpha_j\}\cup\{\beta_j\}$ are exchanged.
- In its turn, this new determinant can be transformed into a generalized version of Slavnov's determinant (which reduces to the usual Slavnov determinant when $p = q$ and when one of the state is an eigenstate)

$\rightsquigarrow$ One can express the form factors of local operators in a form similar to ABA

Remark: Due to the $SU(2)$ symmetry of the XXX spin chain, it is possible to relate the form factors of the antiperiodic chain with the form factors of the σ^z-twisted chain, which can be computed by ABA $\rightsquigarrow$ check of the result

Determinant representations for the scalar products: open XXX chain with non-diagonal boundaries

In the **case with a constraint** (solvable by Bethe ansatz):

- the SOV construction provides the completeness of the Bethe eigenstates: the later are characterized in terms of polynomial solutions
$$Q(\lambda) = \prod_{j=1}^{q} (\lambda^2 - \lambda_j^2), \qquad q \leq N,$$
of the functional T-Q equation
- general separate states (associated with arbitrary polynomials) correspond to (off-shell) Bethe states
- the scalar products of two arbitrary separate states, associated with polynomials $\alpha(\lambda) = \prod_{j=1}^{n_\alpha}(\lambda^2 - \alpha_j^2)$ and $\beta(\lambda) = \prod_{j=1}^{n_\beta}(\lambda^2 - \beta_j^2)$ (with $n_\beta \geq n_\alpha$) can be reformulated in terms of a generalized Slavnov determinant of size n_β
- the determinant simplifies if one of the states is an eigenstate

Remark: A representation in terms of a generalized Slavnov determinant can also be obtained in the case without constraint (most general boundaries) for the scalar products of separate states associated with polynomials (at the price of using the T-Q equation with extra inhomogeneous term)

What about the XXZ cases ?

⋆ **In the antiperiodic XXZ case:** separates states should be associated with functions of the form

$$\alpha(\lambda) = \prod_{j=1}^{p} \sinh\left(\frac{\lambda - \alpha_j}{2}\right), \quad \beta(\lambda) = \prod_{j=1}^{q} \sinh\left(\frac{\lambda - \beta_j}{2}\right)$$

whereas Sklyanin measure is $V_{\xi} = \prod_{k<j} \sinh(\xi_j - \xi_k)$

$\rightsquigarrow$ the naive generalization of the algebraic identities used in the XXX case does not enable us to transform the determinant for $\langle \alpha | \beta \rangle$

⋆ **In the XXZ open case:** separates states should be associated with polynomials of the form

$$\alpha(\lambda) = \prod_{j=1}^{p} [\cosh(2\lambda) - \cosh(2\alpha_j)], \quad \beta(\lambda) = \prod_{j=1}^{q} [\cosh(2\lambda) - \cosh(2\beta_j)]$$

and Sklyanin measures is $V_{\xi} = \prod_{k<j} [\cosh(2\xi_j) - \cosh(2\xi_k)]$

$\rightsquigarrow$ the naive generalization of the algebraic identities used in the XXX case enables us to transform the determinant for $\langle \alpha | \beta \rangle$ into a generalized Slavnov's one only at the price of an additional constraint between the boundary parameters

There is nevertheless the possibility to compute scalar products between two slightly different types of separate states, constructed from two slightly different versions of T-Q equations ($\rightarrow$ 2 different rewriting of the same eigenstate)...

Conclusion

- SOV provides by construction a complete description of the spectrum and eigenstates, as well as determinant representations for the scalar products of separate states and form factors of local operators

- however, it needs a reformulation for the consideration of the homogeneous / thermodynamic limit:
  - ⋆ from the characterization of the spectrum / eigenstates in terms of discrete equations involving the inhomogeneity parameters of the model to a description in terms of solutions of a continuous version of these equations (functional T-Q equation $\rightsquigarrow$ Bethe equations)
  - ⋆ from determinant representations for the form factors involving the inhomogeneity parameters to some more convenient representations in terms of the Bethe roots

- Interesting open problems:
  - ⋆ solution of the functional T-Q equation for the general open chain (case without constraint) ?
  - ⋆ how to transform the scalar product determinant when $Q(\lambda)$ is not a polynomial (cf. antiperiodic XXZ, but also antiperiodic XYZ. . .) ?