Real Normalized Differentials: Degenerations and Applications

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Plan and introduction

- The goal of the talk is to present some results and conjectures that have arisen in the framework of an ongoing project (in part jointly with S.Grushevsky) devoted to applications of Whitham perturbation theory of the soliton equations to the study of geometry of the moduli spaces $\mathcal{M}_{g,n} = \{\Gamma, P_\alpha\}$ of *smooth* genus g algebraic curves with punctures
- Much of the recent progress in the study of tautological ring $R^*(\mathcal{M}_{g,n})$ has been motivated by Faber's conjectures inspired by breakthrough in the intersection theory of Mumford-Morita classes (Witten, Kontsevich, Okounkov).

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Algebraic-geometrical solutions of soliton equations.

General algebraic geometrical integration of soliton equations based on a concept of the Baker-Akhiezer functions was proposed at the end of seventies and is based on early theory of finite-gap solutions of the KdV equations (Novikov, Dubrovin, Matveev, Its, Lax, McKean,.....)
It can be seen as a map from the set of algebraic-geometrical data to solutions of soliton equations

$$\widehat{\mathcal{M}}_{g,n} = \{\Gamma, P_\alpha, z_\alpha\} \times J(\Gamma) \longmapsto u(t)$$

$$n = 1 \longmapsto \frac{3}{4} u_{yy} = (u_t - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx})_x \ \ (\mathrm{KP})$$

$$n = 2 \longmapsto 2D \text{ Toda}, \qquad n = 3 \longmapsto \text{BDHE}$$

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For the KP equation the solutions have the form

$$u(x,y,t) = 2\partial_x^2 \ln \theta(Ux + Vy + Wt + Z|B), \tag{1}$$

where $B = \{B_{ij} = B_{ji},\ ImB > 0$ is the matrix of b-periods of normalized holomorphic differentials on Γ and

$$\theta(z|B) = \sum_{m\in\mathbb{Z}_g} e^{2\pi i(z,m)+\pi i(m,Bm)}, \quad z = (z_1, \dots, z_g)$$

Known algebraic-geometrical applications

Famous Novikov's conjecture:

that an indecomposable symmetric matrix B with positive definite imaginary part is the matrix of a smooth genus g algebraic curve if and only if there exist vectors U $(\neq 0)$, V and W such that u given by (1) satisfies the KP equation.

proven by Shiota (86) until recently has remained the most effective solution of the classical Riemann-Schottky problem on characterization of Jacobians of algebraic curves.

Much stronger characterization was conjectured by Welter's : *undecomposable principally polarized abelain variety X is the Jacobian of a smooth genus g algebraic curve iff its Kummer variety $K(X)$ admits one (!) trisecant line.*

- Trisecant is a projective line intersecting Kummer variety $K(B)$ at three points.
- For any B the corresponding Kummer variety is defined as image of the, so-called Kummer map

$$X = \mathbb{C}^g/(n + Bm) \longmapsto K(X) \subset \mathbb{CP}^{2^g-1}$$

- Three particular cases of the Welter's conjecture was proved (Kr 2005-2008) using three basic soliton equations (KP,2D Toda,BDHE)
- Characterization problem of Prym varieties has required to introduce a new integrable equation – discrete analog of Novikov-Veselov equation (Grushevsky-Kr, 2008)

A perturbation theory

- In any perturbation theory integrals become adiabatic integrals
- The universal Whitham hierarchy describes slow modulations of integrals of soliton equations, i.e. a hierarchy of commuting flows on $\widehat{\mathcal{M}}_{g,n}$.
- They are integrable by generalized hodograph transform, i.e. their solutions are given explicitly in the implicit function form
- Example: Riemann-Hopf equation $u_t = uu_x$, whose solutions are given by the equation $u = f(x + ut)$, ($f(x)$ is an arbitrary function of one variable.

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Vanishing properties of $\mathcal{M}_{g,k}$

The moduli spaces $\mathcal{M}_{g,k}$ of *smooth* genus g Riemann surfaces with punctures have curious vanishing properties.

- Diaz' theorem (1986):
  There does not exist a complete (complex) cycle in $\mathcal{M}_g$ of dimension greater than $g-2$

Note, that is the upper bound. The know constructions give complete cycles of dimension of order $\log_3 g$, only.

- Looijenga theorem (1995):
  The tautological ring $R^*(\mathcal{M}_{g,k})$ vanishes in dimensions greater then $g-2+k$

The tautological ring $R^*(\mathcal{M}_{g,k})$ is generated by classes

$$\psi_i = c_1(L_i), \;\; \kappa_i = p_*(\psi_1^{i+1}) \in H^*(\mathcal{M}_g).$$

Here L_j are canonical line bundles over $\mathcal{M}_{g,k}$.

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Conjectural geometric explanations

Widely accepted by experts "geometric explanation" of vanishing properties of $\mathcal{M}_{g,k}$ is the existence of its stratification by certain number of affine strata or the existence of a cover of $\mathcal{M}_{g,k}$ by certain number of open affine sets.

Historically, Arbarello first realized that a stratification of $\mathcal{M}_g$ could be useful for a study of its geometrical properties. He studied the stratification (known already for Rauch)

$$\mathcal{W}_2 \subset \mathcal{W}_3 \subset \cdots \subset \mathcal{W}_{g-1} \subset \mathcal{W}_g = \mathcal{M}_g,$$

where $\mathcal{W}_n$ if the locus of curves having a Weierstrass point of order at most n.

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Alternative geometric explanation

An alternative approach for geometrical explanation of the vanishing properties of $\mathcal{M}_{g,k}$ was proposed by S. Grushevsky and the author. It was motivated by certain constructions of the Whitham perturbation theory of integrable systems. The key elements of the alternative geometrical explanation are:

- the moduli space $\mathcal{M}_{g,k}^{(n)}$, $n = (n_1, \ldots, n_k)$ of smooth genus g Riemann surfaces with fixed singular parts of meromorphic differentials of order $n_\ell + 1$ and pure imaginary residues at the marked points p_ℓ is the total space of a *real-analytic* foliation, whose leaves $\mathcal{L}$ are locally smooth *complex subvarieties* of real codimension $2g$;
- on $\mathcal{M}_{g,k}^{(n)}$ there is an ordered set of $(\dim_{\mathbb{R}} \mathcal{L})$ continuous functions, which restricted onto the leaves of the foliation are piecewise harmonic. Moreover, the first of these function restricted onto $\mathcal{L}$ is a subharmonic function.

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Recall that:
the singular part of a meromorphic differential at a point p on a Riemann surface C is the equivalence class of meromorphic differentials ω *in a neighborhood of p, with the equivalence* $\omega \sim \omega'$ *if and only if* $\omega' - \omega$ *is holomorphic at p.*

The datum of a singular part with no residue is equivalent to the datum of a jet of a local coordinate, in which the meromorphic differential can be written in the standard form as $z^{-n_\alpha - 1} dz$.

Results

- Proof of Arbarello's conjecture

Theorem

Any compact complex cycle in $\mathcal{M}_g$ of dimension $g - n$ must intersect $\mathcal{W}_n$.

- New upper bound for dimensions of complete (complex) cycles in the moduli space $\mathcal{M}_g^{ct}$ of stable curves of compact type.

Theorem

There do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ having non empty intersection with $\mathcal{M}_g$ of dimension greater than $g - 1$.
For $g \geq 2$ the maximum dimension of complete complex subvarieties in $\mathcal{M}_g^{ct}$ is $\frac{3}{2}g - 2$.

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Previously known bounds

- Diaz:
  there is no compact cycle in $\mathcal{M}_g^{ct}$ of dimension greater that $2g-3$.
- Keel and Sadun:
  for $g \geq 3$ there do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ of dimension greater than $2g-4$.

The proof is by easy induction arguments starting from the base $g=3$. The proof of the base statement is a corollary of remarkable vanishing result:

- *there do not exist a complete complex subvarieties of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of codimension* g.

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Real normalized differentials

The foliation structure arises through identification of $\mathcal{M}_{g,k}^{(n)}$ with the moduli space of curves with fixed *real-normalized* meromorphic differential. By definition a real normalized meromorphic differential is a differential whose period over any cycle on the curve is *real*.

As easily follows from the positive-definiteness of the imaginary part of the period matrix

Lemma

For any $X \in \mathcal{M}_{g,k}^{(n)}$ there exists a unique meromorphic differential $\Psi_X \in H^0(C, K_C + \sum_\ell (n_\ell + 1)p_\ell)$ with prescribed singular part σ_ℓ at p_ℓ, and with all periods real.

Foliation

Definition

A leaf $\mathcal{L}$ of the foliation on $\mathcal{M}_{g,k}^{(n)}$ defined to be the locus along which the periods of the corresponding differentials remain (covariantly) constant.

The leaves $\mathcal{L}$ of the foliation can be regarded as a generalization of the Hurwitz spaces of $\mathbb{P}^1$ covers.

It is basic fact of the Whitham theory:

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Coordinates along a leaf

A set of holomorphic coordinates on $\mathcal{M}_{g,k}^{(n)}$ are "critical" values of the corresponding abelian integral $F(p) = c + \int^p \Psi$, $p \in \Gamma$:

At the generic point, where zeros q_s of Ψ are distinct, the coordinates on $\mathcal{L}$ are the evaluation of F at these critical points:

$$\varphi_s = F(q_s), \quad \Psi(q_s) = 0, \quad s = 0, \ldots, d = \dim \mathcal{L}, \tag{2}$$

normalized by the condition $\sum_s \varphi_s = 0$.

A direct corollary of the real normalization is the statement that:

- *imaginary parts $f_s = \Im \varphi_s$ of the critical values depend only on labeling of the critical points*

They can be arranged into decreasing order

$$f_0 \geq f_1 \geq \cdots \geq f_{d-1} \geq f_d.$$

After that f_j can be seen as a well-defined continuous function on $\mathcal{M}_{g,k}^{(n)}$, which restricted onto $\mathcal{L}$ is a piecewise harmonic function. Moreover, f_0 restricted onto $\mathcal{L}$ is a *subharmonic function*, i.e, f_0 *has no local maximum on $\mathcal{L}$ unless it is constant*.

Elliptic families of solutions of the KP equation

Let $\mathcal{M}_{g,1}^{\leq n,\tau} := \{C, \Psi, \Psi_1\}$ be the moduli space of smooth genus g algebraic curves C with a pair of real normalized differentials having pole of order *at most* $n+1$ at one marked point p_0, whose singular parts σ and σ_1 satisfy the equation $\sigma = \tau\sigma_1$ with $\tau \in \mathbb{C}$, $Im\,\tau > 0$.

The (local) period map is defined as

$$\Pi : \mathcal{M}_{g,1}^{\leq n,\tau} \to \left(\oint_{\gamma_i} \Psi, \oint_{\gamma_i} \Psi_1 \right) \in \mathbb{R}^{2g} \oplus \mathbb{R}^{2g}$$

Note, that if $\Pi(C, \Psi, \Psi_1) \in \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g}$ then the holomorphic differential $dz = \Psi - \tau\Psi_1$ defines $(N:1)$ map

$$z : C \to E_\tau = \mathbb{C}/(1, \tau)$$

where

$$N = \sum_{i=1}^{g} \oint_{a_i} \Psi \oint_{b_i} \Psi_1 - \oint_{a_i} \Psi_1 \oint_{b_i} \Psi$$

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Idea of the proof.

For any $Y \subset \mathcal{M}_g$ the preimage of Y under the forgetful map $\mathcal{M}_{g,1}^{\leq n,\tau} \to \mathcal{M}_g$ will be denoted by $Y^{\leq n,\tau}$. It is of dimension $\dim Y + n + 1$

Lemma

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If the statement of the theorem holds for $g' \leq g-1$, then simple dimension counting shows that a complete cycle $Z \subset \mathcal{M}_g^{ct}$ has dimension at most g.

If Z is of dimension g, then by Diaz' theorem it intersects at least one of the divisors $\mathcal{M}_{i,1} \times \mathcal{M}_{g-i,1} \subset \mathcal{M}_g^{ct}$.

Lemma

Let $Z \in \mathcal{M}_g^{ct}$ be a complete cycle of dimension g. Then the period map

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is injective at a generic point.

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Contradiction !

If the statement of the theorem holds for $g' \leq g-1$, then simple dimension counting shows that a complete cycle $Z \subset \mathcal{M}_g^{ct}$ has dimension at most g.

If Z is of dimension g, then by Diaz' theorem it intersects at least one of the divisors $\mathcal{M}_{i,1} \times \mathcal{M}_{g-i,1} \subset \mathcal{M}_g^{ct}$.

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Limits of real-normalized differentials

In the recent paper by S.Grushevsky, C. Norton and Kr a "compactification" of the moduli spaces $\mathcal{M} := \mathcal{M}_{g,k}^{\leq n}$ was constructed:

Theorem (Rougth version)

For any sequence of $X_k = (C_k, \Psi_k) \in \mathcal{M}$ such that C_k converges to a stable curve C, there exists a subsequence along which there is a non-zero ! multi-scale limit of Ψ_k on each irreducible component C^v of the normalization $\tilde{C}$ of C.

The multi scale limit means the following:

- (i) $\tilde{C} = \mathcal{C}^{(0)} \cup \mathcal{C}^{(1)} \cdots \cup \mathcal{C}^{(L)}$;
- (ii) there is a set of decreasing scales $\lim \mu_k^{(\lambda+1)}/\mu_k^{(\lambda)} \to 0$;
- on each irreducible component $C^v \in \mathcal{C}^{(\lambda)}$ there is a non-zero real normalized differential $\Phi^v \neq 0$ such that $\mu^{(\lambda)}\Psi_k \to \Phi^v$ uniformly on any compact containing no preimages of the nodes.

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Plumbing construction

Let $q_1, q_2 \in C$ (with C a possibly disconnected Riemann surface) be two distinct points. Let z_1, z_2 be local coordinates on C near q_1, q_2 such that $z_j(q_j) = 0$ and the nighborhooods $V_j := \{|z_j| < 1\} \subset C$ of q_j. are disjoint. Then for any $s \in \mathbb{C}$ with $|s| < 1$ we denote $U_j = U_j^s := \{|z_j| < \sqrt{|s|}\} \subset V_j$ the corresponding disks, and denote $\gamma_j := \partial U_j$ their boundary circles, which we orient negatively with respect to U_j. The *standard plumbing C_s with parameter s* is the Riemann surface

$$C_s := [C \setminus (U_1 \sqcup U_2)] / (\gamma_1 \sim \gamma_2)$$

where γ_1 is identified with γ_2 via the diffeomorphism $I(z_1) := s/z_1$. The structure of a Riemann surface on C_s is defined by saying that a function on C_s is holomorphic, if it is holomorphic on the complement of the *seam* γ (the image of γ_1 and γ_2) and continuous along the seam.

Plumbing coordinates

For a stable curve $C \in \overline{\mathcal{M}}_g$, its *dual graph* Γ has vertices v that correspond to *normalizations* C^v of irreducible components of C, edges $|e|$ that correspond to nodes $q_{|e|}$ of C, oriented edges e that correspond to preimages q_e of the nodes (as points on the normalization $\tilde{C}$ of C). So E_v is the set of all preimages of the nodes that are contained in C^v, and q_e and q_{-e} are the two preimages on $\tilde{C}$ of a node $q_{|e|}$ of C.

The plumbing coordinates in the neighborhood of C are coordinates u on the Cartesian product of the moduli spaces for $(C^v, \{q_e, e \in E_v\})$ together with a set of gluing parameters $s_{|e|}$ for each node.

The jump problem

The jump problem is posed on $\widehat{C}_{u,s}$ — a Riemann surface with $\#E$ boundary components. The initial data is a set ϕ of complex-valued smooth 1-forms ϕ_e on γ_e, which we call *jumps*. The jumps are required to satisfy $\phi_e = -I_e^*(\phi_{-e})$ and $\int_{\gamma_e} \phi_e = 0$ for all $e \in E$. The *jump problem* is to find a holomorphic 1-form ω_s on the interior of $\widehat{C}_{u,s}$ that extends continuously to every boundary component γ_e of $\widehat{C}_{u,s}$, and such that the boundary extensions have jumps ϕ_e, i.e. satisfy for any e the equation

$$\omega_s|_{\gamma_e} - I_e^* \left(\omega_s|_{\gamma_{-e}}\right) = \phi_e.$$

Theorem

There exists a constant t independent of u, such that for any $|s| < t$ and any ϕ, the jump problem has a unique solution ω_s on $\widehat{C}_{u,s}$ satisfying

- $\int_{\gamma_e} \omega = 0$ *for any e;*
- $\int_{\gamma} \omega \in \mathbb{R}$ *for any cycle $\gamma \in H_1(C_u, \mathbb{Z})$.*

Moreover, if the initial data are of the form $\phi_e := (f_e - I_e^* f_{-e})\,|_{\gamma_e^{s_e}}$ where $f_e dz_e$ is a 1-forms in V_e, then there exists a constant M such that

$$||\omega_s||_{\widehat{C}_s^v} \leq M|f|(\sqrt{|s_v|})^{\mathrm{ord}\, f+1}, \quad |s_v| := \max_{e \in E_v} |s_e|$$

Approximation versus gluing

A meromorphic differential Φ on $\widehat{C}_s$ (which is the shorthand for a collection of meromorphic differentials Φ^v on $\widehat{C}_s^v$) glues to define a meromorphic differential on C_s if and only if

$$\Phi^{v(e)}\Big|_{\gamma_e} = I_e^*\left(\Phi^{v(-e)}\Big|_{\gamma_{-e}}\right) \tag{3}$$

for all e.

Of course not every differential Φ on $\widehat{C}_s$ satisfies (3) and glues to a differential on C_s. One standard setup is for differentials with simple poles at preimages of the nodes, with opposite residues. Choosing coordinate z_e near q_e such that locally $\Phi^{v(e)} = a_e dz_e/z_e$, with $a_e = -a_{-e}$, and performing plumbing in these coordinates, one constructs a glued differential on C_s. However, since the local coordinates z_e depend on the differential, it is hard to ensure from this viewpoint that all suitable differentials on all smooth Riemann surfaces in a neighborhood can be obtained in this way.

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Our approach is direct and analytic. We start with any collection of fixed local coordinates z_e near q_e (for any u), and thus with fixed plumbing coordinates on the moduli space. Given any Φ on $\widehat{C}_s$, we will subtract from it another differential ω on $\widehat{C}_s$ such that their difference satisfies (3), and thus defines a differential on C_s. The condition for ω must then be that its "jumps" on γ_e are the same as for Φ, and we construct it by explicitly solving the jump problem.