A probabilistic approximation of the Cauchy problem solution for an evolution equation with the differential operator of the order greater than 2

Platonova Mariia

St. Petersburg State University

Symposium on Probability Theory and Random Processes, Saint Petersburg

5–9 June 2017

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{c_m}{m!}\frac{\partial^m u}{\partial x^m}, \quad u(0,x) = \varphi(x) \tag{1}$$

where

$$c_m = \begin{cases} \pm 1, & m = 2k+1, \\ (-1)^{k+1}, & m = 2k. \end{cases}$$

This problem can be written in terms of the Fourier transform (with respect to x)

$$\frac{\partial \widehat{u}(t,p)}{\partial t} = \frac{c_m}{m!}(-ip)^m \widehat{u}(t,p), \quad \widehat{u}(0,p) = \widehat{\varphi}(p). \tag{2}$$

Then

$$\widehat{u}(t,p) = \widehat{\varphi}(p)\exp\left(\frac{c_m}{m!}t(-ip)^m\right).$$

If $m = 2$ then the solution (1) can be represented in the form

$$u(t, x) = \mathbf{E}\varphi(x - w(t)) \tag{3}$$

where $w(t)$ is a standard Wiener process.

Yu. Daletskii; E. Orsingher, X. Zhao, 1999; L. Debbi, 2006 – theory of pseudoprocesses.
T. Funaki, 1979; S. Mazzucchi, 2013; S. Mazzucchi, S. Bonaccorsi, 2015 – a construction of a complex-valued stochastic process;

Let $\eta(t)$ be a stochastic stable process with independent increments and the Lévy measure $\Lambda(dx)$ satisfying the property

$$\int_{\mathbf{R}} \min(|x|, 1)\Lambda(dx) < \infty. \tag{4}$$

We define a generalized function I that acts on a test function φ as

$$(I, \varphi) = \int_{\mathbf{R}} \big(\varphi(x) - \varphi(0)\big)\Lambda(dx).$$

Then a characteristic function of $\eta(t)$ is given by

$$f_{\eta(t)}(p) = \exp\left[t\left(I_x, e^{ipx}\right)\right] = \exp\left[t\int_{\mathbf{R}} \left(e^{ipx} - 1\right)\Lambda(dx)\right]. \tag{5}$$

If the Lévy measure $\Lambda(dx)$ satisfies the property

$$\int_{\mathbf{R}} \min(x^2, 1)\Lambda(dx) < \infty \tag{6}$$

then the corresponding generalized function I acts on a test function φ as

$$(I, \varphi) = \int_{\mathbf{R}} \Big(\varphi(x) - \varphi(0) - \varphi'(0)x \cdot \mathbf{1}_{[-1,1]}(x)\Big)\Lambda(dx).$$

and the characteristic function of $\eta(t)$ can be written as

$$f_{\eta(t)}(p) = \exp\left[t\,(I_x, e^{ipx})\right] = \exp\left[t \int_{\mathbf{R}} (e^{ipx} - 1 - ipx \cdot \mathbf{1}_{[-1,1]}(x))\Lambda(dx)\right].$$

If we consider the Lévy measure (in the general case) as a generalized function then we can consider a generalized function $\widetilde{I} = I + a\delta^{(1)} + \frac{b^2}{2}\delta^{(2)}$ and the characteristic function of an arbitrary Lévy process $\eta(t)$ can be represented as

$$f_{\eta(t)}(p) = \exp\left[t\,(\widetilde{I}_x, e^{ipx})\right].$$

For every Lévy process $\eta(t)$ we can consider a semigroup P^t that is defined by

$$P^t \varphi(x) = \mathbf{E}\varphi(x - \eta(t)). \tag{7}$$

By $\mathcal{A}$ we define a generator of the semigroup P^t. Then a function $u(t,x) = P^t\varphi(x)$ solves the Cauchy problem

$$\frac{\partial u}{\partial t} = \mathcal{A}u, \;\; u(0,x) = \varphi(x). \tag{8}$$

The generator $\mathcal{A}$ of the semigroup P^t is a convolution with a generalized function $\widetilde{l}$

$$\mathcal{A}\varphi(x) = \left(\varphi * \widetilde{l}\right)(x) \tag{9}$$

where we use a standard regularization of the generalized function $\widetilde{l}$.

A generator of the semigroup $P^t = \exp\left(t\frac{c_m}{m!}\frac{d^m}{dx^m}\right)$ is a convolution with a generalized function $\frac{c_m}{m!}\delta^{(m)}$. A generalized function $I = (-1)^m\frac{\delta^{(m)}}{m!}$ can be got as a limit $I_\varepsilon \to I$ as $\varepsilon \to 0$

$$(I_\varepsilon, \varphi) = \int_{\varepsilon}^{e\varepsilon}\left(\varphi(y) - \sum_{j=0}^{m-1}\frac{\varphi^{(j)}(0)y^j}{j!}\right)\frac{dy}{y^{1+m}}.$$

Let $\nu(dx, dt)$ be a Poisson random measure on $[0, T] \times \mathbf{R}$ with intensity measure $\mathbf{E}\nu(dt, dx) = dt \cdot \Lambda(dx) = \frac{dt \cdot dx}{|x|^{1+m}}$, $m \in \mathbf{N}$.
For $\varepsilon > 0$ by $\xi_\varepsilon(t)$, $t \in [0, T]$ we denote the random process

$$\xi_\varepsilon(t) = \iint\limits_{[0,t]\times[\varepsilon,e\varepsilon)} x\nu(ds, dx). \tag{10}$$

For the case $m = 4k + 1$, $m = 4k + 2$ and the case $m = 4k - 1$, $m = 4k$ we use different approaches.

The case $m = 4k + 2$

For $\varepsilon > 0$ we define a function $u_\varepsilon(t, x)$

$$u_\varepsilon(t, x) = \mathbf{E}\Big[(\varphi * \omega_\varepsilon^t)(x - \xi_\varepsilon(t))\Big],$$

where $\omega_\varepsilon^t(x)$ is defined by its Fourier transform

$$\widehat{\omega}_\varepsilon^t(p) = \exp\Big(- t \int_\varepsilon^{e\varepsilon} \Big(\sum_{j=1}^{m-1} \frac{(ipy)^j}{j!} \Big) \frac{dy}{y^{1+m}} \Big).$$

Theorem

Suppose that $\varphi \in W_2^{l+m+1}(\mathbf{R})$, $l \geqslant 0$ and let $u(t, x)$ be a solution of the Cauchy problem (1). Then there exists $C = C(m) > 0$ such that

$$\|u_\varepsilon(t, \cdot) - u(t, \cdot)\|_{W_2^l(\mathbf{R})} \leqslant Ct \, \|\varphi\|_{W_2^{l+m+1}(\mathbf{R})} \, \varepsilon.$$

By $P_\pm$ we denote the Riesz projectors that act from $L_2(\mathbf{R})$ to Hardy spaces $H^2_\pm$. For every $\varphi \in L_2(\mathbf{R})$ we have

$$\varphi(x) = P_+\varphi(x) + P_-\varphi(x) = \varphi^+(x) + \varphi^-(x),$$

where the support of the Fourier transform of the function φ^+ belongs to the negative semi-axis and the Fourier transform of the function φ^- belongs to the positive semi-axis.
For any $M > 0$ by P_M denote the projector in $L_2(\mathbf{R})$ on the subspace of the functions such that $\operatorname{supp} \widehat{\psi} \subset [-M, M]$.
Now we consider the complex-valued process $\sigma\xi_\varepsilon(t)$ where σ is a complex constant. For $\sigma\xi_\varepsilon(t)$ we have

$$\mathbf{E} \exp(ip\sigma\xi_\varepsilon(t)) = \exp\left(t \int\limits_{\varepsilon}^{e\varepsilon} (e^{i\sigma py} - 1)\frac{dy}{y^{1+\alpha}}\right).$$

Set $\sigma_+ = \exp(\frac{i\pi}{m})$ and $\sigma_- = \exp(-\frac{i\pi}{m})$. Note that σ_+ belongs to upper half-plane and σ_- belongs to lower half-plane and we have

$$\sigma_+^m = \sigma_-^m = -1.$$

For $\varepsilon > 0$ define a function $u_\varepsilon(t,x)$ by

$$u_\varepsilon(t,x) = \mathbf{E}\Big[(\varphi_M^- * \omega_\varepsilon^t)(x - \sigma_+\xi_\varepsilon(t)) + (\varphi_M^+ * \omega_\varepsilon^t)(x - \sigma_-\xi_\varepsilon(t))\Big]$$

where $\omega_\varepsilon^t(x)$ is defined by its Fourier transform

$$\widehat{\omega}_\varepsilon^t(p) = \begin{cases} \exp\Big(- t\int\limits_{\varepsilon}^{e\varepsilon}\Big(\sum\limits_{j=1}^{m-1} \frac{(ip\sigma_+ y)^j}{j!}\Big) \frac{dy}{y^{m+1}}\Big), & p \geqslant 0, \\ \exp\Big(- t\int\limits_{\varepsilon}^{e\varepsilon}\Big(\sum\limits_{j=1}^{m-1} \frac{(ip\sigma_- y)^j}{j!}\Big) \frac{dy}{y^{m+1}}\Big), & p < 0. \end{cases}$$

Theorem

Suppose $\varphi \in W_2^{l+m+1}(\mathbf{R})$, $l \geqslant 0$, $M(\varepsilon) = (e\varepsilon)^{-1}$ *and let* $u(t,x)$ *be a solution of the Cauchy problem* (1)*.Then there exists* $C = C(m) > 0$ *such that*

$$\|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbf{R})} \leqslant C(t+\varepsilon^m)\,\|\varphi\|_{W_2^{l+m+1}(\mathbf{R})}\,\varepsilon.$$

Thus we have constructed a probabilistic approximation of a solution of the Cauchy problem (1) in the case $m = 4k+2$

$$u(t,x) = \lim_{\varepsilon\to 0} \mathbf{E}\big[(\varphi * \omega_\varepsilon^t)(x - \xi_\varepsilon(t))\big],$$

and in the case $m = 4k$ we have

$$u(t,x) = \lim_{\varepsilon\to 0} \mathbf{E}\big[(\varphi_M^- * \omega_\varepsilon^t)(x - \sigma_+\xi_\varepsilon(t)) + (\varphi_M^+ * \omega_\varepsilon^t)(x - \sigma_-\xi_\varepsilon(t))\big].$$

Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. nonnegative random variables. Suppose that the moments of the order up to $m+1$ exist and

$$\mu_m = \int_0^{+\infty} y^m d\mathcal{P}(y) = 1. \tag{11}$$

Let $\eta(t)$, $t \in [0, \infty)$ be a standard Poisson process independent of $\{\xi_j\}_{j=1}^{\infty}$. Define a random process $\zeta_n(t)$, $t \in [0, T]$ by

$$\zeta_n(t) = \frac{1}{n^{1/m}} \sum_{j=1}^{\eta(nt)} \xi_j. \tag{12}$$

The case $m = 4k + 2$

For $n \in \mathbf{N}$ we define a function

$$u_n(t,x) = \mathbf{E}\Big[(\varphi * \varkappa_n^t)(x - \zeta_n(t))\Big],$$

where

$$\widehat{\varkappa}_n^t(p) = \exp\Big(- nt \sum_{j=1}^{m-1} \frac{\mu_j (ip)^j}{j! n^{j/m}} \Big).$$

Theorem

Let $\varphi \in W_2^{l+m+1}(\mathbf{R})$, $l \geqslant 0$ and let $u(t,x)$ be a solution of the Cauchy problem (1). Then there exists $C = C(m) > 0$ such that

$$\|u_n(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbf{R})} \leqslant \frac{Ct}{n^{1/m}} \|\varphi\|_{W_2^{l+m+1}(\mathbf{R})}.$$

For $n \in \mathbf{N}$ we define a function

$$u_n(t,x) = \mathbf{E}\Big[(\varphi_M^- * \varkappa_n^t)(x - \sigma_+\zeta_n(t)) + (\varphi_M^+ * \varkappa_n^t)(x - \sigma_-\zeta_n(t))\Big],$$

where

$$\widehat{\varkappa}_n^t(p) = \begin{cases} \exp\Big(-nt\Big(\sum_{j=1}^{m-1} \frac{\mu_j (ip\sigma_+)^j}{j!\, n^{j/m}}\Big)\Big), & p \geqslant 0, \\ \exp\Big(-nt\Big(\sum_{j=1}^{m-1} \frac{\mu_j (ip\sigma_-)^j}{j!\, n^{j/m}}\Big)\Big), & p < 0. \end{cases}$$

Theorem

Let $\varphi \in W_2^{l+m+1}(\mathbf{R})$, $l \geqslant 0$, $M(n) = \delta_0 n^{1/m}$ and let $u(t,x)$ be a solution of the Cauchy problem (1). Then there exists $C = C(m) > 0$ such that

$$\|u_n(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbf{R})} \leqslant \frac{C}{n^{1/m}}\Big(t + \frac{1}{n}\Big)\|\varphi\|_{W_2^{l+m+1}(\mathbf{R})}.$$