The critical locus of integrable systems

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- Rigid body: moments of inertia I_1, I_2, I_3

- angular velocity $(\omega_1, \omega_2, \omega_3)$ relative to principal axes

- Euler's equations:

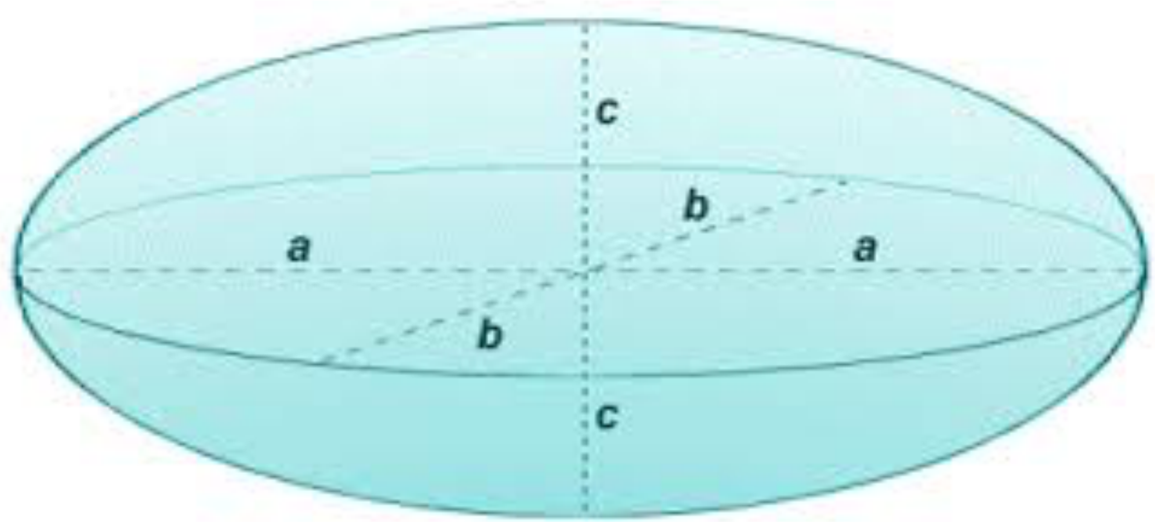
$$
\begin{aligned}
I_1 \dot{\omega}_1 &= \omega_2 \omega_3 (I_2 - I_3) \\
I_2 \dot{\omega}_2 &= \omega_3 \omega_1 (I_3 - I_1) \\
I_3 \dot{\omega}_3 &= \omega_1 \omega_2 (I_1 - I_2)
\end{aligned}
$$

Jacobi $\rightarrow$ Bessel

"The day before yesterday, I reduced to quadrature the problem of geodesic lines on an ellipsoid with three unequal axes. They are the simplest formulas in the world, Abelian integrals, which become the well known elliptic integrals if 2 axes are set equal."

Königsberg, 28th Dec. 1838.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

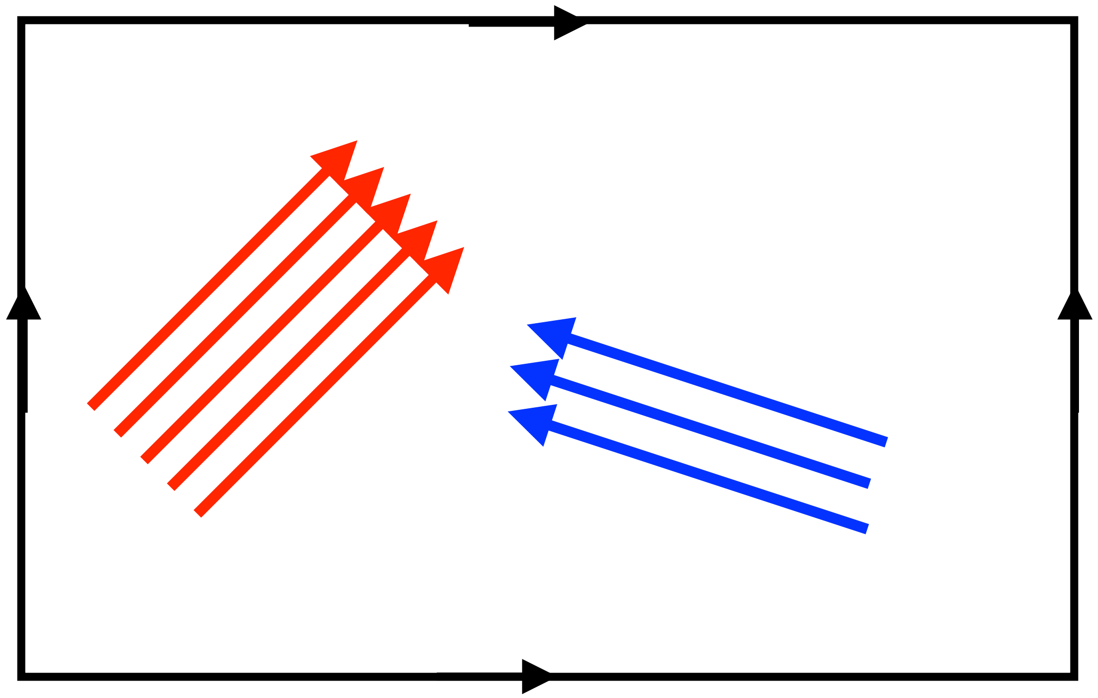
geodesics on the ellipsoid

- T^*S^2 symplectic form ω
- metric $\rightarrow$ function H_1
- Hamiltonian vector field $i_{X_1}\omega = dH_1$ geodesic flow
- second function H_2 such that $[X_1, X_2] = 0$

- $h = (H_1, H_2) : T^*S^2 \to \mathbf{R}^2$

- generic fibre torus $T^2 \cong$ Jacobian of a (real) genus 2 curve

- X_1, X_2 linear vector fields on T^2



- critical locus = points where X_1, X_2 are linearly dependent

- = critical point for some combination of H_1, H_2

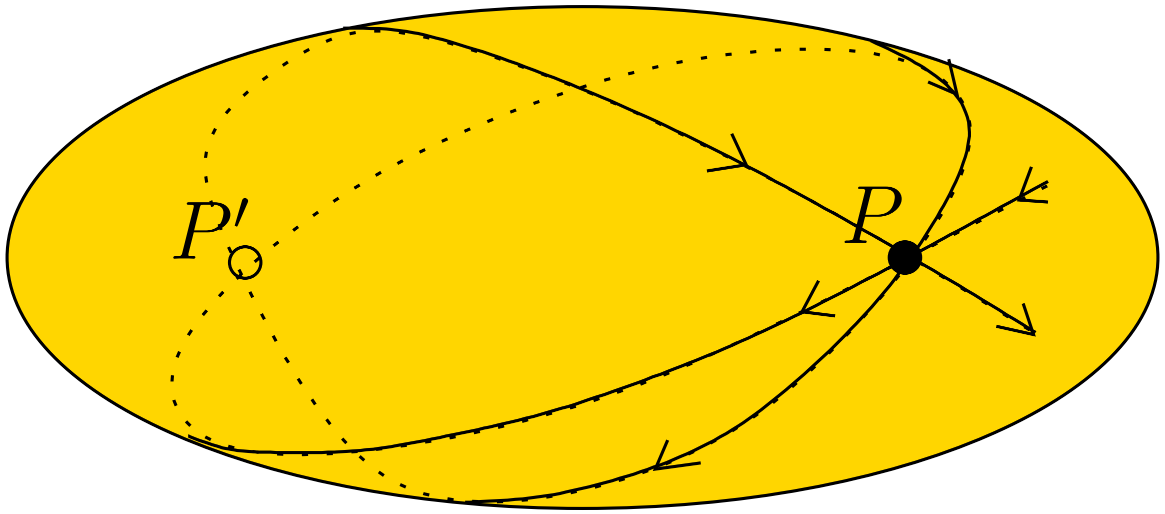
- fibre is singular on critical locus

- fibre = degenerate torus

- 4 umbilical points $P, Q, P' = -P, Q' = -Q$

- singular fibre $\sim$ geodesics through P or Q

- $=$ two copies of T^2, intersecting in the geodesic through P and Q (two orientations)



- ellipsoid $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- $a < b < c$ umbilical points lie on plane $y = 0$

- geodesic = ellipse

- arc length $\sim$ elliptic curve

INTEGRABLE SYSTEMS

- M^{2m} complex symplectic manifold
- $f_1, \ldots, f_m$ holomorphic functions
- proper map $h : M \to \mathbf{C}^m$
- Hamiltonian vector fields $[X_i, X_j] = 0$

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- Hamiltonian vector fields $[X_i, X_j] = 0$

- (or proper map $h : M \to N^m$, local coordinates f_i)

- if Dh_x is surjective for each $x \in h^{-1}(a)$
- then the fibre is an m-dimensional complex torus
- and $X_1, \ldots, X_m$ are tangential to the torus
- $f_1, \ldots, f_m$ constants of the flow

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NONDEGENERATE LOCUS

SUR CERTAINS SYSTEMES DYNAMIQUES SEPARABLES.

By J. VEY.*

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1. Algèbres de Liouville. Soit X une variété symplectique, disons analytique réelle, de dimension $2m$; et soient $f_1, \ldots, f_m$ m fonctions analytiques réelles sur X, commutant deux à deux pour le crochet de Poisson. Si les fonctions f_i sont indépendantes, et si les variétés

$$f_1 = C_1, \ldots, f_m = C_m$$

Mathematische Zeitschrift

Math. Z. 206, 363–407 (1991)

Action-angle coordinates at singularities for analytic integrable systems*

Hidekazu Ito
Mathematical Institute, Tôhoku University, Sendai 980, Japan

- Hamiltonian vector fields $X_1, \ldots, X_d$ vanish at $x \in M^{2m}$

- action on tangent space T_x preserving symplectic form ω

- $L \subset T_x$ span of all X_i, $\dim L = m - d$

- Hamiltonian vector fields $X_1, \ldots, X_d$ vanish at $x \in M^{2m}$
- action on tangent space T_x preserving symplectic form ω
- $L \subset T_x$ span of all X_i, $\dim L = m - d$
- $\omega(X_i, X_j) = 0 \Rightarrow L \subset L^{\perp}$
- $L^{\perp}/L$ symplectic, dimension $2d$

- $X_1, \ldots, X_d$ span a commutative subalgebra of $\mathfrak{sp}(2d, \mathbf{C})$.

- **Defn**: The point x is called *nondegenerate* if this is a Cartan subalgebra.

- $\mathfrak{sp}(2d, \mathbf{C}) \cong$ quadratic functions on $\mathbf{C}^{2d}$

- commutative subalgebra spanned by the Hessians of $h_1, \ldots, h_d$

- nondegeneracy + analytic $\Rightarrow$ local normal form

- $C_d = \{x \in M : \dim \ker Dh_x = d\}$ is a submanifold

- dimension $2(m-d)$ and symplectic

- ... and is a fibration by tori of dimension $(m-d)$

HIGGS BUNDLES

- compact Riemann surface Σ, genus > 1

- holomorphic vector bundle V rank n

- Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

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- stability $\Rightarrow$ moduli space $\mathcal{M}$

- $\mathcal{M}$ is symplectic $((V, \Phi)$ "conjugate variables"$)$

- fibration $h : \mathcal{M} \to \mathcal{A} = \bigoplus_{k=1}^{n} H^0(\Sigma, K^k)$

$$h(V, \Phi) = (b_1, \ldots, b_n) = \left(\operatorname{tr}\Phi, \frac{1}{2}\operatorname{tr}\Phi^2, \ldots, \frac{1}{n}\operatorname{tr}\Phi^n\right)$$

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- $f \in \mathcal{A}^*$ defines a function on $\mathcal{M}$

- $\Rightarrow$ Hamiltonian vector field X_f

- $[X_f, X_g] = 0$: integrable system

- $\det(x - \Phi) = x^n + a_1 x^{n-1} + \ldots + a_n$

- a_k holomorphic section of K^k

- $\ldots$ polynomial in $\operatorname{tr}\Phi^\ell$

- $\det(x - \Phi) = 0$ algebraic curve S

- $a \in \mathcal{A}$ defines S

- a regular value of $h \Rightarrow S$ is smooth

- fibre $h^{-1}(a) \cong \mathrm{Jac}(S)$

- $\sim$ holomorphic line bundles on S

- $S : x^n + a_1 x^{n-1} + \ldots + a_n = 0$

- $\pi : S \to \Sigma$ n-fold branched covering

- L line bundle on S

  Direct image: $U \subset \Sigma$ $H^0(U, \pi_* L) \stackrel{def}{=} H^0(\pi^{-1}(U), L)$

- $\pi_* L = V$ rank n vector bundle

- x single valued section of π^*K on S

- $x : H^0(\pi^{-1}(U), L) \to H^0(\pi^{-1}(U), L \otimes \pi^*K)$

- $= \Phi : H^0(U, V) \to H^0(U, V \otimes K)$

- critical locus $\subset$ singular fibre $\sim$ singular curve S

- singular fibre $\sim$ rank one torsion-free sheaves on S

- nondegenerate critical points?

CRITICAL POINTS FOR $SL(2, \mathbb{C})$

- $\Lambda^2 V$ trivial, $\operatorname{tr}\Phi = 0$

- spectral curve $x^2 - q = 0$, $q \in H^0(\Sigma, K^2)$

- involution $\sigma(x) = -x$

- $\pi_* L = V$ where $\sigma^* L \cong L^* \otimes \pi^* K^{1/2}$ (Prym variety)

- S singular if q has zeros of multiplicity > 1

  local form of equation $x^2 = z^m$

- $H^0(\pi^{-1}(U), \mathcal{O}) : f_0(z) + xf_1(z)$

- if $V = \pi_* L$ for a line bundle $(f_0(z), f_1(z))$ local section

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- if $V = \pi_* L$ for a line bundle $(f_0(z), f_1(z))$ local section

- $x(f_0(z) + xf_1(z)) = z^m f_1(z) + xf_0(z) \Rightarrow$

$$\text{Higgs field} \quad \Phi = \begin{pmatrix} 0 & z^m \\ 1 & 0 \end{pmatrix}.$$

- torsion-free sheaf, not line bundle $\Rightarrow$

- direct image $\nu_*(L')$ of a line bundle on a (partial) normalization $\nu : S' \to S$

- $x^2 = z^{2m}$ normalized by two disjoint components $x = \pm z^m$

  z local coordinate on each

$$\text{Higgs field} \qquad \Phi = \begin{pmatrix} z^m & 0 \\ 0 & -z^m \end{pmatrix}.$$

- $x^2 = z^{2m+1}$ put $x = t^{2m+1}, z = t^2$, t local coordinate

- $f(t) = f_0(z) + tf_1(z)$ and

  $x(f_0(z) + tf_1(z)) = z^{m+1}f_1(z) + tz^m f_0(z)$

- Higgs field $\quad \Phi = \begin{pmatrix} 0 & z^{m+1} \\ z^m & 0 \end{pmatrix}.$

Prop: The nondegenerate critical locus C_d consists of Higgs bundles (V, Φ) where Φ vanishes at d distinct points with multiplicity one and with semisimple derivative.

- S has d ordinary double points
- $1 \leq d \leq 2g - 2$
- each C_d is a subintegrable system

- Φ vanishes on $D \Rightarrow \phi = \Phi/s$ is a $K(-D)$-twisted Higgs field.

  N.Nitsure, *Moduli space of semistable pairs on a curve*, Proc LMS, **62** (1991)

- $K(-D)$ spectral curve = normalization of $x^2 - q = 0$

- torus dimension $= 3g - 3 - d$

- = Prym variety of $K(-D)$ spectral curve

- Prym variety tangent space $H^1(\Sigma, K^{-1}(D))$

- tangent space of base $= H^1(\Sigma, K^{-1}(D))^* = H^0(\Sigma, K^2(-D))$

- Prym variety tangent space $H^1(\Sigma, K^{-1}(D))$
- tangent space of base $= H^1(\Sigma, K^{-1}(D))^* = H^0(\Sigma, K^2(-D))$
- base = Severi variety of nodal curves in total space of K
- tangent space $= H^0(K, \mathcal{I}_Z(\pi^* K^2))/(x^2 - q)$, Z = nodes
- $\cong H^0(\Sigma, K^2(-D))$

THE EXTREME CASE $d = 2g - 2$

- $\phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K(-D))$

- $\mathrm{tr}\, \phi^2 \in H^0(\Sigma, K^2(-2D)) \Rightarrow K^2(-2D)$ trivial

THE EXTREME CASE $d = 2g - 2$

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- $\operatorname{tr} \phi^2 \in H^0(\Sigma, K^2(-2D)) \Rightarrow K^2(-2D)$ trivial
- $K(-D)$ trivial $\Rightarrow$ automorphisms $\Rightarrow$ non-stable
- $U = K(-D)$ U^2 trivial – 2^{2g} choices

- spectral curve $S' =$ unramified double cover of Σ defined by $U \in H^1(\Sigma, \mathbf{Z}_2)$

- base of integrable system $= H^0(\Sigma, KU) \cong \mathbf{C}^{g-1}$

- S' fixed, so

$$C_{2g-2} = H^1(S', \mathcal{O}^*)^- \times H^0(\Sigma, KU)$$

- fixed point set of $(V, \Phi) \mapsto (V \otimes U, \Phi)$

EXAMPLE $g = 2$

- genus 2 curve Σ: $y^2 = z(z-1)(z-r)(z-s)(z-t)$

- $T^*\mathsf{P}^3 \subset \mathcal{M}$ open set, V stable

- affine coordinates (u_0, u_1, u_2) on P^3

- $\eta_0 du_0 + \eta_1 du_1 + \eta_2 du_2$ on $T^*\mathsf{P}^3$

- $F : \mathcal{M} \to H^0(\Sigma, K^2)$

  $$F(u, \eta) = (h_0 + h_1 z + h_2 z^2)\frac{dz^2}{y}$$

$$
\begin{aligned}
h_0 &= rst[\eta_0(u_0^2-1)+\eta_1(u_0u_1+u_2)+\eta_2(u_2u_0+u_1)]^2 - \\
&\quad st[\eta_0(u_0u_1-u_2)+\eta_1(u_1^2+1)+\eta_2(u_1u_2+u_0)]^2 + \\
&\quad 4rs(\eta_0u_0+\eta_1u_1)^2 - rt[\eta_0(u_0^2+1)+\eta_0(u_0u_1+u_2) \\
&\quad +\eta_2(u_2u_0-u_1)]^2 \\
h_1 &= t(u_0^2+u_1^2+u_2^2+1)[(\eta_0^2+\eta_1^2+\eta_2^2)+(\eta_0u_0+\eta_1u_1+\eta_2u_2)^2] + \\
&\quad st(u_0^2-u_1^2+u_2^2-1)[(\eta_0^2-\eta_1^2+\eta_2^2)-(\eta_0u_0+\eta_1u_1+\eta_2u_2)^2] + \\
&\quad 4r(u_0u_2-u_1)[\eta_0\eta_2+(\eta_0u_0+\eta_1u_1+\eta_2u_2)\eta_1] + \\
&\quad 4sr(u_2u_0+u_1)[\eta_2\eta_0-(\eta_0u_0+\eta_1u_1+\eta_2u_2)\eta_1] + \\
&\quad 4s(u_1u_2+u_0)[\eta_1\eta_2-(\eta_0u_0+\eta_1u_1+\eta_2u_2)\eta_0] + \\
&\quad 4rt(u_0u_1+u_2)[\eta_0\eta_1-(\eta_0u_0+\eta_1u_1+\eta_2u_2)\eta_2] \\
h_2 &= s[\eta_0(u_2u_0+u_1)+\eta_1(u_1u_2+u_0)+\eta_2(u_2^2-1)]^2 - \\
&\quad [\eta_0(u_2u_0-u_1)+\eta_1(u_1u_2+u_0)+\eta_2(u_2^2+1)]^2 - \\
&\quad t[\eta_0(u_0u_1+u_2)+\eta_2(u_1u_2-u_0)+\eta_1(u_2^2+1)]^2 \\
&\quad +4r(\eta_1u_1+\eta_2u_2)^2
\end{aligned}
$$

- $V = L \oplus L^*$: Kummer surface

- $t(s-1)(u_0^4 + u_1^4 + u_2^4 + 1) - 8(r(s-t+1) - s)u_0u_1u_2$
  $-2(st + t - 2s)(u_1^2u_2^2 + u_0^2) - 2(s-1)(2r-t)(u_2^2u_0^2 + u_1^2)$
  $+t(2r - (s+1))(u_0^2u_1^2 + u_2^2) = 0.$

F.Loray & V.Heu,, *Hitchin Hamiltonians in genus 2,* "Analytic and Algebraic Geometry" (A.Aryasomayajula et al (eds.)), Springer Nature Singapore Pte Ltd. and Hindustan Book Agency, (2017) 153–172. arXiv: 1506.02404v1

THE LOCUS C_1

- quadratic differential $\quad (z-r)(az+b)\dfrac{dz^2}{y}$

- ... double zero at ramification point $z=r$

- $z-r=s^2,\ s\in H^0(\Sigma,K^{1/2})$

THE LOCUS C_1

- quadratic differential $(z-r)(az+b)\dfrac{dz^2}{y}$
- ... double zero at ramification point $z=r$
- $z-r=s^2$, $s\in H^0(\Sigma,K^{1/2})$
- $K^2(-2D)\cong K$
- spectral curve S': $\tilde{x}^2=(az+b)$, $\tilde{x}\in H^0(S',\pi^*K^{1/2})$
- genus 4

- $y \in H^0(\Sigma, K^3), y/s \in H^0(\Sigma, K^{5/2})$
- $\tilde{y} = \tilde{x}y/s$, $\tilde{y} \in H^0(S', \pi^* K^3)$ $\tilde{y}^2 = z(z-1)(az+b)(z-s)(z-t)$
- ... genus 2 curve Σ_1, $\pi_1 : S' \to \Sigma_1$
- Prym$(S', \Sigma) \cong$ Jac(Σ_1), $V = \pi_* \pi_1^* L$

- integrable system C_1: 2-parameter family of abelian varieties
- ... base $(a, b) \in \mathbf{C}^2$
- = Jacobians of $y^2 = z(z-1)(z+b/a)(z-s)(z-t)$
- pencil of Kummer surfaces:

$$
\begin{aligned}
&t(s-1)(u_0^4+u_1^4+u_2^4+1) - 8(r(s-t+1)-s)u_0u_1u_2 \\
&-2(st+t-2s)(u_1^2u_2^2+u_0^2) - 2(s-1)(2r-t)(u_2^2u_0^2+u_1^2) \\
&+t(2r-(s+1))(u_0^2u_1^2+u_2^2) = 0.
\end{aligned}
$$

THE LOCUS C_2

- fixed points of involution:

  $(u_0, u_1, u_2, \eta_0, \eta_1, \eta_2) \mapsto (u_0, -u_1, -u_2, \eta_0, -\eta_1, -\eta_2)$

- two lines in P^3

- base multiples of $(z - r)\dfrac{dz^2}{y}$

- fibre elliptic curve:

$$y^2 = st(x^2 - 1)^2 + 4sx^2 - t(x^2 + 1)^2.$$

HECKE CURVES

- if $V = \pi_* L$ for a line bundle $(f_0(z), f_1(z))$ local section

- $x(f_0(z) + x f_1(z)) = z^m f_1(z) + x f_0(z) \Rightarrow$

$$\text{Higgs field} \qquad \Phi = \begin{pmatrix} 0 & z^m \\ 1 & 0 \end{pmatrix}.$$

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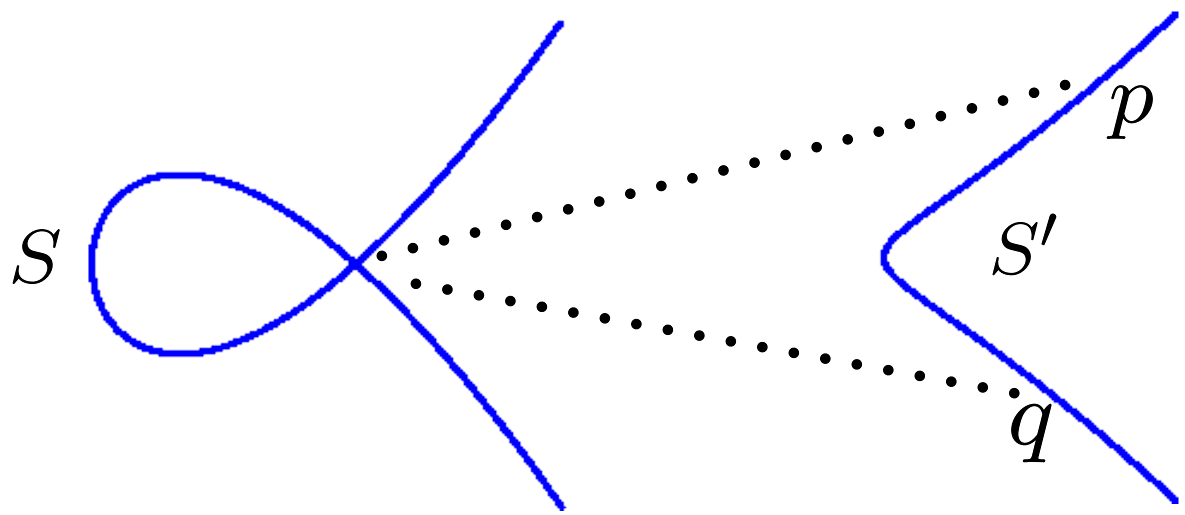

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- $+$ identification of fibres $L_p \cong L_q$
- $=$ copy of $\mathbf{C}^*$ in singular fibre
- fibre is compact

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- $+$ identification of fibres $L_p \cong L_q$
- $=$ copy of $\mathbf{C}^*$ in singular fibre
- fibre is compact
- what is its compactification?

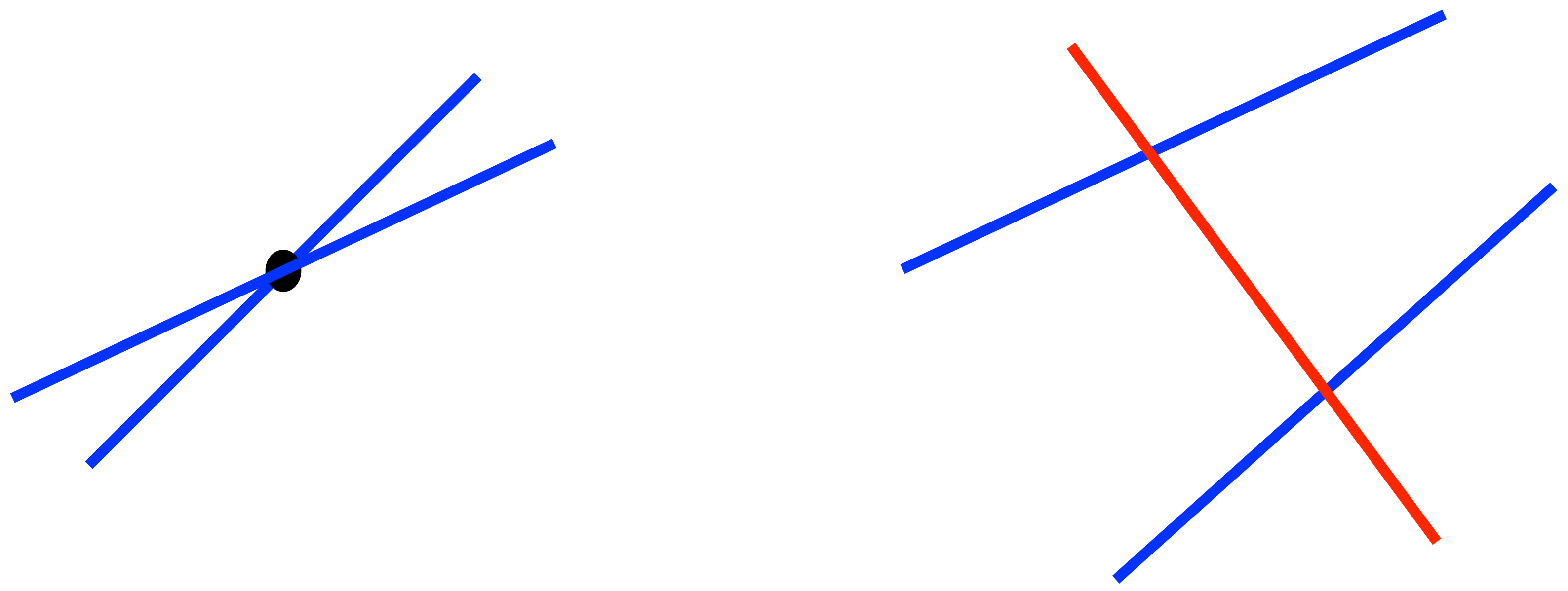
- projective bundle $\pi : \mathrm{P}(V) \to \Sigma$
- Higgs field $\Phi \in H^0(\Sigma, S^2V^* \otimes K)$
- $\sim$ section s of line bundle $H^2\pi^*K$ on $\mathrm{P}(V)$
- divisor curve C of eigenspaces
- if smooth $C \cong S =$ curve of eigenvalues

- Φ vanishes at $x \in \Sigma$
- $\Rightarrow C$ is reducible
- one component fibre $\pi^{-1}(x)$
- other $\cong S'$

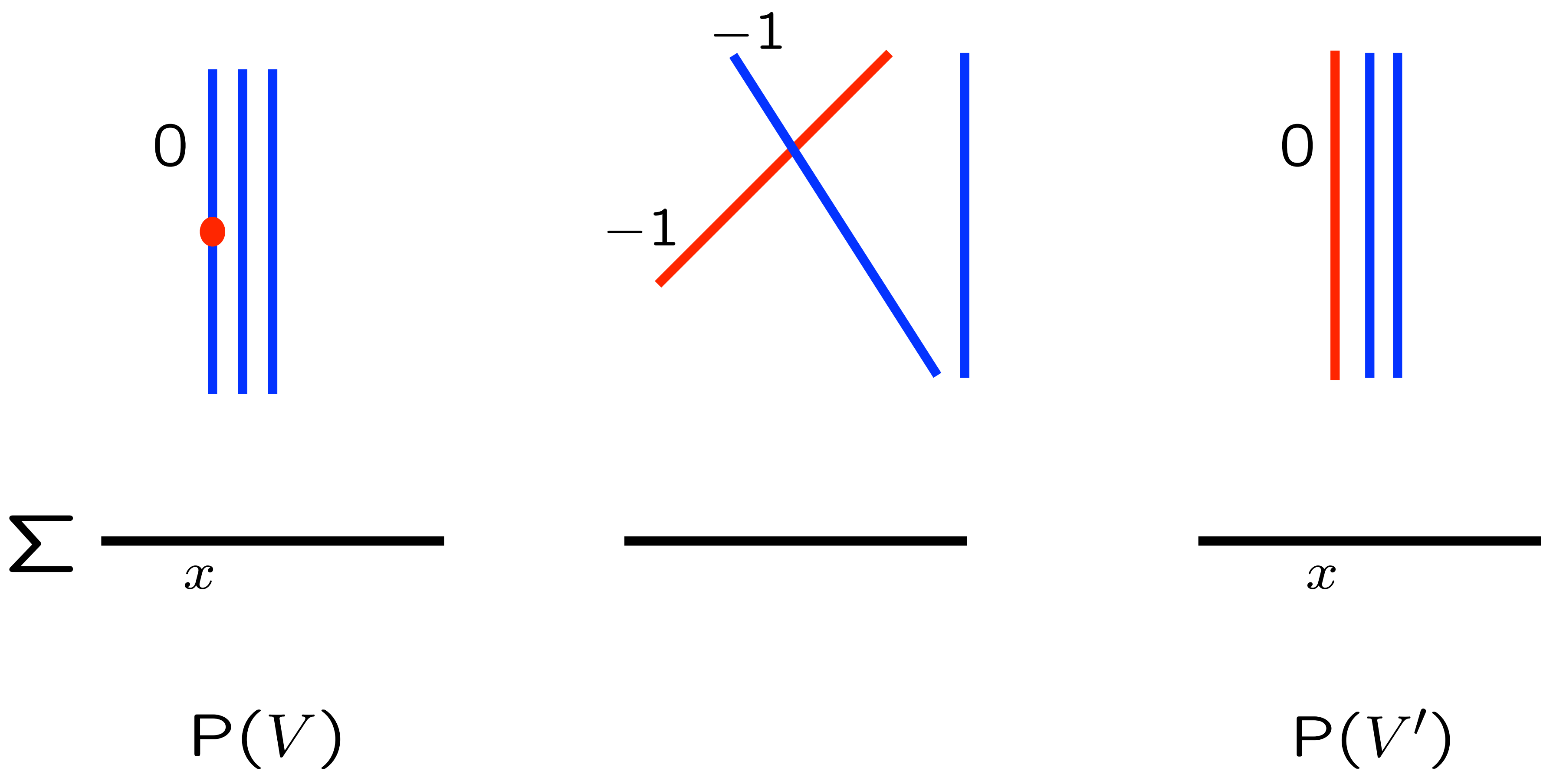
- projective bundle $\pi : \mathrm{P}(V) \to \Sigma$

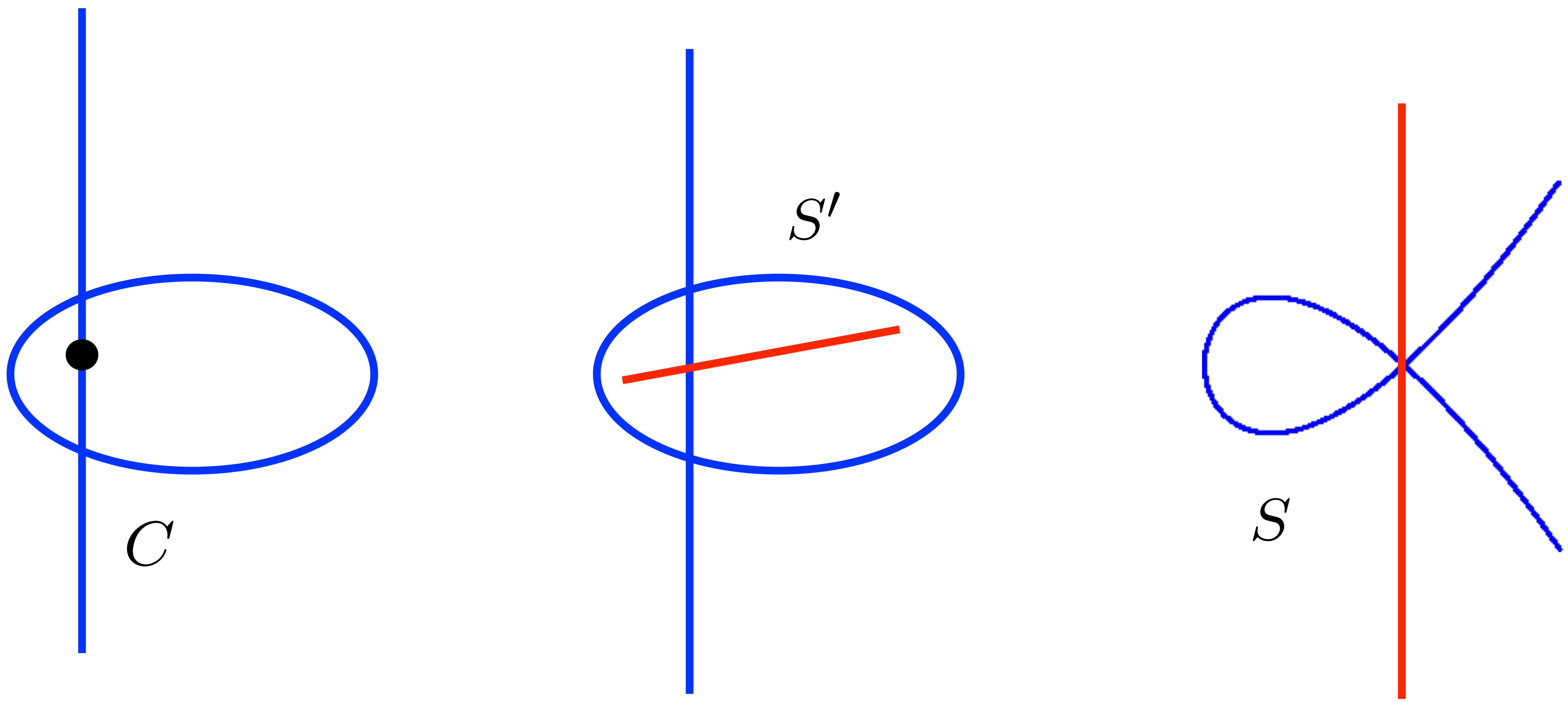
- algebraic surface

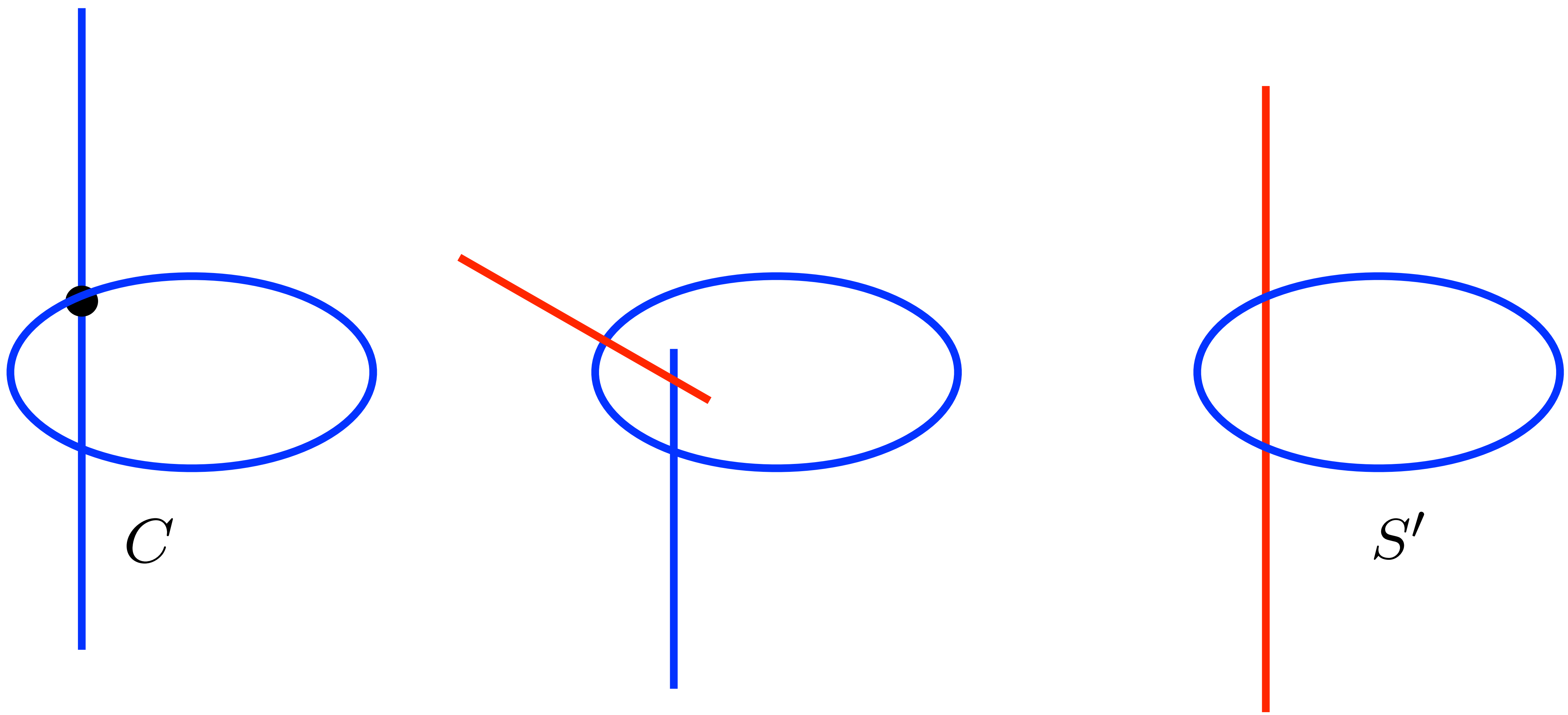
- blow up a point $\sim$ replace by P^1 of tangent directions



- Hecke transform = blowing up and down




C
S'
S



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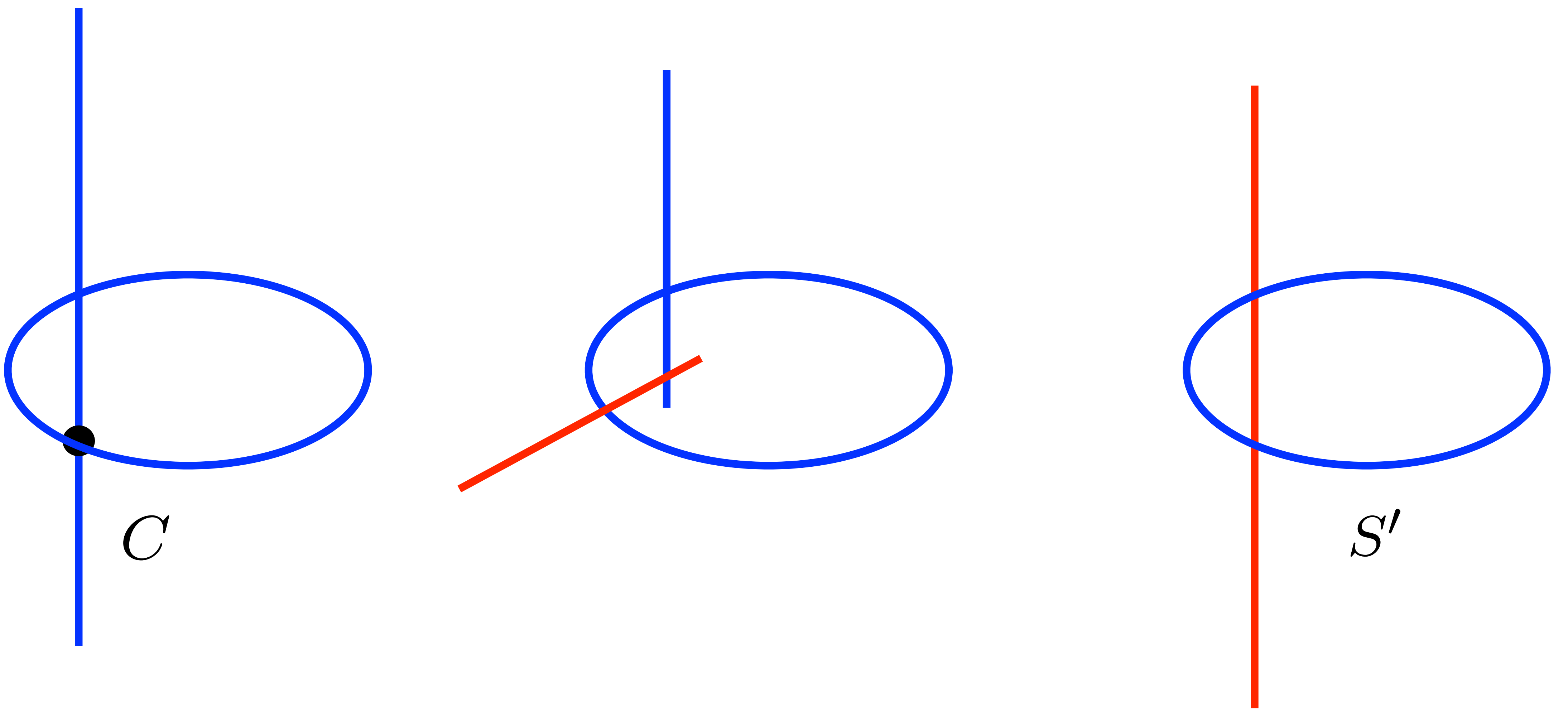
C
S'

BACK TO THE ELLIPSOID

- ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

- $\Sigma = \mathsf{P}^1, V$ trivial rank 2 bundle

- Φ has simple poles at $a, b, c \in \mathbf{C}$ and a double pole at ∞

$$\Phi = \frac{A dz}{z-a}+\frac{B dz}{z-b}+\frac{C dz}{z-c}+D dz$$

$$\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K(5)) = H^0(\Sigma, \mathrm{End}\, V(3))$$

A.Beauville, *Jacobiennes des courbes spectrales et systèmes hamiltoniennes complètement intégrables*, Acta Math. **164** 211-235 (1990).

- A, B, C and D nilpotent

- $\Rightarrow$ $\mathbf{C}^*$-action

- $\Phi \in H^0(\Sigma, \mathrm{End}\, V(3)) \Rightarrow \mathrm{tr}\, \Phi^2$ degree 6 polynomial $p(z)$

- spectral curve S hyperelliptic $x^2 = p(z)$

- critical locus: Φ vanishes at some point

- $\Phi = (z-a)\phi$, $\phi \in H^0(\Sigma, \mathrm{End}\, V(2))$

- $\mathrm{tr}\, \Phi^2 = (z-a)^2 q(z)$

- spectral curve S': $x^2 = q(z)$ elliptic.

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- $\mathrm{tr}\, \Phi^2 = (z-a)^2 q(z)$
- spectral curve S': $x^2 = q(z)$ elliptic.
- $\sim$ Euler's equations