Probabilistic models for nonlinear parabolic systems with cross-diffusion

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Aim : Probabilistic approaches to investigation of nonlinear parabolic systems arising in physics, biology, financial mathematics,

$$u_t - div(F(u)\nabla u) = f(u), \quad F(u) = (F_{ij}^{lm}),$$

$$u(0,x) = u_0(x) \in R^{d_1}, x \in R^d$$

and / or

$$u_s + div(M(u)\nabla u) + f(u) = 0, \quad M(u) = (M_{ij}),$$

$$u(T,x) = u_0(x), \quad 0 \le s \le t \le T.$$

$$i,j = 1,\dots,d, \quad l,m = 1,\dots,d_1.$$

Classification of nonlinear systems

1) Systems with diagonal principal part

PDE-approach – Ladyzenskaya, Solonnikov, Uraltzeva (1967) - quasilinear case.

Probabilistic approach – Linear systems Stroock, Dalecky 1965-1970, nonlinear – Dalecky, Bel (1980 -1990)- classical solutions Bel.,Woyczynski (2012) – weak and viscosity solutions.

$(\mathcal{L}^u u)^m = \frac{1}{2} Tr A^u \nabla^2 u^m (A^u)^* + \langle a^u, \nabla \rangle u^m,$

$$\mathbf{u^m_s} + \mathcal{L}^\mathbf{u}\mathbf{u^m} + \sum_{\mathbf{l=1}}^{\mathbf{d_1}} \sum_{\mathbf{i=1}}^{\mathbf{d}} \mathbf{B^u_{iml}} \nabla_\mathbf{i} \mathbf{u^l} + \sum_{\mathbf{l=1}}^{\mathbf{d_1}} \mathbf{c^u_{ml}} \mathbf{u^l} = 0 \quad (1)$$

$u^m(T,x) = u_0^m(x), \quad A^u(x) = A(x, u(x)).$

Stochastic problem has the form

$$d\xi(\theta) = a(\xi(\theta), u(\theta, \xi(\theta)))d\theta + A(\xi(\theta), u(\theta, \xi(\theta)))dw(\theta), \quad (2)$$

$$\xi(s) = x \in R^d, \quad s \le \theta \le T,$$

$$d\eta(\theta) = c(\xi(\theta), u(\theta, \xi(\theta)))\eta(\theta)d\theta$$

$$+C(\xi(\theta), u(\theta, \xi(\theta)))(\eta(\theta), dw(\theta)), \quad (3)$$

$$\eta(s) = h \in R^{d_1},$$

$$\langle h, u(s,x)\rangle = E[\langle \eta(T), u_0(\xi_{s,x}(T))\rangle], \quad (4)$$

where $B_i^{ml} = C_j^{lm} A_{ji}$ and summing over repeated indices is assumed.

Quasilinear case

Consider the Cauchy problem

$$u_s^m + \mathcal{L}(x,u,\nabla u)u^m + B_k^{ml}(x,u,\nabla u)\nabla u^l \qquad (5)$$

$$+c^{ml}(x,u,u_x)u^l = 0, \quad u(T,x) = u_0(x),$$

where

$$\mathcal{L}(x,u,\nabla u)u = \langle a(x,u,\nabla u)\nabla u$$

$$+\frac{1}{2} TrA(x,u,\nabla u)\nabla^2 uA(x,u,\nabla u)$$

Set $v(s,x) = \nabla u(s,x)$, derive the equation for $V = (u, \nabla u)$ called a differential prolongation of the original equation and note that the systems w.r.t. V is similar to (1).

2) Reaction-diffusion systems. Consider the Cauchy problem

$$u_s^m + \mathcal{L}_m(x,u)u^m + \sum_l c_{ml}(x,u)u^l = 0, \qquad (6)$$

$$u_m(T,x) = u_{0m}(x),$$

$$\mathcal{L}_m(x,u)v = \langle a_m(x,u), \nabla v\rangle + \frac{1}{2} Tr A_m(x,u)\nabla^2 v A_m^*(x,u),$$

Set $u_m(s,x) = u(s,x,m)$ and consider a Markov chain $\gamma(\theta) \in V = \{1,2,\ldots,d_1\}$ such that

$$P(\gamma(t+\Delta t) = m|\gamma(t) = l, (\xi(\theta),\gamma(\theta)), \theta \le t)$$

$$= c^{ml}(\xi(t), u(t,\xi(t),\gamma(t)))\Delta t + o(\Delta t). \qquad (7)$$

A stochastic counterpart for (6)

$$d\xi(\theta) = a^u(\xi(\theta),\gamma(\theta))d\theta + A^u(\xi(\theta),\gamma(\theta))dw(\theta), \xi(s) = x \quad (8)$$

$$d\gamma(\theta) = \int_V g^u(\xi(\theta),\gamma(\theta),z)p(d\theta,dz), \xi(s) = x, \quad (9)$$

$$u(s,x,m) = E[u_0(\xi_{s,x}(T),\gamma_{s,m}(T))], \quad (10)$$

$$g^v(x,l,z) = \sum_{m=1}^{d_1}(m-l)I_{\{z\in\Delta_{lm}(x,v)\}},$$

$$|\Delta_{lm}(x,v)| = c_{lm}(x,v).$$

3. Consider the Cauchy problem for the MHD-Burgers system

$$\frac{\partial v}{\partial t}+\langle v,\nabla\rangle v=\frac{1}{2}\nu^2\Delta v+(\nabla\times B)\times B,\qquad v(0,x)=v_0(x) \tag{11}$$

$$\frac{\partial B}{\partial t}=\frac{\sigma^2}{2}\Delta B+\nabla\times(v\times B),\qquad B(0,x)=B_0(x), \tag{12}$$

where $\times$ - vector product, $v\in R^3$ – fluid velocity, $B\in R^3$ – magnetic field, μ and σ – fluid viscosity and conductivity.

When $x, v, B \in R^1$

$$\frac{\partial u_1}{\partial t} + \frac{\partial(u_1 u_2)}{\partial x} = \frac{\sigma^2}{2}\frac{\partial^2 u_1}{\partial x^2}, \qquad u_1(0,x) = u_{10}(x) \tag{13}$$

$$\frac{\partial u_2}{\partial t} + \frac{1}{2}\frac{\partial(u_1^2 + u_2^2)}{\partial x} = \frac{\nu^2}{2}\frac{\partial^2 u_2}{\partial x^2}, \qquad u_2(0,x) = u_{20}(x). \tag{14}$$

Note that in Elsesser variables $e^{\pm} = u_1 \pm u_2$ the system (13),(14) is reduced to

$$\frac{\partial e^{\pm}}{\partial t} + \frac{\partial(e^{\pm})^2}{\partial x} = \frac{\mu^2 + \sigma^2}{4}\frac{\partial^2 e^{\pm}}{\partial x^2} + \frac{\mu^2 - \sigma^2}{4}\frac{\partial^2 e^{\mp}}{\partial x^2}$$

that is a **cross-diffusion system**.

4. Cross-diffusion systems a) Chemotaxis models -

$$u_t = \mathrm{div}(a(u,v)\nabla u + b(u,v)\nabla v) + g(u,v), \quad (15)$$

$$v_t = \mathrm{div}(\alpha(u,v)\nabla v + \beta(u,v)\nabla u) + \gamma(u,v), \quad (16)$$

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x).$$

b) Modeling spatial segregation phenomena of competing species in population dynamics, Shigesada, Kawasaki and Teramoto proposed [1] in 1979 to study some nonlinear parabolic systems which include the following problem

$$\begin{cases} u_t = \Delta[(\alpha_1 + \alpha_{11} u + \alpha_{12} v) u] + u(a_1 - b_1 u - c_1 v), \\ v_t = \Delta[(\alpha_2 + \alpha_{21} u + \alpha_{22} v) v] + v(a_2 - b_2 u - c_2 v), \\ u(0,x) = u_0(x), \quad v(0,x) = u_v(x). \end{cases} \tag{17}$$

a_q, b_q, c_q – positive constants, α_{ql} – nonnegative constants. Lotka-Volterra system with cross diffusion.

The aim of this talk is to construct a probabilistic representation of a generalized solution of

$$\mathbf{u^q_t = \Delta[(u^1 + u^2)u^q] + u^q(a_1 - b_1 u^1 - c_1 u^2)}, \tag{18}$$

$$u^q(0,x) = u_{m0}(x).$$

A pair (u^1, u^2) is called a generalized solution of (18) if 1)$u^q \in L^\infty_{loc}([0,\infty) \times L^\infty(R^d))\ \ q = 1,2$;
2) $\nabla(u^1 + u^2) \in L^2_{loc}((0,\infty) \times R^d)$;
3) For any test function $h \in C_0^\infty(R^d)$

$$\frac{\partial}{\partial t}\int_{R^d} u^q(t,x)h(x)dx + \frac{1}{2}\int_{R^d} \nabla[u^q(t,x)M_q^2(u)]\cdot\nabla h(x)dx \tag{19}$$

$$= \int_{R^d} u^q(t,x)m_u^q(x)h(x)dx, \qquad q = 1,2,$$

where

$$\frac{1}{2}M^2(u) = u^1(\theta) + u^2(\theta),$$

$$m_u^q(x) = a_q - b_q u^1(x) - c_q u^2(x), \qquad u = (u^1, u^2).$$

3') For any test function $h \in C_0^\infty([0,\infty) \times R^d)$

$$\int_{R^d} u^q(t,x)h(t,x)dx - \int_{R^d} u^q(0,x)h(0,x)dx =$$

$$\int_0^t \int_{R^d} u^q(\theta,x)\left\{h_\theta(\theta,x) + \frac{1}{2}M_q^2(u)\Delta h(\theta,x)\right\} dxd\theta +$$

$$\int_0^t \int_{R^d} u^q(\theta,x)m_u^q(\theta,x)h(\theta,x)dxd\theta, \qquad q = 1,2. \tag{20}$$

Main results :

Set $\mathcal{W} = \{v = (v^1, v^2) : v^q \in L^\infty((0,\infty); L^\infty(R^d)), \nabla v^q \in L^2_{loc}((0,\infty) \times R^d), v^q > 0\}$.

Let $u^q \in \mathcal{W}$ solve (18). Consider SDE

$$\mathbf{d}\xi(\theta) = \mathbf{M_u}(\xi(\theta))\mathbf{dw}(\theta), \quad \xi(\mathbf{s}) = \kappa,$$

$$\hat{\xi}(t) = \xi(T-t), \quad \xi_{0,\kappa}(t) = \varphi_{0,t}(\kappa),$$

$$\psi_{0,t} \circ \varphi_{0,t}(\kappa) = \kappa$$

Theorem

A weak solution $u(t,x) \in \mathcal{W}$ of the Cauchy problem (18) admits a probabilistic representation

$$u^q(t,x) = E\left[\hat{\eta}^q(t) u_0^q(\hat{\xi}_{0,x}(t))\right], \qquad (21)$$

where $\xi^q(\theta), \eta^q(\theta)$ satisfy

$$\begin{cases} d\xi_{0,\kappa}(\theta) = M(u(\theta, \xi_{0,\kappa}(\theta)))dw(\theta), \; \xi_{0,\kappa}(0) = \kappa, \\ d\eta^q(\theta) = \tilde{m}_q^u(\xi_{0,\kappa}^q(\theta))\eta^q(\theta)d\theta + C_q^u(\xi_{0,\kappa}^q(\theta))\eta^q(\theta)dw(\theta) \end{cases} \qquad (22)$$

$\eta^q(0) = 1.$

$$d\hat{\xi}(\theta) = [M_u \nabla M_u](\hat{\xi}(\theta))d\theta + M_u(\hat{\xi}(\theta))d\tilde{w}(\theta), \ \hat{\xi}(0) = x, \quad (23)$$

with $M_u(x) = \sqrt{2[u^1(t,x) + u^2(t,x)]}$ and $\tilde{w}(\theta) = w(T-\theta) - w(T)$ for a fixed $T > \theta$.

Denote by $\mathbf{J}_t \equiv \mathbf{J}_{0,t} = \nabla\varphi_{0,y}(t)$ the Jacobian matrix of the map $\varphi_{0,t} : R^n \to R^d$, and set $J_{0,t}(\omega) = \det \mathbf{J}_{0,t}(\omega)$.

We call $r^q(\theta) = \eta^q(\theta)h(\xi(\theta))J(\theta)$ a **stochastic test function**.

Assume that (u^1, u^2) is a weak solution of (18) and u^q are strictly positive bounded $\mathcal{W}^1$ functions. Then the SDE

$$d\xi(\theta) = M_u(\xi(\theta))dw(\theta), \quad \xi(s) = \kappa,$$

has a unique solution which is C^1-smooth in κ. Let $\xi^q(\theta) = \varphi_{s,t}(\kappa)$, $\partial_\kappa \varphi_{s,t}(\kappa) = J_{s,t}$ be its Jacobian. Set $\hat{\xi}^q(\theta) = \psi_{t,s}(x)$.

Given the stochastic test function

$$r^q(\theta) = \eta^q(\theta) h(\xi_{0,\kappa}(\theta)) J^q(\theta)$$

we show that if u^1, u^2 are generalized solutions to (18), then

$$E\langle\langle u^q(t), r^q(t)\rangle\rangle = \langle\langle u^q(0), h\rangle\rangle$$
$$+E\int_0^t \langle\langle u^q(\theta), [M_u^2(\xi(\theta))\Delta h(\xi(\theta))$$
$$+m_u^q(\theta) h(\xi(\theta))] J(\theta)\eta^q(\theta)\rangle\rangle d\theta.$$

holds.

First we compute $dr^q(t)$

$$dr^q(t) = d\eta^q(\theta)h(\xi_{s,\kappa}(\theta))J(\theta) + \eta^q(\theta)dh(\xi^q_{s,\kappa}(\theta))J^q(\theta)$$

$$+\eta^q(\theta)h(\xi_{s,\kappa}(\theta))dJ(\theta) + d\eta^q(\theta)dh(\xi_{s,\kappa}(\theta))J(\theta)$$

$$+d\eta^q(\theta)h(\xi_{s,\kappa}(\theta))dJ(\theta) + \eta^q(\theta)dh(\xi_{s,\kappa}(\theta))dJ(\theta)$$

Taking into account expressions for $d\eta^q(\theta)$, $dh(\xi^q_{s,\kappa}(\theta))$ and $dJ^q(\theta)$ we can deduce

$$dr^q(\theta) = [[\mathcal{L}_0(u)]^* + \tilde{c}^q_u(\xi_{s,\kappa}(\theta))]\, h(\xi_{s,\kappa}(\theta))d\theta \quad (24)$$

$$+[C^q_u M_q(u)](\xi_{s,\kappa}(\theta))\nabla h(\xi_{s,\kappa}(\theta))J(\theta)\eta^q(\theta)d\theta$$

$$-C^q_u \cdot \nabla M_u](\xi_{s,\kappa}(\theta))h(\xi_{s,\kappa}(\theta))\eta^q(\theta)J(\theta)d\theta$$

$$+[\nabla M_q(u)(\xi_{s,\kappa}(\theta))]M_u(\xi_{s,\kappa}(\theta))\nabla h(\xi_{s,\kappa}(\theta))\eta^q(\theta)J(\theta)d\theta$$

$$+J(\theta)\eta^q(\theta)K^q_u(\theta)dw(\theta).$$

Now it remains to specify coefficients of the SDE for the process for $\eta^q(t)$ satisfying (22) which ensure the required integral identities. To this end

Lemma

Assume that $\tilde{m}$ and C^q in (22) have the form

$$\tilde{m}_u^q = m_u^q(\xi(\theta)) - \langle \nabla M_u(\xi(\theta)), \nabla M_u(\xi(\theta))\rangle,$$

$$C_u^q(\xi(\theta)) = -\nabla M_u(\xi(\theta)) = -\frac{\nabla u^1 + \nabla u^2}{\sqrt{2[u^1+u^2]}}. \quad (27)$$

Then we get

$$dr_0^q(\theta, y) = [\mathcal{L}_u^q]^* h(\xi_{0,y}(\theta))\eta^q(\theta) J^q(\theta) d\theta + d\, mart \quad (28)$$

As soon as we get (28) we have to use the change of variables formula under the integral sign, namely,

$$\int u(\psi_{0,t}(x))h(x)dx = \int u(y)h(\varphi_{0,t})J_{0,t}(y)dy$$

Since

$$\psi_{0,t}(x) = y, x = \varphi_{0,t}(y)$$

and hence

$$u(t) \circ \psi_{0,t} = \hat{\eta}(t)u(0) \circ \psi_{0,t},$$

where

$$\hat{\eta}(t) = exp\{\int_0^t \bar{m}^q(\psi_{\theta,t}(x)d\theta + \int_0^t C^q(\psi_{\theta,t}(x)dw(\theta)\}$$

To obtain a closed stochastic system we onsider a stochastic equation of the form

$$d\eta^q(\theta) = [\bar{m}^q]^*(\xi(\theta))\eta^q(\theta)d\theta + [\bar{C}^q]^*(\xi(\theta))\eta^q(\theta)dw(\theta), \quad (29)$$

$$\eta^q(s) = \gamma^q$$

with respect to the vector $\eta^q(\theta) = \begin{pmatrix} \zeta_1^q(\theta) \\ \eta_2^q(\theta) \end{pmatrix}$ with coefficients $\bar{m}^q$ and $\bar{C}^q$ to be chosen below. Let $\zeta^q(\theta)$ maps γ^q to $\eta^q(\theta)$, that is $\zeta^q(\theta) = \begin{pmatrix} \zeta_{11}(\theta) & \zeta_{12}(\theta) \\ \zeta_{21}(\theta) & \zeta_{22}(\theta) \end{pmatrix}$

$$\bar{C}^i_{11} = -\nabla_i M_u I, \quad \bar{C}^i_{12} = 0, \quad \bar{c}^q_{11} = c^q_u + \|\nabla M_u\|^2 I. \tag{30}$$

Next we choose

$$\bar{C}^i_{21} = -\nabla_i M_u I, C^i_{22} = \nabla_i M_u \delta^j_k - \nabla_i M_u I, \tag{31}$$

$$\bar{m}^{iq}_{21} = \nabla_i m^q_u + [\nabla_i M_u]^2 I,$$

$$\bar{m}_{22} = [m_u]_{jk} - \nabla_j M_u \nabla_k M_u + \|\nabla M_u\|^2 I. \tag{32}$$

Theorem

Under assumptions of theorem 1 both the functions $u^q(t,x)$ admit stochastic representations (21) and functions $V_i^q = (u^q, \nabla_i u^q)$ admit stochastic representations

$$\begin{pmatrix} u^q(t,x) \\ \nabla_i u^q(t,x) \end{pmatrix} = E\left[\begin{pmatrix} \hat{\zeta}^q_{11}(\theta) & 0 \\ \hat{\zeta}^q_{21}(\theta) & \hat{\zeta}^q_{22}(\theta) \end{pmatrix} \begin{pmatrix} u^q_0(\hat{\xi}_{0,x}(\theta)) \\ v^q_{i0}(\hat{\xi}_{0,x}(\theta)) \end{pmatrix} \right].$$

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