Tail analysis for recurrent Markov chains with asymptotically zero drift.

Vitali Wachtel
Universität Augsburg

(joint work with Denis Denisov (University of Manchester)
and Dima Korshunov (University of Lancaster))

Let $\{X_n,\, n \geq 0\}$ be a time homogeneous Markov chain with values in $\mathbb{R}_+$.

Denote by $\xi(x)$ a random variable corresponding to the jump of the chain at point x, that is,

$$\mathbf{P}(\xi(x) \in B) = \mathbf{P}(X_{n+1} - X_n \in B | X_n = x) = \mathbf{P}_x(X_1 \in x + B).$$

Let $m_k(x)$ denote the kth moment of the chain at x, i.e.,

$$m_k(x) := \mathbf{E}\xi^k(x).$$

We consider the case of the **asymptotically zero drift**:

$$m_1(x) \to 0 \quad \text{as } x \to \infty.$$

Recurrence, positive recurrence and transience were studied by Lamperti(1960, 1963):

- If $2xm_1(x) + m_2(x) \leq -\varepsilon$, then the chain is positive recurrent;
- If $\frac{2xm_1(x)}{m_2(x)} < 1 - \varepsilon$, then the chain is recurrent;
- If $\frac{2xm_1(x)}{m_2(x)} > 1 + \varepsilon$, then the chain is transient.

Critical Lamperti problem

We shall assume that

$$m_1(x) \sim \frac{-\mu}{x} \quad \text{and} \quad m_2(x) \sim b.$$

Then we have

- If $\mu > b/2$ then the chain is positive recurrent;
- If $-b/2 < \mu < b/2$ then the chain is null recurrent;
- If $\mu < -b/2$ then the chain is transient.

Was can be said about stationary measures in the recurrent case?

Examples of chains with calculable stationary measures.

- **Diffusion with the drift $m_1(x)$ and the diffusion coefficient $m_2(x)$.**

  The invariant density function solves the Kolmogorov forward equation

$$0 = -\frac{d}{dx}(m_1(x)p(x)) + \frac{1}{2}\frac{d^2}{dx^2}(m_2(x)p(x)),$$

  which has the following solution:

$$p(x) = \frac{C}{m_2(x)} \exp\left\{\int_0^x \frac{2m_1(y)}{m_2(y)} dy\right\}.$$

- **Markov chains on $\mathbb{Z}_+$ with $|\xi(x)| \le 1$.**

  Set $p_-(x) = \mathbf{P}(\xi(x) = -1)$, $p_+(x) = \mathbf{P}(\xi(x) = 1)$ and $1 - p_-(x) - p_+(x) = \mathbf{P}(\xi(x) = 0)$.

  Then the stationary probabilities $\pi(x)$, $x \in \mathbb{Z}_+$ satisfy

  $$\pi(x) = \pi(x-1)p_+(x-1) + \pi(x)(1 - p_-(x) - p_+(x)) + \pi(x+1)p_-(x+1).$$

  Consequently,

  $$\pi(x) = \pi(0) \prod_{k=1}^{x} \frac{p_+(k-1)}{p_-(k)}.$$

- **Random walks with delay.**

  Let $\xi_1, \xi_2, \ldots$ be independent identically distributed random variables with zero mean. Then the random walk $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ is oscillating. This, in its turn, implies that the chain

$$X_n = (X_{n-1} + \xi_n)^+$$

  is recurrent. For its stationary measure we then have

$$\pi(dx) = \sum_{n=1}^{\infty} \mathbf{P}_0(X_n \in dx, \tau_0 > n) = \sum_{n=1}^{\infty} \mathbf{P}_0 \left(S_n \in dx, \min_{k \le n} S_k > 0 \right)$$

$$= \sum_{k=1}^{\infty} \mathbf{P}(\chi_1 + \chi_2 + \ldots + \chi_k \in dx) = H(dx).$$

Therefore, X_n is null-recurrent for every oscillating random walk S_n and

$$\pi(0,x] = H(0,x].$$

If, for example, $\mathbf{E}\xi^2 < \infty$, then $\mathbf{E}\chi < \infty$ and, consequently,

$$\pi(0,x] \sim \frac{x}{\mathbf{E}\chi}, \quad \text{as } x \to \infty.$$

Menshikov and Popov (1995) investigated Markov chains on $\mathbb{Z}_+$ with bounded jumps: For every $\varepsilon > 0$ there exist constants $c_\pm(\varepsilon)$ such that

$$c_-(\varepsilon)x^{-2\mu/b-\varepsilon} \le \pi(\{x\}) \le c_+(\varepsilon)x^{-2\mu/b+\varepsilon}.$$

Korshunov (2011) has shown that if $\{(\xi^+(x))^{2+\gamma}, x \ge 0\}$ and $\{(\xi^-(x))^2, x \ge 0\}$ are uniformly integrable, then the moment of order γ of the distribution π is finite for $\gamma < 2\mu/b - 1$, and infinite for $\gamma > 2\mu/b - 1$. Consequently, for every $\varepsilon > 0$ there exists $c(\varepsilon)$ such that

$$\pi(x, \infty) \le c(\varepsilon)x^{-2\mu/b+1+\varepsilon}.$$

Denisov, Korshunov and Wachtel (2013) have considered posiive recurrent chains satisfying

$$\frac{2m_1(x)}{m_2(x)} = -r(x) + O(x^{-2-\delta}).$$

for some $r(x) > 0$ such that $r'(x) \sim -\frac{2\mu}{bx^2}$.

It has been shown that if

$$\sup_x \mathbf{E}|\xi(x)|^{3+\delta} < \infty, \quad \mathbf{E}[\xi^{2\mu/b+3+\delta}(x); \xi(x) \geq Ax] = O(x^{2\mu/b})$$

and

$$m_3(x) \to m_3 \in (-\infty, \infty)$$

then there exists a constant $c > 0$ such that

$$\pi(x,\infty) \sim cxe^{-\int_0^x r(y)dy} = cx^{-2\mu/b+1}\ell(x).$$

Define

$$m_k^{[s(x)]}(x) := \mathbf{E}[\xi^k(x); |\xi(x)| \le s(x)]$$

and assume that

$$m_1^{[s(x)]}(x) \sim -\frac{\mu}{x} \quad \text{and} \quad m_2^{[s(x)]}(x) \sim b$$

for some $b > 0$ and $\mu > -b/2$.

We shall also assume that there exist a dereasing diffrentiable function $r(x)$ and a decreasing integrable function $p(x)$ such that

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = -r(x) + O(p(x)).$$

Obviously,

$$r(x) \sim \frac{2\mu}{bx}.$$

Define a monotone function

$$R(x) := \int_0^x r(y)dy, \quad x > 0,$$

$R(x) = 0$ for $x \le 0$. Since $xr(x) \to 2\mu/b > -1$,

$$\frac{R(x)}{\log x} \to \frac{2\mu}{b} > -1 \quad \text{as } x \to \infty.$$

Define also

$$U(x) := \int_0^x e^{R(y)}dy \to \infty \quad \text{as } x \to \infty.$$

According to our assumptions,

$$r(x) = \frac{2\mu}{b}\frac{1}{x} + \frac{\varepsilon(x)}{x},$$

where $\varepsilon(x) \to 0$ as $x \to \infty$. Then there exists a slowly varying at infinity function $\ell(x)$ such that $e^{R(x)} = x^{\rho-1}\ell(x)$ and $U(x) \sim x^{\rho}\ell(x)/\rho$ where $\rho = 2\mu/b + 1 > 0$.

Theorem 1. Let X_n be a recurrent Markov chain and let $\pi(\cdot)$ be its stationary measure. Let π have right-unbounded support, that is, $\pi(x,\infty)>0$ for all x. Assume that, for some increasing $s(x)=o(x)$,

$$\begin{aligned}
\mathbf{P}\{\xi(x)<-s(x)\} &= o(p(x)/x),\\
\mathbf{E}\big\{U(x+\xi(x));\ \xi(x)>s(x)\big\} &= o(p(x)/x)U(x),\\
\mathbf{E}\big\{|\xi(x)|^3;\ |\xi(x)|\le s(x)\big\} &= o(x^2p(x)) \quad \text{as } x\to\infty.
\end{aligned}$$

Then

$$\pi(x_1,x_2] \quad\sim\quad c\int_{x_1}^{x_2}\frac{y}{U(y)}dy$$

as $x_1,x_2\to\infty$ in such a way that

$$1<\liminf\frac{x_2}{x_1}\le\limsup\frac{x_2}{x_1}<\infty.$$

Corollary 2. If X_n is positive recurrent, $2\mu > b$, and conditions of Theorem 1 hold, then

$$\pi(x,\infty) \sim \frac{c}{\rho - 2}\frac{x^2}{U(x)} \quad \text{as } x \to \infty.$$

If X_n is null recurrent, $2\mu \in (-b, b)$, and conditions of Theorem 1 hold, then

$$\pi(0,x) \sim \frac{c}{2 - \rho}\frac{x^2}{U(x)} \quad \text{as } x \to \infty.$$

Corollary 3. Assume that the conditions of Theorem 1 are valid. Then the integrability of $y/U(y)$ is necessary and sufficient for the Markov chain X_n on $\mathbb{R}_+$ to be positive recurrent.

Let $B = [0, x_0]$ be such that $\pi(B) > 0$ and set $\tau_B := \min\{k \geq 1 : X_k \in B\}$. For the measure π we have

$$\pi(dx) = \int_B \pi(dz) \sum_{n=1}^{\infty} \mathbf{P}_z(X_n \in dx, \tau_B > n), \ x > x_0.$$

If we find a positive function $V(x)$ such that $V(x) = \mathbf{E}_x[V(X_1), \tau_B > 1]$, then we can perform the following change of measure:

$$\pi(dx) = \frac{1}{V(x)} \int_B \pi(dz)V(z) \sum_{n=1}^{\infty} \mathbf{P}_z(\widehat{X}_n \in dx) =: c_0 \frac{H(dx)}{V(x)},$$

where $\widehat{X}_n$ is a Markov chain with the following transition kernel

$$\mathbf{P}_x(\widehat{X}_1 \in dy) = \frac{V(y)}{V(x)} \mathbf{P}_x(X_1 \in dy, \tau_B > 1)$$

and initial distribution

$$\mathbf{P}(\widehat{X}_0 \in dz) = \frac{1}{c_0} \pi(dz)V(z),\ z \in B.$$

This function has been constructed in Denisov, Korshunov and Wachtel (2013). It was also shown there that $V(x) \sim U(x)$. But this is not enough to study asymptotic properties of $\widehat{X}_n$, one needs also to obtain an asymptotic expansion

$$V(x) = U(x) + (c + o(1))\frac{U(x)}{x}.$$

For this representation one has to assume the convergence of third moments $m_3(x)$.

Let us perform of a measure change by a smooth Lyapunov function, which is almost harmonic.

The main advantage of this approach is the fact that this function can chosen as smooth as one wishes. As a compensation we get a non-probabilistic transition kernel and we have to control total masses.

Consider

$$R_p(x) = \int_0^x (r(y) - p(y))dy \quad \text{and} \quad U_p(x) = \int_0^x e^{R_p(y)} dy.$$

Then one can show that

$$-(2\mu + b)\frac{p(x)}{x} \le \frac{\mathbf{E}U_p(x + \xi(x)) - U_p(x)}{U_p(x)} \le 0, \quad x \ge x_0.$$

Therefore, the measure

$$Q(x, dy) := \frac{U_p(y)}{U_p(x)}\mathbf{P}_x(X_1 \in dy, \tau_B > 1)$$

is substochastic and

$$q(x) := -\log Q(x, \mathbb{R}_+) = O(p(x)/x).$$

Let $\widehat{X}_n$ be a Markov chain with transition kernel $Q(x,dy)/Q(x,\mathbb{R}_+)$. Then

$$\begin{aligned} \pi(dx) &= \int_B \pi(dz) \sum_{n=1}^{\infty} \frac{U_p(x)}{U_p(x)} \mathbf{E}_z \left[e^{-\sum_{j=0}^{n-1} q(\widehat{X}_j)}; \widehat{X}_n \in dx \right] \\ &\approx \frac{1}{U_p(x)} \int_B \pi(dz) U_p(z) f(z) \sum_{n=1}^{\infty} \mathbf{P}_z(\widehat{X}_n \in dx) \end{aligned}$$

Having this representation it remains to prove that

$$\sum_{n=1}^{\infty} \mathbf{P}_z(\widehat{X}_n \le x) \sim c x^2.$$