Tau-functions in Strebel combinatorial model of $M_{n,g}$ and Kontsevich-Witten cycles

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"Russian-Chinese conference on integrable systems",
St.Petersburg, August 22, 2018

References

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Combinatorial model $\overline{M}_{g,n}^{comb}[\mathbf{p}]$ of $\overline{M}_{g,n}$ based on Jenkins-Strebel differentials

The model depends on the choice of vector $\mathbf{p} \in \mathbb{R}_+^n$. Definition of JS quadratic differential Q: second order poles at n marked points $\{y_j\}$ and all periods of v on canonical cover $\widehat{C}$

$$v^2 = Q$$

of genus $\hat{g} = 4g - 3 + n$ are real; near y_j

$$Q(x) = -\frac{p_i^2}{4\pi^2} \frac{(dx)^2}{x^2} \, .$$

Ribbon graph Γ on C: vertices at zeros of Q, edges - horizontal trajectories, each face contains 1 marked point.

Generic stratum W of $\overline{M}_{g,n}^{comb}[\mathbf{p}]$: all vertices are 3-valent (zeros of Q are simple); consists of cells labeled by topologically different 3-valent Γ. Other strata are obtained by degeneration of edges.

Combinatorial model: continuation

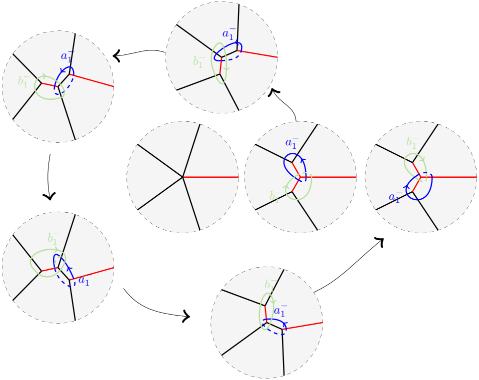
Witten-Kontsevich cycles: some vertices of Γ have odd valency different from 3 (if some vertices have even valencies - these are not cycles). For WK cycles all zeros of Q are branch points of $\widehat{C}$.

In real codimension 2 there are two WK cycles:

- $W_{1,1}$ - Kontsevich boundary (2 simple poles of Q i.e. 2 one-valent vertices of Γ). Obtained by collapsing two edges with two common endpoints.
- W_5 - Witten's cycle (one 5-valent vertex i.e one triple zero of Q). Obtained by collapsing two edges with one common endpoint.

Denote by $\tilde{W}$ the union of all cells of codimension 0 and 1. The fundamental group of $\tilde{W}$ is generated by paths around $W_{1,1}$ (combinatorial Dehn's twists) and W_5 (pentagon moves).

Pentagon move



Combinatorial Dehn's twist

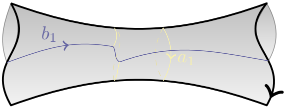


Figure: Action of combinatorial Dehn twist along non-separating loop on a pair of canonical cycles from $H_1(C)$.

Orientation of cycles, Kontsevich and homological symplectic forms

Kontsevich's symplectic form:

$$\Omega = \sum_{f\in F(\Gamma)} \eta_f \ , \qquad \eta_f = p_f^2\omega_f = \sum_{\substack{e_j,e_k\in\partial f\\ 1\le j<k<n_f}} \mathrm{d}\ell_j \wedge \mathrm{d}\ell_k$$

$e_1,\dots e_{n_f}$ are the edges bounding face f ordered counter clockwise (the form is independent of the choice of the "first" edge on each face), and ℓ_j denotes the length of the edge e_j (form w_f represents the corresponding ψ-class).

Theorem.

$$\Omega = \sum_{j=1}^{g_-} dA_j \wedge dB_j$$

where (A_j, B_j) are periods of v over symplectic basis in $H_-(\widehat{C})$.

Main result: relations between tautological classes from tau-functions

In $Pic(\overline{M}_{g,n}^{comb}[\mathbf{p}], \mathbb{Q})$:

$$\lambda_1 + \frac{1}{12}\sum_{i=1}^{n}\psi_i = \frac{1}{144}W_5 + \frac{13}{144}W_{1,1}$$

$$\lambda_2^{(n)} + \frac{1}{12}\sum_{i=1}^{n}\psi_i = \frac{13}{144}W_5 + \frac{25}{144}W_{1,1}$$

implying

$$\lambda_2^{(n)} - 13\lambda_1 = \sum_{i=1}^{n}\psi_i - W_{1,1}$$

where $W_{1,1}$ - "Kontsevich's boundary".

Basic notations

C - Riemann surface of genus g; (a_α, b_α) - canonical basis in $H_1(C, \mathbb{Z})$, $\alpha = 1, \ldots, g$; v_α - normalized basis of holomorphic 1-forms: $\oint_{a_\alpha} v_\beta = \delta_{\alpha\beta}$.
Canonical bimeromorphic differential $B(x, y)$; normalization $\oint_{a_\alpha} B(\cdot, y) = 0$. Bergman projective connection: as $y \to x$,

$$B(x,y) = \left(\frac{1}{(\xi(x) - \xi(y))^2} + \frac{1}{6} S_B(\xi(x)) + \ldots \right) d\xi(x) d\xi(y)$$

Warm-up example: spaces of holomorphic differentials

- $H_g(m_1, \ldots, m_n)$ with $m_1 + \cdots + m_n = 2g - 2$ - space of pairs (C, w) where w - holomorphic 1-form with zeros of multiplicity $m_1, \ldots, m_n$. Dimension is $2g + n - 1$; for $n = 2g - 2$ dimension equals $4g - 3$.
- Homological coordinates on $H_g(m_1, \ldots, m_n)$: $\oint_{a_\alpha} w$, $\oint_{b_\alpha} w$; $\int_{l_i} w$, $l_i = [x_1, x_i]$. Here $(a_\alpha, b_\alpha, l_i)$ - basis in relative homologies $H_1(C, \{x_k\})$.
- Dual basis in $H_1(C \setminus \{x_k\})$: $(b_\alpha, -a_\alpha, s_k)$. Here s_k - small positively oriented contour around x_m.

Variational formulas

- For basic holomorphic differentials:

$$\frac{\partial v_\alpha(x)}{\partial(\int_{c_i} w)} = \int_{c_i^*} \frac{v_\alpha(y)B(x,y)}{w(y)}$$

- For canonical bimeromorphic differential:

$$\frac{\partial B(x,y)}{\partial(\int_{c_i} w)} = \frac{1}{2\pi i}\int_{c_i^*} \frac{B(x,y)B(x,z)}{w(z)}$$

- For Bergman projective connection:

$$\frac{\partial S_B(x)}{\partial(\int_{c_i} w)} = \frac{1}{12\pi i}\int_{c_i^*} \frac{B^2(x,z)}{w(z)}$$

Bergman tau-function

- Introduce another projective connection:

$$S_w = \frac{w''}{w} - \frac{3}{2}\left(\frac{w'}{w}\right)^2$$

- Define tau-function $\tau(C, w, \{a_\alpha, b_\alpha\})$:

$$\frac{\partial \log \tau(C, w)}{\partial(\int_{c_i} w)} = -\frac{2}{\pi i}\int_{c_i^*} \frac{S_B - S_w}{w}$$

  Compatibility follows from variational formulas for S_B.

- Meaning of τ: $\tau = Z^{24}$, where Z - chiral partition function of free bosons on C.

Properties of the tau-function

- Under transformation of canonical basis of cycles by symplectic matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ τ transforms as follows:

$$\tau(C, w, \{a'_\alpha, b'_\alpha\}) = \det{}^{24}(C\mathbf{B} + D)\,\tau(C, w, \{a_\alpha, b_\alpha\})$$

  where $\mathbf{B}$ - matrix of b-periods of C.
- Under rescaling of w:

$$\tau(C, \epsilon w) = \epsilon^{2\left(2g-2+n-\sum_{k=1}^{n}\frac{1}{m_k+1}\right)}\,\tau(C, w)$$

- Therefore, on open part of projectivized moduli space τ is a section of line bundle

$$\lambda^{24} \otimes L^{-2\left(2g-2+n-\sum_{k=1}^{n}\frac{1}{m_k+1}\right)}$$

  where λ - Hodge line bundle, L-tautological line bundle

Explicit formula for tau-function

$$\tau(C,w,\{a_\alpha,b_\alpha\}) = \frac{\left(\left(\sum_{i=1}^g v_i(\zeta)\frac{\partial}{\partial s_i}\right)^g \theta(s;\mathbf{B})\Big|_{s=K^\zeta}\right)^{16}}{W(\zeta)^{16}}$$

$$\times \frac{\prod_{k<l} E(x_k,x_l)^{4m_k m_l}}{\prod_k E(\zeta,x_k)^{8(g-1)m_k}} w^{8(g-1)}(\zeta)$$

where K^ζ - vector of Riemann constants, $E(x,y)$ - prime-form, W - Wronskian determinant of normalized holomorphic differentials.

For genus 1 the tau-function coincides with Dedekind eta-function:

$$\tau(A,B) = \eta^{48}(B/A)$$

where A and B are periods of w.

Tau-function on compactification of $H_g(1,\dots,1)$

Boundary components of $H_g(1,\dots,1)$:

- D_{deg} - two zeros of w merge to form zero of second order
- Component D_0 of Deligne-Mumford boundary - pinching of C along homologically non-trivial cycle
- Components D_j of Deligne-Mumford boundary, $j = 1,\dots,[g/2]$ - pinching of C along homologically trivial cycle to get two Riemann surfaces of genera j and $g - j$.

Asymptotics of τ near all boundary components can be found explicitly, which gives the formula for Hodge class in terms of classes of bondary divisors and tautological class ψ in rational Picard group of projectivization of $H_g(1,\dots,1)$:

$$\lambda = \frac{g-1}{4}\psi + \frac{1}{24}\delta_{deg} + \frac{1}{12}\delta_0 + \frac{1}{8}\sum_{j=1}^{[g/2]}\delta_j$$

Spaces of holomorphic quadratic differentials

- $H_{g,2}$ - space of pairs (C, q), where q - holomorphic quadratic differential with simple zeros.
- $\dim H_{g,2} = 6g - 6$
- For fixed C, dimension of linear vector space V_2 of holomorphic quadratic differentials equals $3g - 3$. Denote corresponding vector bundle over moduli space of Riemann surfaces by Λ_2.
  Determinant line bundle: $\lambda_2 = \det \Lambda_2$.
- Relations between classes of λ_2 and λ_1 (Mumford 1977):

$$\lambda_2 - 13\lambda_1 = -\Delta$$

where $\Delta = \sum_{j=0}^{[g/2]} D_j$.

Canonical covering

- $\hat{C}$ - double covering of C; on $\hat{C}$ we have that $w = q^{1/2}$ is a well-defined holomorphic 1-form. Branch points of $\hat{C}$ - zeros x_i ($i = 1, \ldots, 4g-4$) of W. Genus $\hat{g}$ of $\hat{C}$ equals $4g-3$
- w - holomorphic 1-form on $\hat{C}$ with zeros of order 2 at x_i: $w \in H_{\hat{g}}(2, \ldots, 2)$.
- Denote the involution on $\hat{C}$ by $*$. Under the action of $*$ we have the splitting of homologies

  $$H_1 = H_+ \oplus H_-$$

  where $\mathrm{dim} H_+ = 2g$ and $\mathrm{dim} H_- = 6g-6$.

Isomorphizm between H^{-} and V_2

- Holomorphic part of cohomologies:

$$H^{(1,0)} = H^{+} \oplus H^{-}$$

  where $\dim H^{+} = g$ and $\dim H^{-} = 3g - 3$.
- Isomorphizm between V_2 and H_{-}:

$$q \quad \in \quad V_2 \qquad \text{then} \qquad v = \frac{q}{w} \quad \in \quad H^{-}$$

- The link between corresponding determinant line bundles: λ_2 and λ_{-}:

$$\lambda_{-} = \lambda_2 - \frac{3}{2}(g-1)\psi$$

Tau-functions on spaces of quadratic differentials

- $\hat{B}(x, y)$ - canonical bimeromorphic differential on $\hat{C}$;

$$B_{\pm}(x, y) = \hat{B}(x, y) \pm \hat{B}(x, y^*) \ .$$

- Corresponding projective connections $S_B^{\pm}$
- Homological coordinates: $\int_s w$ for $s \in H_-$ (exactly $6g - 6$ independent).
- Tau-functions $\tau_{\pm}$:

$$\frac{\partial \log \tau_{\pm}(C, q)}{\partial (\int_s w)} = -\frac{2}{\pi i} \int_{s^*} \frac{S_B^{\pm} - S_w}{w}$$

where s^* - cycle dual to s.

Properties of Hodge (τ_+) and Prym (τ_-) tau-fuctions

- Relation with $\hat{\tau}$:
$$\tau_+\tau_- = \hat{\tau}^2(\hat{C}, w)$$
where $\hat{\tau}$ - tau-function on $H_{\hat{g}}(2, \dots, 2)$.
- On open part of $M_{2,g}$ the tau-function $\tau_\pm$ is a section of line bundle
$$\lambda_\pm \otimes L^{\kappa_\pm}$$
where $\kappa_+ = 5/36(g-1)$; $\quad \kappa_- = 11/36(g-1)$ and L is the tautological line bundle.

Line bundle λ_- on compactification of Q_g

- Boundary of Q_g: 1. (pullback of) Deligne-Mumford boundary Δ_{DM} of M_g; 2. D_{deg} where two zeros of q merge.
- Asymptotics of $\tau_\pm$ near different boundary components implies

$$\lambda_+ = \frac{5(g-1)}{36}\psi + \frac{1}{72}\delta_{deg} + \frac{1}{12}\delta_{DM}$$

$$\lambda_- = \frac{11(g-1)}{36}\psi + \frac{13}{72}\delta_{deg} + \frac{1}{12}\delta_{DM}$$

- Excluding δ_{deg} we get

$$\lambda_- - 13\lambda_+ = -\delta_{DM} - \frac{3(g-1)}{2}\psi$$

- Using the link between λ_- and λ_2: Mumford's relation

$$\lambda_2 - 13\lambda_1 = -\delta_{DM}$$

Resolution of 5-valent vertex of Strebel graph

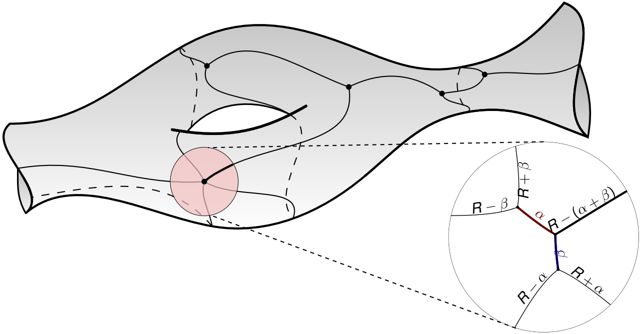
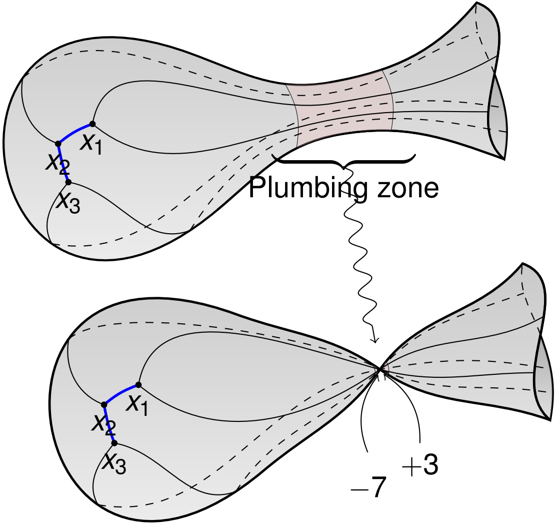


Figure: The Riemann surface $\mathcal{C}^{\alpha,\beta}$ obtained by replacing the five-valent vertex with three regular vertices and two new edges of length α and β between them with appropriate adjustment of the lengths of the original fatgraph. As $\alpha, \beta \to 0$ we recover the original Riemann surface.

Plumbing picture



Resolution of a pair of 1-valent vertices ("Kontsevich boundary")

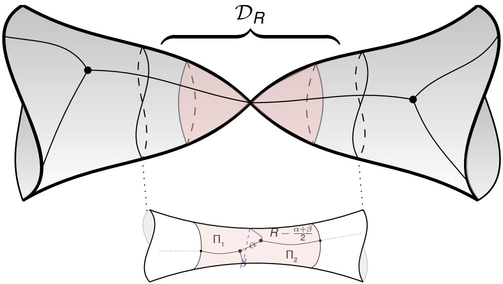


Figure: Resolution of a double point $x^0_{1,2}$ on $\mathcal{C}$ by insertion of an annulus with two simple zeros x_1 and x_2

Resolution of Kontsevich's boundary in plumbing picture

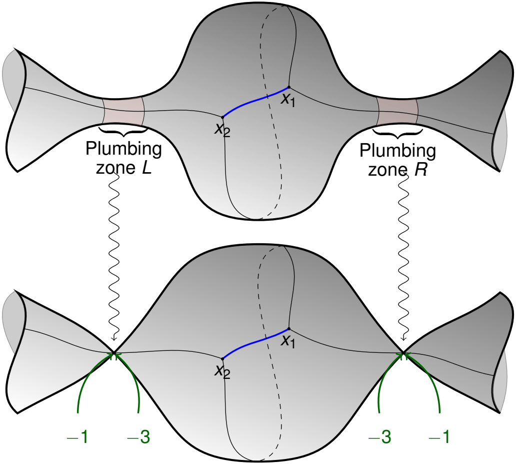


Figure: A Riemann sphere with two three-valent vertices is glued between one-valent vertices by introducing two plumbing zones

Moduli of curves from $\mathcal{Q}_0^{\mathbb{R}}(-7)$

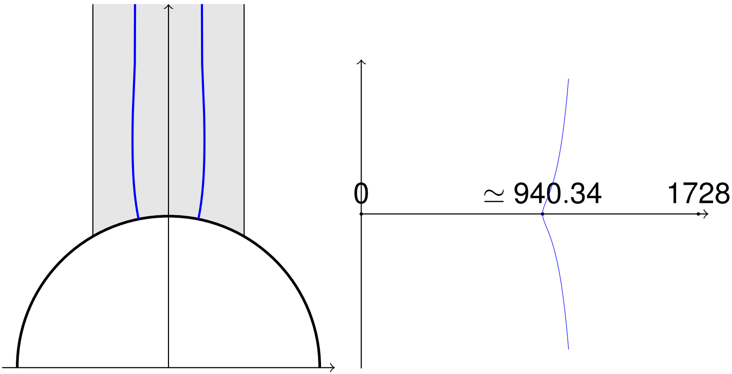


Figure: Left: The space $\mathcal{Q}_0^{\mathbb{R}}(-7)$ is fibered over the blue curve. Right: the same set in the plane of J-invariant.

Moduli or curves from $\mathcal{Q}_0^{\mathbb{R}}([-3]^2)$

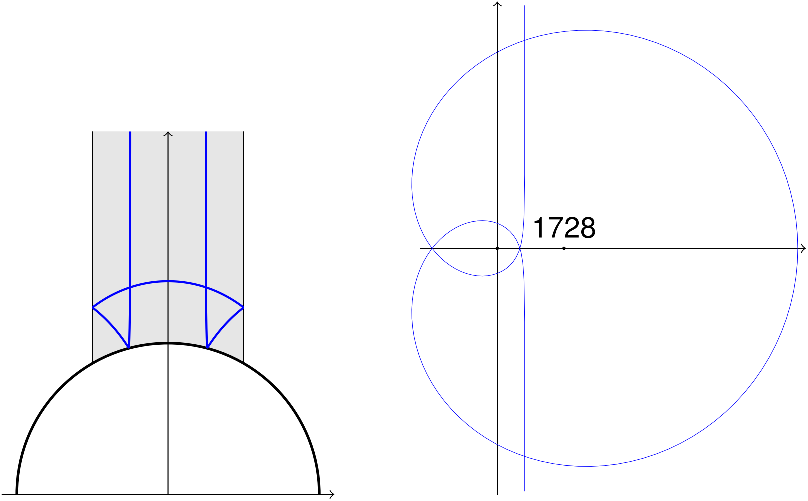
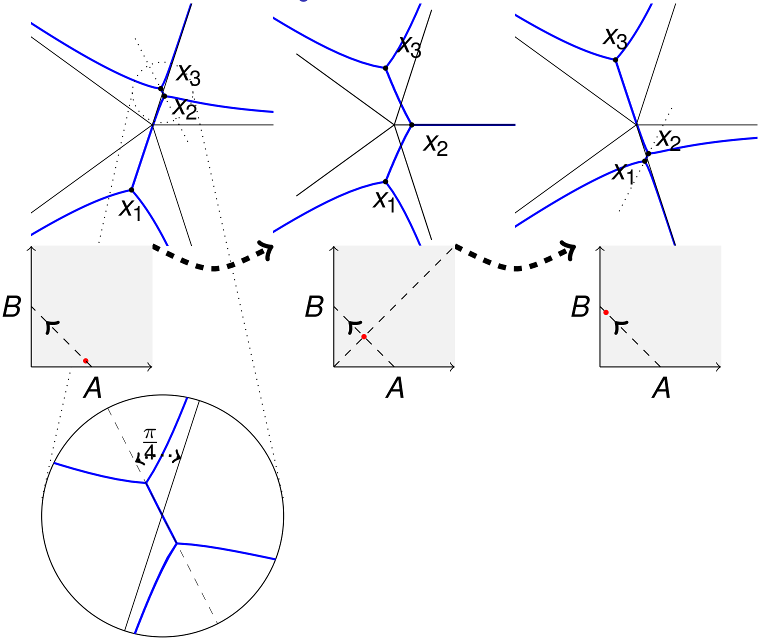
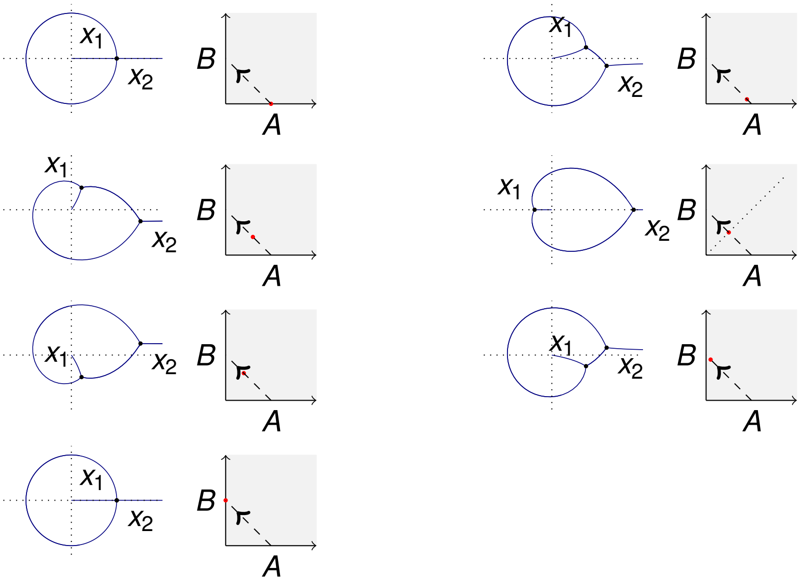


Figure: Left: the space $\mathcal{Q}_0^{\mathbb{R}}([-3]^2)$ is fibered over the blue curve in the moduli space of elliptic curves with the fiber $\mathbb{R}_+$. Right: the same curve in the plane of J-invariant.

Traversing a cell of $\mathcal{Q}_0^{\mathbb{R}}(-7)$



Traversing a cell of $\mathcal{Q}_0^{\mathbb{R}}([-3]^2)$



Relations between classes on $\overline{M}_{g,n}$

Known relations:

$$\lambda_2^{(n)} - 13\lambda = \sum_{i=1}^{n} \psi_i - \delta_{DM}$$

$$\kappa_1 = 12\lambda + \sum_{i=1}^{n} \psi_i - \delta_{DM}$$

$$\kappa_1 = \lambda_2^{(n)} -$$

Hyperbolic combinatorial model (Penner)

$$12\kappa_1 = W_5^{hyp} + W_{1,1}^{hyp}$$

Tau-functions provide analogs of these relations in JS combinatorial model!