L_2-small ball behavior for dependent sequences and for weighted stationary processes

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1. El. Comm. in Probab. **21** (2016), paper N41
(with S.Y. Hong and M.A. Lifshits)
2. Preprint https://arxiv.org/abs/1705.00422
(with M.A. Lifshits)

We explore logarithmic L_2-small deviation probabilities

$$\log\left[\mathbb{P}\Big(\int_T |Y(t)|^2\mathfrak{m}(dt) \le \varepsilon^2\Big)\right], \qquad \text{as } \varepsilon \to 0$$

where $(Y(t))_{t\in T}$ is a stationary Gaussian process, $\mathfrak{m}$ is a measure on T. Our goal is to relate the asymptotics of small deviation probabilities with that of the spectrum.

Our results are tightly related with those on fractional Brownian motion and its relatives.

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In terms of such processes with stationary increments our message is that the spectral asymptotics is relevant to the small deviation behavior but the self-similarity is not.

We find logarithmic asymptotics of L_2-small deviation probabilities for weighted stationary Gaussian processes having power-type spectrum:

- periodic processes;

- processes with continuous spectrum;

- stationary sequences.

Our results are based on the spectral theory of pseudo-differential operators with anisotropic-homogeneous symbols.

[BS] M.Š. Birman, M.Z. Solomjak, Vestnik LGU (1977), no. 13, 13–21; Vestnik LGU (1979), no. 3, 5–10 (Russian).

[BKS] M.Sh. Birman, G.E. Karadzhov, M.Z. Solomyak, In: Adv. Soviet Math., **7**, AMS, Providence, R.I., 1991, 85–106.

The spectral results that we use are not sensible to the symmetry of the spectral measure. Therefore, it is very natural to apply them to the complex-valued processes. In this context *proper* Gaussian processes are particularly convenient because their distributions are determined by the spectra of the corresponding covariance operators.

1. Periodic stationary processes

Let $X = \{X(t), t \in \mathbb{R}\}$ be a complex-valued 2π-periodical centered second order mean-square continuous stationary process. Then its covariance function admits a spectral representation

$$K_X(s) := \mathrm{cov}(X(\cdot), X(\cdot + s)) = \sum_{k\in\mathbb{Z}} \mu_k\, e^{\mathbf{i}ks}, \ s \in \mathbb{R},$$

where $\mu := (\mu_k)_{k\in\mathbb{Z}}$ is a finite measure on $\mathbb{Z}$ called the spectral measure of X.

We are interested in the small ball behavior of the *weighted* L_2-norm

$$\int_0^{2\pi} q(t)|X(t)|^2 dt = ||\sqrt{q}X||_2^2$$

with some weight $q \in L^1[0, 2\pi]$. The covariance operator admits decomposition

$$\mathcal{K}_{\sqrt{q}X} = \mathcal{Q}\mathcal{D}\mathcal{D}\mathcal{Q} =: \mathcal{T}^*\mathcal{T}, \qquad \mathcal{T} = \mathcal{D}\,\mathcal{Q},$$

where $\mathcal{Q}$ stands for the operator of multiplication by $\sqrt{q} \in L^2[0, 2\pi]$, $\mathcal{D}$ can be interpreted as a convolution operator with the kernel

$$D(s) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \sqrt{\mu_k}\, e^{\mathbf{i}ks}.$$

For the study of logarithmic asymptotics of small deviation probabilities, we need to know the one-term asymptotic behavior of eigenvalues of $\mathcal{K}_{\sqrt{q}X}$ that coincide with its singular values $s_n(\mathcal{K}_{\sqrt{q}X})$.

From now on we assume that the spectral measure has a power-like decay

$$\mu_k \sim M_\pm |k|^{-r}, \qquad \text{as } k \to \pm\infty,$$

with some $r > 1$ and $M_\pm \geq 0$, $M_+ + M_- > 0$. This assumption is typical of the literature on small deviations of Gaussian processes.

We study the asymptotics at zero of the *distribution function of singular values*

$$N_{\mathcal{K}_{\sqrt{q}X}}(\lambda) := \#\{n : s_n(\mathcal{K}_{\sqrt{q}X}) \geq \lambda\}.$$

This is indeed an equivalent setting because

$$N_{\mathcal{K}_{\sqrt{q}X}}(\lambda) \sim \Delta \cdot \lambda^{-\frac{1}{r}},\ \lambda \to 0_+ \ \Leftrightarrow\ s_n(\mathcal{K}_{\sqrt{q}X}) \sim \Delta^r \cdot n^{-r},\ n \to \infty.$$

We can consider $\mathcal{K}_{\sqrt{q}X}$ as an operator in $L_2(\mathbb{R})$

$$(\mathcal{K}_{\sqrt{q}X} u)(s) = b(s) \int_{\mathbb{R}} K_X(s-t) b(t) u(t)\, dt,$$

where $b = \sqrt{q} \cdot \mathbf{1}_{[0,2\pi]}$.

To apply the results of **[BS]** we need to eliminate the singularities of the covariance out of diagonal $s = t$. To do this, we introduce the cut-off function $h(s) = h(-s)$ equals to one on $[-\pi, \pi]$ and vanishes outside of the interval $[-\frac{3\pi}{2}, \frac{3\pi}{2}]$. We decompose the kernel and the operator

$$K_X(s) = K_X(s)\Big[h(s) + (1 - h(s))\Big] =: K_1(s) + K_2(s)$$

$$\mathcal{K}_{\sqrt{q}X} = \mathcal{K}_1 + \mathcal{K}_2$$

and prove that $\lim_{\lambda \to 0_+} \lambda^{\frac{1}{r}} \cdot N_{\mathcal{K}_2}(\lambda) = 0$. On the other hand, the function K_1 satisfies

$$\mathcal{F}K_1(\xi) \sim M_{\pm}|\xi|^{-r} \quad \text{as } \xi \to \pm\infty,$$

where $\mathcal{F}K_1$ denotes the Fourier transform of K_1. According to **[BS]**, this implies

$$\Delta_{\frac{1}{r}} := \lim_{\lambda \to 0_+} \lambda^{\frac{1}{r}} N_{\mathcal{K}_1}(\lambda) = \frac{M_-^{\frac{1}{r}} + M_+^{\frac{1}{r}}}{(2\pi)^{\frac{r-1}{r}}} \int_0^{2\pi} q(t)^{\frac{1}{r}}\, dt.$$

By the classical Weyl Theorem, we obtain

$$N_{\mathcal{K}_{\sqrt{q}X}}(\lambda) \sim N_{\mathcal{K}_1}(\lambda) \sim \Delta_{\frac{1}{r}} \lambda^{-\frac{1}{r}}, \quad \text{as } \lambda \to 0_+.$$

This gives the asymptotics of $s_n(\mathcal{K}_{\sqrt{q}X}) = \lambda_n(\mathcal{K}_{\sqrt{q}X})$.

Let the spectral measure of X be as above, and let $q \in L^1[0, 2\pi]$. Then

$$\lambda_n(\mathcal{K}_{\sqrt{q}X}) \sim \left(\frac{M_-^{\frac{1}{r}} + M_+^{\frac{1}{r}}}{2\pi} \int\limits_0^{2\pi} q(t)^{\frac{1}{r}}\, dt \right)^r \frac{2\pi}{n^r}, \qquad \text{as } n \to \infty.$$

provided that λ_n are numbered in the non-increasing order, according to their multiplicities.

If X is a real-valued Gaussian process, we have the Karhunen–Loève expansion

$$||\sqrt{q}X||_2^2 = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K}_{\sqrt{q}X})\, |\xi_n|^2 \ ,$$

where $(\xi_n)_{n\in\mathbb{N}}$ are i.i.d. standard Gaussian r.v's. Notice that the spectral measure is symmetric, i.e. $M_+ = M_- =: M$. Using the eigenvalues asymptotics and well-known Zolotarev's result we obtain the following Theorem.

Let $\{X(t), t \in \mathbb{R}\}$ be a 2π-periodical real centered mean-square continuous stationary Gaussian process. Assume that its spectral measure satisfies

$$\mu_k \sim M|k|^{-r}, \qquad \text{as } |k| \to \infty,$$

with some $r > 1, M > 0$. Let q be a summable weight. Then we have, as $\varepsilon \to 0$,

$$\log \mathbb{P}\Big(\int_0^{2\pi} q(t)|X(t)|^2 dt \le \varepsilon^2 \Big) \sim -\Big(\frac{M^{\frac{1}{r}}}{r \sin(\pi/r)} \int_0^{2\pi} q(t)^{\frac{1}{r}} dt \Big)^{\frac{r}{r-1}} \frac{(r-1)(2\pi)^{\frac{1}{r-1}}}{2\,\varepsilon^{\frac{2}{r-1}}}.$$

Proper complex processes

If X is a complex-valued centered Gaussian process then we still have the Karhunen–Loève expansion but, unfortunately, unlike the real case, the variables ξ_n need not be independent, although they are uncorrelated. By this reason, we need to restrict the consideration to an important subclass of the variables and processes where uncorrelated variables are independent.

A complex-valued random process $(X(t))_{t\in T}$ is called centered *proper* (or *circularly*) Gaussian if
• The coordinate vector $\left(X^{(re)}(t_1), X^{(im)}(t_1), \ldots, X^{(im)}(t_n)\right)$ is a centered Gaussian vector in $\mathbb{R}^{2n}$ for any $t_1, \ldots, t_n \in T$;
• $\mathbb{E}X(t_1)X(t_2) = 0$ for all $t_1, t_2 \in T$.
We clearly have $\mathbb{E}X(t) = 0, \forall t \in T$. Moreover, $\mathbb{E}X(t)^2 = 0$ yields that the distribution of $X(t)$ in the complex plane is spherically symmetric.

For a proper Gaussian process X, the Karhunen–Loève expansion reads as follows:

$$||\sqrt{q}X||_2^2 = \frac{1}{2}\sum_{n=1}^{\infty} \lambda_n(\mathcal{K}_{\sqrt{q}X})\ \left(\xi_{n,1}^2 + \xi_{n,2}^2\right)\ ,$$

where $(\xi_n)_{n\in\mathbb{N}}$ are i.i.d. standard Gaussian r.v's. Using the eigenvalues asymptotics and Zolotarev's result we obtain the following Theorem.

Let $\{X(t), t\in\mathbb{R}\}$ be a 2π-periodical complex centered mean-square continuous stationary proper Gaussian process. Assume that its spectral measure has power-like decay with some $r > 1$. Let q be a summable weight. Then we have, as $\varepsilon \to 0$,

$$\log \mathbb{P}\Big(\int_0^{2\pi} q(t)|X(t)|^2 dt \le \varepsilon^2\Big)$$
$$\sim -\Big(\frac{M_-^{\frac{1}{r}} + M_+^{\frac{1}{r}}}{2r\sin(\pi/r)}\int_0^{2\pi} q(t)^{\frac{1}{r}} dt\Big)^{\frac{r}{r-1}} \frac{(r-1)(2\pi)^{\frac{1}{r-1}}}{\varepsilon^{\frac{2}{r-1}}}.$$

2. Stationary sequences

Let a real stationary centered Gaussian sequence $(U_k)_{k\in\mathbb{Z}}$ admit a representation

$$U_k = \sum_{m=-\infty}^{\infty} a_m X_{k-m},$$

where $(a_m) \in \ell_2(\mathbb{Z})$, and (X_j) are i.i.d. standard Gaussian (this representation exists iff (U_k) has a spectral density). Let the coefficients $(d_k)_{k\in\mathbb{Z}}$ have the asymptotics

$$d_k \sim d_\pm\, |k|^{-p}, \quad \text{for some } p > \frac{1}{2}, \quad k \to \pm\infty,$$

where at least one of the numbers $d_\pm$ is strictly positive. Then, as $\varepsilon \to 0$,

$$\log \mathbb{P}\Big(\sum_{k\in\mathbb{Z}} d_k^2 U_k^2 \le \varepsilon^2 \Big) \sim -\Big(\frac{d_-^{\frac{1}{p}} + d_+^{\frac{1}{p}}}{4\, p \sin\left(\frac{\pi}{2p}\right)} \int_0^{2\pi} |\mathfrak{a}(t)|^{\frac{1}{p}}\, dt \Big)^{\frac{2p}{2p-1}} \frac{2p-1}{2\, \varepsilon^{\frac{2}{2p-1}}},$$

where $\mathfrak{a}(t) = \sum_{k\in\mathbb{Z}} a_k\, e^{\mathrm{i}\,kt}$.

Sketch of the proof: We have to study the norm of the random vector $Z \in \ell_2(\mathbb{Z})$ defined by its coordinates $Z_k = d_k U_k$, $k \in \mathbb{Z}$. It turns out that the corresponding covariance operator $\mathcal{K}_Z$ is unitary equivalent to

$$\mathcal{K}_Z = \mathcal{D}\mathcal{A}\mathcal{A}^*\mathcal{D},$$

where $\mathcal{A}$ stands for the multiplication by $\mathfrak{a} \in L^2[0, 2\pi]$ while $\mathcal{D}$ is a convolution with the kernel $D(s) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} d_k\, e^{\mathrm{i}ks}$.

We see that the elements of decomposition of $\mathcal{K}_Z$ are the same as in the previous problem but the order of use of operators is different. However, a well-known theorem in operator theory implies the coincidence of non-zero eigenvalues for operators $\mathcal{T}\mathcal{T}^*$ and $\mathcal{T}^*\mathcal{T}$ for any bounded linear operator $\mathcal{T}$. This implies that spectral asymptotics of the same type holds for the operator $\mathcal{K}_Z$ (with the natural replacement $r \to 2p$, $M_\pm \to d_\pm^2$, $\sqrt{q} \to \mathfrak{a}$). $\square$

3. Stationary processes with continuous spectra

Let $X(t), t \in \mathbb{R}$, be a centered second order complex stationary process on $\mathbb{R}$. Then

$$K_X(s) := \mathrm{cov}(X(\cdot), X(\cdot + s)) = \int_{\mathbb{R}} e^{\mathbf{i}us}\, \mu(du), \; s \in \mathbb{R},$$

where μ is the spectral measure of X.

Assume that μ has a density $m \in L_1(\mathbb{R})$. Then the covariance operator admits decomposition

$$\mathcal{K}_{\sqrt{q}X} = \widetilde{\mathcal{T}}^* \widetilde{\mathcal{T}}, \qquad \widetilde{\mathcal{T}} = \mathcal{M}\mathcal{F}\mathcal{Q},$$

where $\mathcal{M}$ and $\mathcal{Q}$ stand for the multiplication by $\sqrt{m} \in L^2(\mathbb{R})$ and $\sqrt{q} \in L^2(\mathbb{R})$, respectively, while $\mathcal{F}$ is the Fourier transform.

Suppose that m has a power-like decay

$$m(u) \sim M_{\pm} |u|^{-r}, \qquad \text{as } u \to \pm\infty,$$

with some $r > 1$ and $M_{\pm} \geq 0$, $M_+ + M_- > 0$.

If q has bounded support then we again can apply the results of **[BS]**. However, in general case we should use subtle estimates of **[BKS]** and make additional assumption on q.

We consider the sequence of operators $\widetilde{\mathcal{T}}_k = \mathcal{M}\mathcal{F}\mathcal{Q}_k$, $k \in \mathbb{N}$, where $\mathcal{Q}_k$ is multiplication by compactly supported weight

$$b_k(t) = \sqrt{q(t)} \cdot \mathbf{1}_{[-k,k]}(t).$$

Operators $\mathcal{K}_k = \widetilde{\mathcal{T}}_k^* \widetilde{\mathcal{T}}_k$ satisfy the assumptions of **[BS]**. This gives

$$\Delta^{(k)}_{\frac{1}{r}} := \lim_{\lambda \to 0_+} \lambda^{\frac{1}{r}} N_{\mathcal{K}_k}(\lambda) = \frac{M_-^{\frac{1}{r}} + M_+^{\frac{1}{r}}}{(2\pi)^{\frac{r-1}{r}}} \int\limits_{-k}^{k} q(t)^{\frac{1}{r}}\, dt.$$

Thus, we need to justify the passage to the limit as $k \to \infty$. Following **[BKS]**, for $f \in L_2(\mathbb{R})$ we define the numerical sequence

$$v(f) = \{v_\ell(f)\}_{\ell \in \mathbb{Z}}; \qquad v_\ell(f) := \|f\|_{2,[\ell,\ell+1]}.$$

Our assumption on m implies

$$\sup_{\ell \in \mathbb{Z}} \Big(|\ell|^{\frac{r}{2}} \cdot v_\ell(\sqrt{m}) \Big) < \infty$$

(in the notation of **[BKS]**, $v(\sqrt{m}) \in l_{2/r,w}$). Since $\frac{2}{r} < 2$, Subsection 5.7 in **[BKS]** shows that

$$\sup_n \Big(n^{\frac{r}{2}} \cdot s_n(\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}_k^*) \Big)$$
$$\le C \cdot \sup_{\ell \in \mathbb{Z}} \Big(|\ell|^{\frac{r}{2}} \cdot v_\ell(\sqrt{m}) \Big) \cdot \|v(\sqrt{q}) - v(b_k)\|_{l_{2/r}}.$$

Thus, if

$$\|v(\sqrt{q})\|_{l_{2/r}} = \sum_{\ell\in\mathbb{Z}} \|q\|_{1,[\ell,\ell+1]}^{\frac{1}{r}} < \infty, \qquad (*)$$

we obtain

$$\sup_n \left(n^{\frac{r}{2}} \cdot s_n(\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}_k^*)\right) \to 0 \qquad \text{as } n\to\infty.$$

This implies

$$\lim_{\lambda\to 0_+} \lambda^{\frac{2}{r}} N_{\widetilde{\mathcal{T}}_k^*}(\lambda) \to \lim_{\lambda\to 0_+} \lambda^{\frac{2}{r}} N_{\widetilde{\mathcal{T}}^*}(\lambda) \qquad \text{as } k\to\infty.$$

Since $\lambda_n(\mathcal{K}_{\sqrt{q}X}) = s_n^2(\widetilde{\mathcal{T}}^*)$, this implies

$$\Delta_{\frac{1}{r}}^{(k)} \to \lim_{\lambda\to 0_+} \lambda^{\frac{1}{r}} N_{\mathcal{K}_{\sqrt{q}X}}(\lambda) \qquad \text{as } k\to\infty.$$

This gives the asymptotics of $s_n(\mathcal{K}_{\sqrt{q}X}) = \lambda_n(\mathcal{K}_{\sqrt{q}X})$.

Let the spectral density of X be as above. Assume that the weight $q\in L^1(\mathbb{R})$ satisfies $(*)$. Then

$$\lambda_n(\mathcal{K}_{\sqrt{q}X}) \sim \left(\frac{M_-^{\frac{1}{r}} + M_+^{\frac{1}{r}}}{2\pi} \int_{\mathbb{R}} q(t)^{\frac{1}{r}}\,dt\right)^r \frac{2\pi}{n^r}, \qquad \textit{as } n\to\infty.$$

provided that λ_n are numbered in the non-increasing order, according to their multiplicities.

Notice that the final formula is quite similar to previous ones, although intermediate technical details differ.

Let $\{X(t), t \in \mathbb{R}\}$ be a real centered mean-square continuous stationary Gaussian process. Assume that it has a spectral density satisfying

$$m(u) \sim M|u|^{-r}, \qquad \text{as } |u| \to \infty,$$

with some $r > 1, M > 0$. Let q be a summable weight on $\mathbb{R}$ satisfying condition $(*)$. Then we have, as $\varepsilon \to 0$,

$$\log \mathbb{P}\Big(\int_{\mathbb{R}} q(t)|X(t)|^2 dt \le \varepsilon^2 \Big) \sim -\Big(\frac{M^{\frac{1}{r}}}{r \sin(\pi/r)} \int_{\mathbb{R}} q(t)^{\frac{1}{r}} dt \Big)^{\frac{r}{r-1}} \frac{(r-1)(2\pi)^{\frac{1}{r-1}}}{2\, \varepsilon^{\frac{2}{r-1}}}.$$

In particular, this result covers the earlier ones obtained for concrete processes on bounded intervals.

Let $\{X(t), t \in \mathbb{R}\}$ be a complex centered mean-square continuous stationary proper Gaussian process. Assume that it has a spectral density having a power-like decay with some $r > 1$. Let q be a summable weight on $\mathbb{R}$ satisfying $(*)$. Then we have, as $\varepsilon \to 0$,

$$\log \mathbb{P}\Big(\int_{\mathbb{R}} q(t)|X(t)|^2 dt \le \varepsilon^2 \Big) \sim -\Big(\frac{M_-^{\frac{1}{r}} + M_+^{\frac{1}{r}}}{2r \sin(\pi/r)} \int_{\mathbb{R}} q(t)^{\frac{1}{r}} dt \Big)^{\frac{r}{r-1}} \frac{(r-1)(2\pi)^{\frac{1}{r-1}}}{\varepsilon^{\frac{2}{r-1}}}.$$

Besides the weight integration domain, the constants in the limit are the same as in part **1**.