On the recursion operators for the integrable equations

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Outline

- factorized representation $R = L_1^{-1} L_2$ of the recursion operators with weak non-localities
- method for constructing the operators L_1 and L_2
- operators L_1 and L_2 and the Lax pairs

The problem of constructing the recursion operators for the integrable equations has been investigated by many authors. Several methods are worked out to study the task. Some of them use the Lax representation. This way is very effective when the Lax pair is known. If it is not the case then it is reasonable to study directly the defining equation $\frac{d}{dt}R = [F^*, R]$, where R is the recursion operator and F^* is the linearization operator. To solve the equation the most authors use the multi-Hamiltonian approach. Their basic goal is to find two Hamiltonian operators H_1 and H_2 to the given equation. Then the recursion operator is given by the following formula $R = H_2 H_1^{-1}$. In the present article we concentrate on the alternative method for constructing the recursion operator which is based on the symmetries and the formula $R = L_1^{-1} L_2$

We begin with an integrable evolutionary type equation

$$u_t = f(u, u_1, u_2, ..., u_k), \qquad u_j = D_x^j u. \tag{1}$$

admitting a hierarchy of the symmetries

$$u_\tau = g(u, u_1, u_2, ..., u_n). \tag{2}$$

Symmetry means that $\frac{d}{d\tau} f = \frac{d}{dt} g$. Assume that Eq. (1) admits a weakly non-local recursion operator of the form (**Maltsev A Ya and Novikov S P 2001** *Phys. D* 156(1-2) 53–80)

$$R = R_0 + \sum_{i=1}^{m} g^{(k_i)} D_x^{-1} h^{(i)} \tag{3}$$

where R_0 is a differential operator. The non-local part consists of the combinations of the generators of the symmetries $u_{\tau_j} = g^{(j)}$ and the variational derivatives $h^{(j)}$ of the conserved densities.

Recall that operator R converts a symmetry into another symmetry.
Theorem. *Recursion operator*

$$R = R_0 + \sum_{i=1}^{m} g^{(k_i)} D_x^{-1} h^{(i)}$$

is represented as a ratio

$$R = L_1^{-1} L_2$$

of two differential operators L_1 and L_2.
As L_1 we take a differential operator such that the fundamental set of solutions to the equation $L_1 g = 0$ coincides with $g^{(k_1)}, g^{(k_2)}, ..., g^{(k_m)}$
Then it is easily proved that $L_2 := L_1 R$ is a differential operator.

We call the symmetries $g^{(k_1)}, g^{(k_2)}, ..., g^{(k_m)}$ defining the non-local part of the recursion operator

$$R = R_0 + \sum_{i=1}^{m} g^{(k_i)} D_x^{-1} h^{(i)}$$

the seed symmetries.
Below we give the set of seed symmetries for some equations
1) KdV equation $S = \{g^{(1)} = u_x\}$
2) Kaup-Kupershmidt type equations $S = \{g^{(1)} = u_x,\ g^{(2)} = u_t\}$
3) Krichever-Novikov Eq. for R_1, $S = \{g^{(1)} = u_x,\ g^{(2)} = u_t\}$
4) Krichever-Novikov Eq. R_2, $S = \{g^{(1)} = u_x,\ g^{(2)} = u_t,\ g^{(3)} = u_\tau\}$
5) Harry Dym equation $S = \{g^{(2)} = u_t\}$
Thus we have only four different possible choices for S. And therefore four choices for the operator L_1.

Having S we can easily find the operator L_1 up to some additional factor. It is given by the following determinant

$$L_1 U = \rho \begin{vmatrix} D_x^m g^{(k_1)} & D_x^{m-1} g^{(k_1)} & \dots & g^{(k_1)} \\ \dots & \dots & \dots & \dots \\ D_x^m g^{(k_m)} & D_x^{m-1} g^{(k_m)} & \dots & g^{(k_m)} \\ D_x^m U & D_x^{m-1} U & \dots & U \end{vmatrix}, \tag{4}$$

where ρ is a function of the dynamical variables u, u_1, ...
As soon as L_1 is found we can start to search L_2. It can be verified that the order of the operator L_2 is determined as follows

$$m_2 = m + ord\, g^{(k_{m+1})} - ord\, g^{(k_1)}. \tag{5}$$

where m is the order of the operator L_1.

Recursion operator R for the equation

$$u_t = f(u, u_1, u_2, ..., u_k)$$

satisfies the equation

$$\frac{d}{dt}R = [F^*, R],$$

where F^* is the linearization operator

$$F^* = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial u_1}D_x + \frac{\partial f}{\partial u_2}D_x^2 + ... + \frac{\partial f}{\partial u_k}D_x^k.$$

Therefore by using $R = L_1^{-1}L_2$ we derive the equation

$$\frac{d}{dt}L_1^{-1}L_2 = \left[F^*, L_1^{-1}L_2\right]$$

which implies

$$\frac{d}{dt}(L_2)L_2^{-1} + L_2F^*L_2^{-1} = \frac{d}{dt}(L_1)L_1^{-1} + L_1F^*L_1^{-1} =: A.$$

Lemma. *Operator A defined due to the formula*

$$A = \frac{d}{dt}(L_1)L_1^{-1} + L_1F^*L_1^{-1}$$

is a differential operator.

Actually the operators L_1 and L_2 satisfy one and the same equation

$$\frac{d}{dt}(L_j) = AL_j - L_jF^*, \quad j = 1, 2.$$

From this equation we find L_2. Its order m_2 is given by the formula $m_2 = m + ord\, g^{(k_{m+1})} - ord\, g^{(k_1)}$.

Finally we obtain the recursion operator $R = L_1^{-1}L_2$.

Remark. For incorrect choice of the set of seed symmetries S the appropriate L_2 does not exist.

Examples

Note that the discussed method does not use the Lax representation. As an illustrative example we take the KdV equation

$$u_t = u_3 + uu_1. \tag{6}$$

Among the potential values of the parameter m we first choose $m = 1$ and take $S = \{u_x\}$. Thus the first pretender for L_1 is as follows

$$L_1 U = \rho \begin{vmatrix} u_2 & u_1 \\ U_1 & U \end{vmatrix}. \tag{7}$$

For the sake of simplicity we put $\rho = -1$, thus

$$L_1 = u_1 D_x - u_2. \tag{8}$$

Find the linearization operator

$$F^* = D_x^3 + uD_x + u_1 \tag{9}$$

Evaluate the operator A due to formula $A = \frac{d}{dt}(L_1)L_1^{-1} + L_1F^*L_1^{-1}$
The answer is

$$A = D_x^3 - \frac{3u_2}{u_1}D_x^2 + \left(u + \frac{3u_2^2}{u_1^2}\right)D_x + 3\left(\frac{u_4}{u_1} + u_1 - \frac{u_2u_3}{u_1^2}\right). \tag{10}$$

According to the formula for the orders

$$m_2 = m + ord\, g^{(k_{m+1})} - ord\, g^{(k_1)} = 1 + 3 - 1 = 3$$

the corresponding operator L_2 should be of the third order

$$L_2 = \beta^{(3)} D_x^3 + \beta^{(2)} D_x^2 + \beta^{(1)} D_x + \beta^{(0)}. \tag{11}$$

We find the unknown coefficients from the equation

$$\frac{d}{dt}(L_2) = AL_2 - L_2 F^*, \tag{12}$$

The answer is:

$$L_2 = u_1 D_x^3 - u_2 D_x^2 + \frac{2}{3} u u_1 D_x + u_1^2 - \frac{2}{3} u u_2. \tag{13}$$

Thus we get well-known (Gardner, Greene, Kruskal and Miura, 1974) $R = L_1^{-1} L_2 = D_x^2 + \frac{2}{3} u + \frac{1}{3} u_1 D_x^{-1}$.
Recall that the usual representation of R through the Hamiltonian operators $H_1 = D_x$, $H_2 = D_x^3 + 4uD_x + 2u_1$ is as follows

$$R = H_2 H_1^{-1}. \tag{14}$$

Example 2

As the second illustrative example we consider the Kaup-Kupershmidt equation

$$u_t = u_5 + 10uu_3 + 25u_1u_2 + 20u^2u_1. \tag{15}$$

In what follows we will need in its linearization operator

$$F^* = D_x^5 + 10uD_x^3 + 25u_1D_x^2 + \left(25u_2 + 20u^2\right)D_x + 10u_3 + 40uu_1. \tag{16}$$

In order to find the set S of the seed symmetries for the equation (15) we have to examine the possible cases $S_1 = \{u_x\}$, $S_2 = \{u_t\}$, $S_3 = \{u_x, u_t\}$, $S_4 = \{u_x, u_t, u_\tau\}$, where $u_\tau = g^{(3)}$ is the next symmetry of the Kaup-Kupershmidt equation, it is of the seventh order.

Examine the possible cases $S_1 = \{u_x\}$, $S_2 = \{u_t\}$, $S_3 = \{u_x, u_t\}$, $S_4 = \{u_x, u_t, u_\tau\}$. We started with the case $S = S_1$. For the corresponding L_1 we found the operator A, then since $m_2 = 1 + 5 - 1 = 5$ we searched a fifth order differential operator L_2 satisfying the equation $\frac{d}{dt} L_2 = AL_2 - L_2 F^*$ and observed that such operator does not exist. In a similar way we have verified that the case $S = S_2$ also does not fit.

Then we passed to the case $S = S_3$ and succeeded. Operator L_1 is found from the relation

$$L_1 U = \begin{vmatrix} u_3 & u_2 & u_1 \\ u_{2t} & u_{1t} & u_t \\ U_2 & U_1 & U \end{vmatrix} \tag{17}$$

and is of the form

$$L_1 = \alpha D_x^2 + \beta D_x + \gamma, \tag{18}$$

$$L_1 = \alpha D_x^2 + \beta D_x + \gamma,$$

where

1. $\alpha = u_2u_5 + 10uu_2u_3 - u_1u_6 - 35u_1^2u_3 - 10uu_1u_4 - 40uu_1^3,$
2. $\beta = 10uu_1u_5 - u_3u_5 + 45u_1^2u_4 - 10uu_3^2 + 120uu_1^2u_2 + 40u_1^4 + 60u_1u_2u_3 + u_1u_7,$
3. $\gamma = -120uu_1u_2^2 - 10uu_2u_5 + u_3u_6 + 35u_1u_3^2 - 60u_2^2u_3 - 40u_1^3u_2 - u_2u_7 + 10uu_3u_4 + 40uu_1^2u_3 - 45u_1u_2u_4.$

Then we look for the operator

$$A = \sum_{j=0}^{5} A^{(j)} D_x^j \tag{19}$$

from the equation:

$$\frac{d}{dt}(L_1) = AL_1 - L_1F^*. \tag{20}$$

At the next step we look for the eighth order differential operator L_2, $m_2 = m + ord\, g^{(k_{m+1})} - ord\, g^{(k_1)} = 2 + ord\, g^{(3)} - ord\, g^{(1)} = 2 + 7 - 1 = 8$

$$L_2 = \sum_{k=0}^{8} b^{(k)} D_x^k \tag{21}$$

from the equation $\frac{d}{dt}(L_2) = AL_2 - L_2F^*$. It turned out that such operator does exist. Omitting the computations we give only the answer:

$$L_2 = \sum_{k=0}^{8} b^{(k)} D_x^k, \quad L_1 = \alpha D_x^2 + \beta D_x + \gamma$$

1. $b^{(8)} = \alpha, \quad b^{(7)} = \beta, \quad b^{(6)} = 12u\alpha + \gamma, \quad b^{(5)} = 60u_1\alpha + 12u\beta,$
2. $b^{(4)} = (133u_2 + 36u^2)\alpha + 48u_1\beta + 12u\gamma,$
3. $b^{(3)} = (169u_3 + 264uu_1)\alpha + (85u_2 + 36u^2)\beta + 36u_1\gamma,$
4. $b^{(2)} = (132u_4 + 394uu_2 + 381u_1^2 + 32u^3)\alpha + (84u_3 + 192uu_1)\beta + (49u_2 + 36u^2)\gamma,$
5. $b^{(1)} = (63u_5 + 304uu_3 + 852u_1u_2 + 240u^2u_1)\alpha + (48u_4 + 202uu_2 + 189u_1^2 + 32u^3)\beta + (35u_3 + 120uu_1)\gamma,$
6. $b^{(0)} = (17u_6 + 122uu_4 + 444u_1u_3 + 324u_2^2 + 192u^2u_2 + 368uu_1^2)\alpha + (15u_5 + 102uu_3 + 272u_1u_2 + 144u^2u_1)\beta + (13u_4 + 82uu_2 + 69u_1^2 + 32u^3)\gamma.$

Let us find the required recursion operator R. We know that the order of its differential part is $6 = m_2 - m$ and it has two non-local terms, therefore it is of the form:

$$R = \sum_{j=0}^{6} r^{(j)} D_x^j + u_1 D_x^{-1} h^{(1)} + u_t D_x^{-1} h^{(2)}. \tag{22}$$

The relation $L_1 R = L_2$ allows us to find all of the functional parameters in (22):

1. $r^{(6)} = 1, \quad r^{(5)} = 0, \quad r^{(4)} = 12u,$
2. $r^{(3)} = 36u_1, \quad r^{(2)} = 49u_2 + 36u^2,$
3. $r^{(1)} = 35u_3 + 120uu_1, \quad r^{(0)} = 13u_4 + 82uu_2 + 69u_1^2 + 32u^3,$
4. $h^{(1)} = 2u_2 + 8u^2, \quad h^{(2)} = 2.$

So we have the final form of R coinciding with that found earlier by Gürses M, Karasu A and Sokolov V V (1999):

$$\begin{aligned} R = D_x^6 + 12uD_x^4 + 36u_1 D_x^3 + \left(49u_2 + 36u^2\right) D_x^2 + \\ + \left(35u_3 + 120uu_1\right) D_x + 13u_4 + 82uu_2 + 69u_1^2 + \\ +32u^3 + u_1 D_x^{-1}(2u_2 + 8u^2) + 2u_t D_x^{-1}. \end{aligned}$$

Example 3

The next example is connected with the Krichever-Novikov equation

$$u_t = u_3 - \frac{3}{2}\frac{u_2^2}{u_1} + \frac{P(u)}{u_1} \quad \text{with} \quad P'''''(u) = 0. \tag{23}$$

The set S for the equation (23) consists of the generators of two classical symmetries $g^{(1)} = u_x$ and $g^{(2)} = u_t$. Hence $m = 2$ and the operator L_1 is defined by

$$L_1 U = \begin{vmatrix} u_3 & u_2 & u_1 \\ u_{2t} & u_{1t} & u_t \\ U_2 & U_1 & U \end{vmatrix}. \tag{24}$$

$$A = \sum_{j=0}^{3} A^{(j)} D_x^j, \quad L_2 = \sum_{k=0}^{6} b^{(k)} D_x^k$$

The coefficients of the operators A and L_2 are found from equations

$$\frac{d}{dt}(L_1) = AL_1 - L_1 F^* \qquad \frac{d}{dt}(L_2) = AL_2 - L_2 F^*$$

Thus we obtain the final form of the operator R

$$R = D_x^4 - \frac{4u_2}{u_1}D_x^3 + \left(\frac{6u_2^2}{u_1^2} - \frac{2u_3}{u_1} - \frac{4P(u)}{3u_1^2}\right)D_x^2$$
$$+\left(\frac{8u_2u_3}{u_1^2} - \frac{2u_4}{u_1} - \frac{6u_2^3}{u_1^3} + \frac{4P(u)u_2}{u_1^3} - \frac{2P'(u)}{3u_1}\right)D_x + \frac{u_5}{u_1} - \frac{4u_2u_4}{u_1^2}$$
$$-\frac{2u_3^2}{u_1^2} + \frac{8u_2^2u_3}{u_1^3} - \frac{3u_2^4}{u_1^4} + \frac{4P(u)u_2^2}{3u_1^4} + \frac{4P(u)^2}{9u_1^4} - \frac{8P'(u)u_2}{3u_1^2} + \frac{10}{9}P''(u)$$
$$+u_1D_x^{-1}K$$
$$+u_tD_x^{-1}\left(\frac{4u_2u_3}{u_1^3} - \frac{u_4}{u_1^2} - \frac{3u_2^3}{u_1^4} - \frac{P'(u)}{u_1^2} + \frac{2P(u)u_2}{u_1^4}\right)$$

where $K = \frac{6u_2u_5}{u_1^3} - \frac{u_6}{u_1^2} - \frac{5}{9}P'''(u) + \frac{5P''(u)u_2}{3u_1^2} - \frac{10P(u)^2u_2}{9u_1^6}$
$-\frac{10u_2(4u_1u_3-5u_2^2)P(u)}{3u_1^6} - \frac{15u_2(2u_1u_3-u_2^2)(2u_1u_3-3u_2^2)}{2u_1^6}$
$+\frac{5(2P(u)+12u_1u_3-27u_2^2)(3u_4+P'(u))}{18u_1^4}$. It coincides with R found earlier in Sokolov V V 1984 On the Hamiltonian property of the Krichever-Novikov equation *Sov. Math. Dokl.* 30:1 44-46.

Semi-discrete equations

For integrable lattices

$$u_{n,t} = f(u_{n+k}, u_{n+k-1}, \dots, u_{n-k}), \quad \frac{\partial f}{\partial u_{n+k}} \frac{\partial f}{\partial u_{n-k}} \neq 0$$

weakly non-local recursion operators are of the form

$$R = R_0 + \sum_{j=1}^{m} g^{(k_j)} (D_n - 1)^{-1} h^{(j)} \tag{25}$$

Theorem 2. Let R be a weakly nonlocal difference operator of the form (25). Then there exist difference operators L_1 and L_2

$$L_1 = \alpha^{(0)} D_n^m + \alpha^{(1)} D_n^{m-1} + \cdots + \alpha^{(m)}, \tag{26}$$

$$L_2 = \beta^{(p)} D_n^p + \beta^{(p-1)} D_n^{p-1} + \cdots + \beta^{(-q)} D_n^{-q}, \ p > m, \ q > 0 \tag{27}$$

such that the following condition is satisfied $L_1 R = L_2$.

The natural numbers m, p and q in the formulas

$$R = R_0 + \sum_{j=1}^{m} g^{(k_j)}(D_n - 1)^{-1}h^{(j)}$$

$$L_1 = \alpha^{(0)}D_n^m + \alpha^{(1)}D_n^{m-1} + \cdots + \alpha^{(m)},$$

$$L_2 = \beta^{(p)}D_n^p + \beta^{(p-1)}D_n^{p-1} + \cdots + \beta^{(-q)}D_n^{-q}, \; p > m, \, q > 0$$

are related to each other by the formulas

$$p = m + q, \quad q = ord\, g^{(k_{m+1})} - ord\, g^{(k_1)}. \tag{28}$$

Therefore if m is given then p and q are uniquely determined.

By using this representation $L_1 R = L_2$ we can construct recursion operator R for integrable lattices as well. For instance, for the Volterra lattice

$$\frac{d}{dt} u_n = u_n(u_{n+1} - u_{n-1})$$

we have

$$L_1 U_n = -\frac{1}{u_{n,t} u_{n+1,t}} \begin{vmatrix} u_{n+1,t} & u_{n,t} \\ U_{n+1} & U_n \end{vmatrix} \qquad (29)$$

or, the same $L_1 = (D_n - 1)\frac{1}{u_{n,t}}$. We find L_2

$$L_2 = \frac{1}{u_{n+2} - u_n} D_n^2 + \left(\frac{u_{n+2} + u_{n+1}}{u_{n+1}(u_{n+2} - u_n)} - \frac{1}{u_{n+1} - u_{n-1}} \right) D_n +$$

$$\frac{1}{u_{n+2} - u_n} - \frac{u_n + u_{n-1}}{u_n(u_{n+1} - u_{n-1})} - \frac{1}{u_{n+1} - u_{n-1}} D_n^{-1}.$$

A non-autonomous example

Here we consider the following non-autonomous lattice of the relativistic Toda type

$$u_{n,t} = h_n h_{n-1}(a_n u_{n+2} - a_{n-1} u_{n-2}) \tag{30}$$

where $h_n = u_{n+1} u_n - 1$ and the coefficient a_n is an arbitrary periodic function of the period 2, $a_{n+2} = a_n$. This lattice has been found by *(Garifullin and Yamilov, 2012, JPA)*. In (*Garifullin R N, Mikhailov A V and Yamilov R I 2014, TMPh*) the recursion operator for the non-autonomous lattice (30) is constructed by reducing it to an autonomous system found by Tsuchida, for which the recursion operator has already been found earlier. Actually the known recursion operator for the Tsuchida system was recalculated to the scalar form in an appropriate way.
Here we derive the recursion operator directly using the symmetry algorithm discussed above.

As it has been observed in (*Garifullin R N, Mikhailov A V and Yamilov R I 2014, TMPh*) the lattice (30) possesses a rather large hierarchy of the symmetries. Since the lattice is not autonomous then the set S of the seed symmetries obviously might also contain non-autonomous symmetries. The first two members of the symmetry hierarchy are as follows

1. $u_{n,\tau_1} = (-1)^n u_n$,
2. $u_{n,\tau_2} = h_n h_{n-1}(c_n u_{n+2} - c_{n-1} u_{n-2}), \quad c_{n+2} = c_n$,

where c_n is an arbitrary periodic function of n with period equal to two.
As potential sets of seed symmetries, consider the following three sets:

$$S_1 = \{u_{n,\tau_1}\}, \quad S_2 = \{u_{n,\tau_2}\}, \quad S_3 = \{u_{n,\tau_1}; u_{n,\tau_2}\}. \tag{31}$$

We checked that the first two sets do not fit, but the latest is surely the required set of seed symmetries.

The operator L_1 corresponding to S_3 is given by

$$L_1 U_n = \begin{vmatrix} D_n^2(u_{n,\tau_1}) & D_n(u_{n,\tau_1}) & u_{n,\tau_1} \\ D_n^2(u_{n,\tau_2}) & D_n(u_{n,\tau_2}) & u_{n,\tau_2} \\ U_{n+2} & U_{n+1} & U_n \end{vmatrix} \qquad (32)$$

As a result we have

$$L_1 = \alpha D_n^2 + \beta D_n + \gamma, \qquad (33)$$

where

1. $\alpha = (-1)^{n+1} h_n u_{n+1} h_{n-1}(c_n u_{n+2} - c_{n-1} u_{n-2}) +$
$+(-1)^{n+1} h_n u_n h_{n+1}(c_{n-1} u_{n+3} - c_n u_{n-1}),$
2. $\beta = (-1)^{n+1} u_{n+2} h_n h_{n-1}(c_n u_{n+2} - c_{n-1} u_{n-2}) -$
$-(-1)^{n+1} u_n h_{n+1} h_{n+2}(c_n u_{n+4} - c_{n-1} u_n),$
3. $\gamma = (-1)^n h_{n+1} u_{n+1} h_{n+2}(c_n u_{n+4} - c_{n-1} u_n) +$
$+(-1)^n h_{n+1} u_{n+2} h_n(c_{n-1} u_{n+3} - c_n u_{n-1}).$

Find the linearization of the lattice (30)

$$U_{n,t} = F^* U_n, \tag{34}$$

where

$$\begin{aligned} F^* = {} & a_n h_n h_{n-1} D_n^2 + u_n h_{n-1}(a_n u_{n+2} - a_{n-1} u_{n-2}) D_n + \\ & (u_{n+1} h_{n-1} + u_{n-1} h_n)(a_n u_{n+2} - a_{n-1} u_{n-2}) + \\ & u_n h_n (a_n u_{n+2} - a_{n-1} u_{n-2}) D_n^{-1} - a_{n-1} h_n h_{n-1} D_n^{-2}. \end{aligned}$$

In order to find the operator

$$A = A^{(2)} D_n^2 + A^{(1)} D_n + A^{(0)} + A^{(-1)} D_n^{-1} + A^{(-2)} D_n^{-2}. \tag{35}$$

We use the equation

$$\frac{d}{dt} L_1 = A L_1 - L_1 F^*, \tag{36}$$

When A is found we can look for the operator L_2 which has to be of the form

$$L_2 = b^{(4)} D_n^4 + b^{(3)} D_n^3 + b^{(2)} D_n^2 + b^{(1)} D_n + b^{(0)} + b^{(-1)} D_n^{-1} + b^{(-2)} D_n^{-2}.$$

Indeed due to the formulas for the orders we have $p = m + l$, $q = l$ where $m = 2$, $l = 2$.
We substitute the ansatz into the defining equation

$$\frac{d}{dt} L_2 = A L_2 - L_2 F^*$$

and find consecutively the unknown coefficients $b^{(j)}$.

Finally we find the recursion operator

$$R = c_n h_n h_{n-1} D_n^2 + \frac{u_n g_n}{h_n} D_n + \frac{u_{n-1} g_n - u_n g_{n-1}}{h_{n-1}} + c_n h_{n-1} h_{n+1} +$$

$$+ c_{n-1} h_n h_{n-2} + s_n - \frac{u_n g_n}{h_{n-1}} D_n^{-1} + c_{n-1} h_n h_{n-1} D_n^{-2} +$$

$$+ u_{n,\tau_1} (D_n - 1)^{-1} (-1)^{n+1} \left(\frac{g_{n-1}}{h_{n-1}} + \frac{g_{n+1}}{h_n} \right) +$$

$$+ u_{n,\tau_2} (D_n - 1)^{-1} \left(\frac{u_{n-1}}{h_{n-1}} + \frac{u_{n+1}}{h_n} \right).$$

where $g_n = h_n h_{n-1}(c_n u_{n+2} - c_{n-1} u_{n-2})$, the factors u_{n,τ_1} and u_{n,τ_2} are generators of the symmetries. We denoted $s_n = (-1)^{n+1} s_n^{(1)} - s_n^{(2)}$. Obviously in the formula for R function s_n is considered as an arbitrary function satisfying the periodicity condition $s_n = s_{n+2}$. Recursion operator found in (*Garifullin R N, Mikhailov A V and Yamilov R I 2014, TMPh*) can be reduced to this one with $s_n = 0$.

Operators L_1, L_2 and the Lax pairs

The operators L_1 and L_2 from the representation $R = L_1^{-1} L_2$ are connected with the Lax pairs. A pair of the equations

$$L_2 U = \lambda L_1 U, \qquad U_t = F^* U$$

defines the Lax pair for the equation

$$u_t = f(u, u_1, u_2, ..., u_k).$$

This Lax pair does not coincide with the usual one, since it is of higher order, but by appropriate transformation reduces to the usual form.

Examples

For the KdV equation operators L_1 and L_2 given above

$$L_1 = u_1 D_x - u_2,$$

$$L_2 = u_1 D_x^3 - u_2 D_x^2 + \frac{2}{3} u u_1 D_x + u_1^2 - \frac{2}{3} u u_2$$

Equation

$$(L_2 - \lambda L_1) U = 0$$

is a third order ODE for U. Its order is reduced by one

$$U_{xx} = \frac{u_x U_x}{2(u+\lambda)} - \frac{2}{3}(u+\lambda)U + \frac{u_x \sqrt{9U_x^2 + 6(u+\lambda)(U^2+c)}}{6(u+\lambda)}$$

where c is the constant of integration, set $c = 0$. Linearized Eq. $U_t = F^* U$ turns into

$$U_t = \frac{u_{xx} \sqrt{9U_x^2 + 6(u+\lambda)U^2}}{6(u+\lambda)} + \left(\frac{u_{xx}}{2(u+\lambda)} + \frac{u - 2\lambda}{3} \right) U_x.$$

Thus we obtain a nonlinear Lax pair, containing a square root. To get rid the root we change the variables in such a way that U and U_x are some quadratic forms of the new variables φ and ψ

$$U = \frac{2}{\sqrt{6}}\varphi\psi, \quad U_x = \frac{1}{3}\sqrt{u+\lambda}(\varphi^2 - \psi^2).$$

Then we arrive at a linear system for the new variables

$$\begin{cases} \varphi_x = \dfrac{u_x}{4(u+\lambda)}\varphi - \dfrac{1}{\sqrt{6}}\sqrt{u+\lambda}\psi, \\ \psi_x = \dfrac{1}{\sqrt{6}}\sqrt{u+\lambda}\varphi - \dfrac{u_x}{4(u+\lambda)}\psi, \end{cases} \tag{37}$$

Time evolution is linear as well

$$\begin{cases} \varphi_t = K\varphi - \dfrac{\sqrt{6}}{18}(u-2\lambda)\sqrt{u+\lambda}\,\psi, \\ \psi_t = \left(\dfrac{u_{xx}}{\sqrt{6}\sqrt{u+\lambda}} + \dfrac{\sqrt{6}}{18}(u-2\lambda)\sqrt{u+\lambda}\right)\varphi - K\psi. \end{cases} \tag{38}$$

where $K = \frac{3u_{xxx}+u_x(u-2\lambda)}{12(u+\lambda)}$

Changing the variables $\varphi = \alpha p$, $\phi = \alpha^{-1} q$ where $\alpha = (u+\lambda)^{1/4}$ we get

$$\begin{cases} p_x = -\dfrac{1}{\sqrt{6}}q, \\ q_x = \dfrac{1}{\sqrt{6}}(u+\lambda)p. \end{cases} \tag{39}$$

and finally we obtain the well-known Lax pair

$$p_{xx} = -\frac{1}{6}(u+\lambda)p, \qquad p_t = \frac{1}{3}(u-2\lambda)p_x - \frac{1}{6}u_x p.$$

We applied the algorithm above to the following two KdV type equations found by Svinolupov and Sokolov in 1982

$$u_t = u_{xxx} + \frac{1}{2}u_x^3 - \frac{3}{2}u_x \sin^2 u, \tag{40}$$

$$u_\tau = u_{yyy} - \frac{3u_y u_{yy}^2}{2(1+u_y^2)} + \frac{1}{2}u_y^3, \tag{41}$$

Consider the first one. We constructed the operators

$$L_1 = D_x - \frac{u_{xx}}{u_x}$$

$$L_2 = D_x^3 - \frac{u_{xx}}{u_x}D_x^2 + (u_x^2 - \sin^2 u)D_x + \frac{u_{xx}\sin^2 u - 3u_x^2 \sin u \cos u}{u_x}$$

by using the scheme.

Now we can find the recursion operator

$$R := L_1^{-1} L_2 = D_x^2 + u_x^2 - \sin^2 u - u_x D_x^{-1} (\sin u \cos u + u_{xx})$$

Thus we have the Lax pair of the form

$$L_2 U = \lambda L_1 U, \quad U_t = F^* U$$

$U_{xxx} - \frac{u_{xx}}{u_x} U_{xx} + (u_x^2 - \sin^2 u) U_x + \frac{u_{xx} \sin^2 u - 3u_x^2 \sin u \cos u}{u_x} U = \lambda \left(U_x - \frac{u_{xx}}{u_x} U \right)$. The Lax pair admits a reduction in the order

$$U_{xx} = \frac{u_x \sin u \cos u}{\sin^2 u + k} U_x + (\sin^2 u + k) U + \frac{u_x \sqrt{k(k+1)} K}{\sin^2 u + k},$$

$$U_t = \left(\frac{u_{xx} \sin 2u}{2(\sin^2 u + k)} - \frac{\sin^2 u - u_x^2}{2} + k \right) U_x + \frac{u_{xx} \sqrt{k(k+1)} K}{(\sin^2 u + k)}.$$

Here $K = \sqrt{(\sin^2 u + k) U^2 - U_x^2}$.

To get a linear Lax pair we change the variables by appropriate quadratic forms chosen in such a way that the root in K is precisely extracted

$$U = 2\tilde{\varphi}\tilde{\psi}, \quad U_x = \sqrt{\sin^2 u + \lambda}(\tilde{\varphi}^2 + \tilde{\psi}^2).$$

After some transformation we obtain the Lax pair

$$\varphi_{xx} = \frac{(\sin u \cos u - \sqrt{\xi - \xi^2})u_x}{\sin^2 u - \xi}\varphi_x + \frac{1}{4}(\sin^2 u - \xi)\varphi, \quad \xi = -\lambda$$

$$\varphi_t = \left(\frac{(\sin u \cos u - \sqrt{\xi - \xi^2})u_{xx}}{\sin^2 u - \xi} - \frac{1}{2}\sin^2 u + \frac{1}{2}u_x^2 - \xi \right)\varphi_x +$$

$$+\frac{1}{2}\sqrt{\xi - \xi^2}u_x\varphi.$$

Consider now the other equation from Svinolupov-Sokolov list

$$u_\tau = u_{yyy} - \frac{3u_y u_{yy}^2}{2(1+u_y^2)} + \frac{1}{2}u_y^3$$

Find the operators L_1 and L_2

$$L_1 = D_y - \frac{u_{yy}}{u_y}$$

$$L_2 = D_y^3 - \frac{(3u_y^2+1)u_{yy}}{u_y(1+u_y^2)}D_y^2 - \left(\frac{u_y u_{yyy}}{1+u_y^2} - \frac{3u_y^2 u_{yy}^2}{(1+u_y^2)^2} - u_y^2\right)D_y$$

Therefore we have the recursion operator $R = L_1^{-1}L_2$

$$R = D_y^2 - \frac{2u_y u_{yy}}{1+u_y^2}D_y + u_y D_y^{-1}\left(\frac{u_{yyy}}{1+u_y^2} - \frac{u_y u_{yy}^2}{(1+u_y^2)^2} + u_y\right)D_y.$$

Thus we have a third order Lax pair

$$U_\tau = U_{yyy} - \frac{3u_y u_{yy}}{1+u_y^2} U_{yy} + \frac{3}{2}\left(u_y^2 + \frac{(u_y^2-1)u_{yy}^2}{(u_y^2+1)^2}\right) U_y. \quad (42)$$

$$U_{yyy} - \frac{(3u_y^2+1)u_{yy}}{u_y(1+u_y^2)} U_{yy} - \left(\frac{u_y u_{yyy}}{1+u_y^2} - \frac{3u_y^2 u_{yy}^2}{(1+u_y^2)^2} - u_y^2 + \frac{1}{k}\right) U_y + \frac{u_{yy}}{k u_y} U = 0, \quad (43)$$

where k is a parameter.

We reduce the order of the equation (43)

$$U_{yy} = \frac{u_y u_{yy}}{1+u_y^2} U_y + \frac{1+u_y^2}{k} U - \frac{u_y K}{k}. \tag{44}$$

where $K = \sqrt{(k+1)((1+u_y^2)U^2 - kU_y^2) + \frac{l}{k}(1+u_y^2)}$. Thus we have a nonlinear Lax pair

$$U_\tau = U_{yyy} - \frac{3u_y u_{yy}}{1+u_y^2} U_{yy} + \frac{3}{2}\left(u_y^2 + \frac{(u_y^2-1)u_{yy}^2}{(u_y^2+1)^2}\right) U_y.$$

We reduce it to a linear one by the following change of the variables

$$U = 2\sqrt{k}\tilde{\varphi}\tilde{\psi}, \quad U_y = \sqrt{1+u_y^2}(\tilde{\varphi}^2 + \tilde{\psi}^2)$$

Finally we get

$$\varphi_{yy} = \left(\frac{u_y u_{yy}}{1+u_y^2} - \frac{\sqrt{1-\xi}\, u_y}{\sqrt{\xi}} \right) \varphi_y - \frac{1+u_y^2}{4\xi} \varphi, \quad \xi = -k,$$

$$\varphi_t = \left(\frac{u_y u_{yyy}}{1+u_y^2} - \frac{(3u_y^2+1) u_{yy}^2}{2(1+u_y^2)^2} + \frac{\sqrt{1-\xi}\, u_{yy}}{\sqrt{\xi}(1+u_y^2)} + \frac{\xi u_y^2 - 2}{2\xi} \right) \varphi_y - \\ - \frac{\sqrt{1-\xi}\, u_y}{2\xi^{3/2}} \varphi.$$

Publications

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