# Lectures of Summer School
# Various aspects of mathematical physics, 8-11 July 2017

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- **Aleksei Aleksandrov**, (Russia), *Operator modulus of continuity*

Let $f$ be a uniformly continuous function on the real line $\mathbb{R}$. Denote by $\omega_f$ the modulus of continuity of $f$.

We define the operator modulus of continuity as follows

$$\Omega_f(t) = \sup \|f(A) - f(B)\|,$$

where the supremum is taken over all self-adjoint operators $A$ and $B$ on Hilbert space such that $\|A - B\| \leqslant t$.

Clearly, $\Omega_f \geqslant \omega_f$. On the other hand, one can prove that $\Omega_f(t) = +\infty$ for all $t > 0$ if $f(x) = |x|$. It is known that if $\omega_f(t) \leqslant t^\alpha$ for all $t > 0$ and $\alpha < 1$, then $\Omega_f(t) \leqslant c(\alpha)t^\alpha$ for all $t > 0$.

I am going to consider this result and some other results in this direction. Moreover, I am planning to explain that these results can be obtained in essence by methods of the approximation theory.

• **Andrei Badanin**, (Russia), *Third and fourth order operators with periodic coefficients on the line*

We consider the self-adjoint third and fourth order operators with 1-periodic coefficients on the real line. The spectrum of 4-th order operators is absolutely continuous, consists of intervals separated by gaps and has multiplicity 2 or 4. We describe the spectrum in terms of the Lyapunov function, which is analytic on a two-sheeted Riemann surface. Endpoints of the gaps are eigenvalues of periodic or antiperiodic problems, or branch points of the Lyapunov function. We determine asymptotics of the spectrum at high energy. We consider the spectrum in the case of small coefficients.

The spectrum of 3-td order operators is absolutely continuous, covers the real line and has multiplicity 1 or 3. We describe the spectrum in terms of a discriminant, which is an entire function. We prove that there is a finite number of intervals of spectrum of multiplicity 3 and endpoints of these intervals are real zeros of the discriminant. We determine the high energy asymptotics of the periodic, anti-periodic eigenvalues and of branch points of the Lyapunov function, which is analytic on a 3-sheeted Riemann surface. We consider the spectrum in the case of small coefficients.

This is a joint work with E.Korotyaev.

• **Anton Baranov**, (Russia), *Appllications of entire function theory to completeness problems*

We discuss several examples where growth theory for entire functions is used to prove completeness of special families of functions. They include completeness problems for systems of exponentials on an interval and for systems of eigenvectors for certain classes of operators.

• **Maria J. Esteban**, (France) *Spectral estimates for Schrödinger operators on manifolds and with external magnetic fields*

ABSTRACT. In this lecture will be presented various estimates for the principal eigenvalue of Schrödinger operators on bounded and unbounded manifolds and with external magnetic fields. In both cases we will see that there is a strong link with new interpolation inequalities, and that the semi-classical regime is only reached in some asymptotic cases.

Numerical estimates will show that the estimates are surprisingly sharp.

The results presented in this lecture have been done jointly with J. Dolbeault, A. Laptev and M. Loss

• **Setsuro Fujiie** (Japan) *Microlocal method for the semiclassical distribution of quantum resonances*

It is well-known as Weyl's law that the asymptotic distribution of large eigenvalues of the Dirichlet Laplacian on a bounded domain is related with the geometry of the domain. For the Schrödinger operator $-h^2\mathbf{D}+V(x)$ with a small parameter $h$ and a potential $V(x)$, the asymptotic distribution of eigenvalues or resonances (poles of the resolvent) in the semi-classical limit $h \to 0$ near a fixed energy is related with the geometry of the "trapped" trajectories of the underlying classical mechanics. In particular, when the trapped set has a simple geometric structure, such as a single periodic trajectory, a hyperbolic fixed point or a homoclinic trajectory, the quantization condition has been studied to obtain the precise location of eigenvalues or resonances. In this course, we introduce a new microlocal approach to such a problem proposed in [1] and [2]. It will be shown, using for example the standard propagation of singularity theorem, that if an eigenfunction or resonant state is microlocally infinitely small with respect to h on the trapped set, then so is it globally in $R^n$. This fact justifies the formal quantization condition that the asymptotic solution is single-valued on the trapped set at the principal level. Namely, we can conclude, by a simple contradiction argument, that the set of eigenvalues or resonances is "close" to that of energies satisfying the above condition. We will demonstrate the distribution of resonances created by multi-barriers, when the trapped set consists of hyperbolic fixed points and associated homoclinic and heteroclinic trajectories.

References

[1] J.-F. Bony, S. Fujiie, T. Ramond, and M. Zerzeri, Barrier top resonances for non globally analytic potentials, preprint, arXiv:1610.06384 (2016).

[2] J.-F. Bony, S. Fujiie, T. Ramond, and M. Zerzeri, Resonances for ho- moclinic trapped sets, preprint, arXiv:1603.07517 (2016).

• **Fumio Hiroshima** (Japan) *Thresholds and resonances of fractional Schroedinger operators on a lattice.*

Fractional Schrödinger operators are defined on a lattice and the behavior of eigenvalues are discussed. The Birman-Schwinger type estimate is studied. An asymmetry does appear and it can be applied to relativistic Schrödinger operators. We also discuss Schrödinger operators with delta potentials on n- dimensional lattice, which is a joint work with Zahriddin Muminov and Utkir Kuljanovc.

- **Alexander Kiselev** (USA), *Singularity formation in models of fluid mechanics*

I will discuss a family of modified SQG equations that varies between 2D Euler and SQG with patch-like initial data defined on half-plane. The family is modulated by a parameter that sets the degree of the kernel in the Biot-Savart law. The main result I would like to describe is the phase transition in the behavior of solutions that happens right beyond the 2D Euler case. Namely, for the 2D Euler equation the patch solution stays globally regular, while for a range of nearby models there exist regular initial data that lead to finite time blow up. This talk is based on a joint work with Lenya Ryzhik, Yao Yao and Andrej Zlatos.

• Evgeny Korotyaev, (Russia), *Inverse problems and estimates for Schrödinger operators on the circle*

We consider a Schrödinger operator on the circle with a real potential. The spectrum of this operator is a sequence of eigenvalues which go to + infinity. Each eigenvalue has multiplicity 1 or 2. Using these eigenvalues and Dirichlet eigenvalues for the same potential on the unit interval we construct the spectral data. We discuss the inverse problems for this operator.

The inverse spectral problem consists of the following parts:
(i) Direct problem. Describe spectral data for a given potential.
(ii) Uniqueness. Prove that the spectral data uniquely determine the potential.
(iii) Characterization. Give conditions for some data to be the spectral data of some potential.
(iv) Reconstruction. Give an algorithm for recovering the potential from spectral data.
(v) Stability estimates. To obtain two sided estimates of potentials in terms of spectral data.

The proof of (i)-(iii) is based on nonlinear functional analysis in Hilbert space.
The proof of (iv) is based on the Dubrovin equation.
The proof of (v) is based on the trace formulas and the conformal mapping theory.

- **Ari Laptev** (Russia-Sweden-UK) *Spectral inequalities and their applications*

At the beginning I shall describe a theory of coherent state transformations and its applications to the spectral theory. Then we apply this theory to the study of the Weyl type asymptotics and spectral estimates of functional-difference operators associated to mirror curves of special del Pezzo Calabi-Yau threefolds.

- **Jacob Möller** (Aarhus University, Denmark) *Renormalization of linearly coupled models*

Models of quantum matter interacting with bosonic fields are common in physics. Typical examples are models of one or more electron in a crystal, interacting with acoustic or optical excitations of the crystal. There are also phenomenological models of matter-light interactions and of nucleon-meson interactions.

If the dynamics of the quantum matter is governed by the Laplacian, then the socalled Gross-Nelson dressing transform can be used to remove a possible infinite self-energy, and construct a renormalized Hamiltonian without any cutoffs.

If on the other hand the dynamics of the matter particles is non-relativistic, then the Gross-Nelson dressing transform does not apply. One may instead employ a resummation scheme on the level of resolvent expansions, following ideas going back to Hepp and Eckmann.

The talk will revolve around the two renormalization schemes mention above, and will include some recent joint work with Andreas Wunsch from Stuttgart.

• **Segei Naboko** (Russia) *On the spectral analysis of Hermitean Jacobi Matrices: basic facts and examples*

The talk discusses the spectral properties of some classes of Jacobi operators presented by the tridiagonal infinite matrices. Using the results on the asymptotic behaviour of the corresponding orthogonal polynomials the spectral analysis will be carried on for a few examples of mostly unbounded Jacobi Matrices.No special knowledge in the field to be required.

• **Roman G. Novikov** (France-Russia) *An introduction to multidimensional inverse scattering*

e-mail: novikov@cmap.polytechnique.fr We give an introduction into the domain of multidimensional inverse scattering problems by considering such problems for the Schrödinger equation. This lecture is based on classical results going back to M. Born (1926), B. Lippmann, J. Schwinger (1950), L. Faddeev (1956), S. Agmon (1975) and on recent results of R. Novikov (2015, 2016). 1

- **Grigori Rozenblum**, (Sweden), *Eigenvalue estimates: CLR and beyond*

45 years ago an estimate for the eigenvalues of singular differential operators was proved by me, a young 1st year PhD student. A special case of this estimate got the name 'the CLR estimate' and was being used since then in numerous studies in the spectral theory. In my lecture I am going to explain the mathematics and physics lying in the base of this estimate, the approaches for the proof, particular cases, generalizations and applications.

• **Alex Sobolev**, (University College London) *Qualitative and Quantitative Perturbation Theory for Periodic operators*

Under very broad conditions, spectra of elliptic differential operators with periodic coefficients have a band structure, i.e. they represent a union of closed intervals (*bands*), possibly separated by spectrum-free intervals (*gaps*). It was conjectured by H. Bethe and A. Sommerfeld in the 1930's that the number of gaps in the spectrum of the Schrödinger operator $H = -\Delta + V$ with a periodic electric potential $V$ in dimension three must be finite. It is relatively straightforward to see that this conjecture holds for potentials which admit a partial separation of variables. For general potentials this problem turned out to be quite difficult, and the first rigorous results appeared only in the beginning of the 1980's.

The aim of the lecture is to give a short survey of the existing results on the Bethe-Sommerfeld conjecture. The emphasis will be made on the approach (or a variation thereof), initiated by M. Skriganov in the 1980's. The method has two ingredients. First, one observes that the spectrum of the unperturbed operator $H_0 = -\Delta$ can be found explicitly and its study reduces to the classical number-theoretic problem of counting lattice points in balls of a large radius. The second step is to develop an appropriate perturbation theory which would allow to include the perturbation $V$.

- **Natasha Saburova** (Russia), *Schrödiger operators on periodic discrete graphs*

Schrödiger operators on periodic discrete graphs
This lecture provides an introduction to the spectral theory of discrete Laplace and Schrödinger operators on periodic graphs. Such operators can be used to describe the electronic properties of real crystalline structures in the so-called tight-binding approximation. The material is modeled as a periodic discrete graph consisting of vertices, representing positions of atoms and edges, representing chemical bonding of atoms. We discuss some spectral properties of the operators: the band-gap structure of the spectrum; the flat band spectrum, i.e. the eigenvalues of infinite multiplicity, which may appear in gaps or be embedded into the absolutely continuous spectrum; the connection between spectral properties of the operators and geometric parameters of the graphs; the influence of potentials on the spectrum of the Laplacians; the relation between the spectra of the differential Laplacian on equilateral metric graphs (each edge has unit length) and the spectrum of the Laplacian on the corresponding discrete graphs.

- **Andrei Tsyganov** (Russia) *Modern cryptography and classical mechanics*

Algebraic geometry is one of the central subjects of mathematics. All but the most analytic of number theorists use language of algebraic geometry, as do mathematical physicists, complex analysts, homotopy theorists, symplectic geometers, representation theorists, etc. Intersection theory is at the heart of algebraic geometry, which has many applications in computer science.

In 1989, Koblitz proposed Jacobians of hyperelliptic curves of arbitrary genus as a way to construct Abelian groups suitable for public-key cryptography. The most important operations used by hyperelliptic curve cryptosystems are composed of intersection divisor addition, doubling, halving and so on.

On the other hand, intersection divisors naturally appear in classical mechanics, when we study dynamical systems integrable by Abel quadratures. It allows us to study hidden symmetries of these systems hyperelliptic curve cryptosystems. For instance, we can identify well-studied and well-tried cryptographic algorithms, protocols and attacks with various schemes of the discretization of continuous flows in classical mechanics.

In this Lecture we remind main mathematical definitions and try to explain how to use intersection divisors in cryptography and theoretical mechanics.