On the soliton hierarchy with self-consistent sources (IST & Binary Darboux Transf.)

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Outline:

- Background:
  soliton equation with self-consistent sources,

- The KP hierarchy with sources (KPHWS):
  construction, t_n-reduction, τ_k-reduction (k-constraint)

- Generalized dressing approach for the KPHWS:

- Wronskian solutions of the KPHWS

- Bilinear identity of the KPHWS

- Discrete KP equation with self-consistent sources
(sources generated by Binary Darboux Transf.)

- Conclusion and discussions

Background

- Soliton equation with self-consistent sources

  (Physical applications: hydrodynamics, plasma, solid state physic-s)

  KdV case: capillary-gravity waves (Mel'nikov, 1989,...)

  NLS case: electrostatic & acoustic wave (Leon, 1991,...)

  KP case, modified Manakov case...

  (Grinevich, Taimanov, 2008, Pavlov, 1995,...)

Background

- Soliton equation with self-consistent sources

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 (Grinevich, Taimanov, 2008, Pavlov, 1995,...)

- Integration of soliton equation with sources

 Inverse scattering (Mel'nikov 1990; Lin 2001; Gerdjikov 2012...)

 Matrix theory (Mel'nikov, 1989)

 $\overline{\partial}-$method (Doktorov, Shchesnovich, 1996)

 Darboux transf. (Zeng,Ma,Shao,(*binary*)2001; Lin 2006 ...)

 Hirota method (Matsuno,1991; Hu,1991; Chen,Zhang,2003,...)

 Hirota method: *source generalization* (Hu,Wang,Gegenhasi,2006,...)

KdV & KdV equation with sources (KdVES):

KdV:

$$u_t = -(6uu_x + u_{xxx}).$$

KdV & KdV equation with sources (KdVES):

KdV:

$$u_t = -(6uu_x + u_{xxx}).$$

KdVES (Mel'nikov, 1988):

$$u_t = -(6uu_x + u_{xxx}) - 2\frac{\partial}{\partial x}\sum_{j=1}^{N}\phi_j^2,$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \qquad j = 1, \cdots, N.$$

Restricted flows and KdV hierarchy with sources

For N distinct λ_j, $j = 1, ..., N$, the high-order restricted flows of the KdV hierarchy (for $n = 0, 1, \cdots$) is defined as

$$\frac{\delta H_n}{\delta u} - 2\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = 0, \quad \phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \quad \frac{\delta \lambda_j}{\delta u} = \phi_j^2, \qquad j = 1, \cdots, N.$$

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The KdV hierarchy with self-consistent sources (KdVHWS) is

$$u_{t_n} = D\left[\frac{\delta H_n}{\delta u} - 2\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u}\right], \quad \phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \quad \frac{\delta \lambda_j}{\delta u} = \phi_j^2, \quad j = 1, \cdots, N.$$

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For $n = 1$, we have the KdV equation with sources (KdVES)

$$u_{t_1} = -(6uu_x + u_{xxx}) - 2\frac{\partial}{\partial x}\sum_{j=1}^{N} \phi_j^2,$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \qquad j = 1, \cdots, N.$$

Solving KdV hierarchy with sources by inverse scattering method (ISM)

The initial-value problem of the KdVHWS

Assume $u(x,t)$, $\phi_j(x,t), j = 1, ..., N$, vanish rapidly as $|x| \to \infty$,

(a) $u_0(x)$ satisfies: $\int_{-\infty}^{\infty}(|xu_0(x)| + \sum_{j=0}^{2n+1} |u_0^{(j)}(x)|)dx < \infty$;

(b) the Schrödinger equation

$$\psi_{xx} + (\lambda + u_0(x))\psi = 0,$$

has exactly N distinct discrete eigenvalues as

$$\lambda_j = (ik_j)^2 = -k_j^2, \qquad \text{where } k_j > 0, \quad j = 1, \cdots, N.$$

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$$\lambda_j=(ik_j)^2=-k_j^2, \qquad \text{where } k_j>0, \quad j=1,\cdots,N.$$

Let $\beta_j(t), j=1,...,N$, be arbitrary continuous function of t. Using the inverse scattering method, we shall point out the way of constructing the solution $u=u(x,t), \phi_j=\phi_j(x,t), j=1,...,N$, of KdV hierarchy with sources such that

$$u(x,0)=u_0(x), \qquad \frac{1}{8}\int_{-\infty}^{\infty}\phi_j^2(x,t)dx=\beta_j(t), \qquad j=1,\cdots,N.$$

Definition of the scattering data

Denote Jost solutions of Schrödinger equation ($\lambda = k^2$) as

$$f^{-}(x,k,t) \sim e^{-ikx}, \qquad x \to -\infty,$$

$$f^{+}(x,k,t) \sim e^{ikx}, \qquad x \to +\infty.$$

The scattering coefficients for $k \in (-\infty, \infty)$, $k \neq 0$, as

$$f^{-}(x,k,t) = a(k,t)f^{+}(x,-k,t) + b(k,t)f^{+}(x,k,t).$$

Suppose

$$f^{-}(x,ik_j,t) = \widetilde{C}_j(t)f^{+}(x,ik_j,t), \qquad j = 1,\cdots,N.$$

The evolution of scattering data

Using the auxiliary linear problems for KdVHWS, we get the evolution of scattering data.

The evolution of scattering coefficients:

$$\frac{\partial a}{\partial t} = 0, \qquad \frac{\partial b}{\partial t} = 8ik^{2n+1}b.$$

The evolution of discrete spectrum:

$$\frac{dk_j}{dt} = 0, \quad j = 1, \cdots, N,$$

The evolution of normalization constants:

$$\frac{\partial \widetilde{C}_j}{\partial t} = 8\left[(-1)^{n+1}k_j^{2n+1} + \beta_j(t)\right]\widetilde{C}_j, \quad j = 1, \cdots, N.$$

(Lin, Zeng Ma, 2001)

Solving the initial-value problem of KdVHWS

By solving the Gel'fand-Levitan-Marchenko equation

$$K(x,y)+F(x+y)+\int_x^{\infty}K(x,s)F(s+y)ds=0, \qquad y>x,$$

with

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(k)}{a(k)}e^{ikx}dk+\sum_{j=1}^{N}\bar{c}_j^2(t)e^{-k_jx},$$

$$\bar{c}_j^2(t)=-i\widetilde{C}_j(t)\left[\frac{\partial a}{\partial k}(ik_j)\right]^{-1}, \qquad j=1,\cdots,N,$$

one can get the solution to the initial-value problem of KdVHWS:

$$u(x,t)=2\frac{d}{dx}K(x,x).$$

$$\phi_j(x,t)=2\sqrt{2\beta_j(t)}\bar{c}_j(t)\left(e^{-k_jx}+\int_x^{\infty}K(x,s)e^{-k_js}ds\right), \qquad j=1,\cdots,N,$$

Soliton solutions of the KdVES

KdVES:

$$u_t = -(6uu_x + u_{xxx}) - 2\frac{\partial}{\partial x}\sum_{j=1}^{N}\phi_j^2,$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \qquad j = 1, \cdots, N.$$

one-soliton solution of KdVES with $N = 1$, $\lambda_1 = (ik_1)^2$:
(discrete eigenvalues: ik_1; initial normalization const.: $\bar{c}_1^2(0)$)

$$u(x,t) = 2k_1^2 \text{sech}^2(k_1 x - 4k_1^3 t - 4\int_0^t \beta_1(z)dz + x_0),$$

$$\phi_1(x,t) = 2\sqrt{k_1\beta_1(t)}\text{sech}(k_1 x - 4k_1^3 t - 4\int_0^t \beta_1(z)dz + x_0),$$

where $x_0 = \log\frac{\sqrt{2k_1}}{\bar{c}_1(0)}$. (Lin, Zeng Ma, 2001)

2-soliton solution of KdVES with $N = 2$, $\lambda_1 = -4$, $\lambda_2 = -1$:
(discrete eigenvalues: $2i$, i; initial normalization const.: 12, 6)

$$u = \frac{12\left\{3 + 4\cosh[2x - 8t - 8\int_0^t \beta_2(z)dz] + \cosh[4x - 64t - 8\int_0^t \beta_1(z)dz]\right\}}{\Delta^2},$$

$$\phi_1 = 4\sqrt{6\beta_1(t)}\frac{\cosh[x - 4t - 4\int_0^t \beta_2(z)dz]}{\Delta},$$

$$\phi_2 = 4\sqrt{3\beta_2(t)}\frac{\sinh[2x - 32t - 4\int_0^t \beta_1(z)dz]}{\Delta},$$

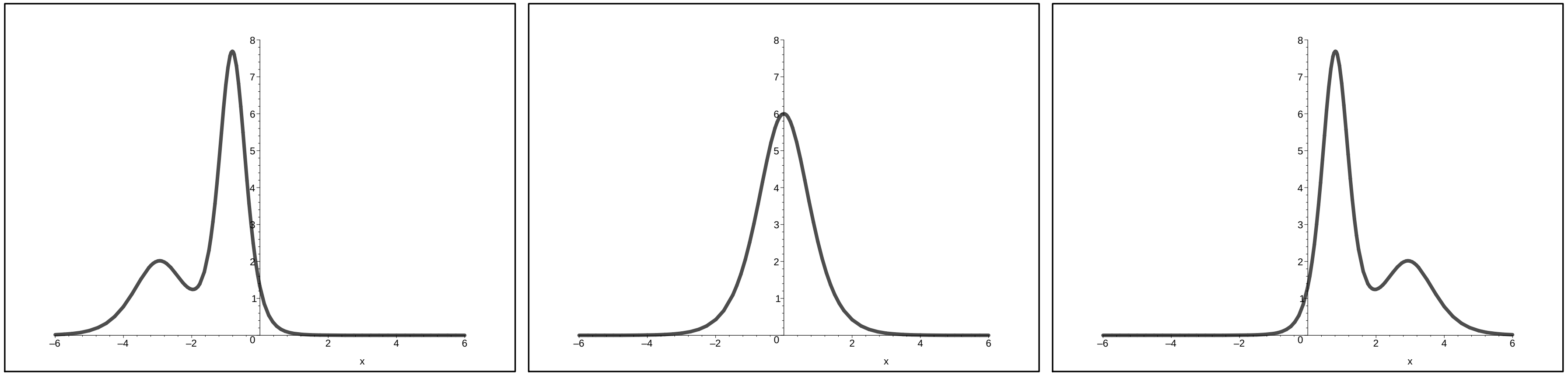
$$\Delta = \cosh[3x - 36t - 4\int_0^t (\beta_1(z) + \beta_2(z))dz] + 3\cosh[x - 28t - 4\int_0^t (\beta_1(z) - \beta_2(z))dz].$$

(Lin, Zeng Ma, 2001)

Varieties of dynamics of soliton solutions

2-soliton solution $u(x,t)$ of KdVES with $\beta_1(z) = 1$, $\beta_2(z) = 9$,

the soliton with smaller amplitude may propagate faster!



$t = -0.06$ $t = 0$ $t = 0.06$

(Lin, Zeng Ma, 2001)

Solving KdV equation with sources by Darboux transformation (DT)

(Two kinds of DT's)

Recall: Darboux transformation for KdV

KdV:

$$u_{t_1} = -(6uu_x + u_{xxx}),$$

Lax pair for KdV:

$$\frac{\partial^2}{\partial x^2}\Psi + (\lambda + u)\Psi = 0,$$

$$\frac{\partial}{\partial t_1}\Psi = u_x\Psi + (4\lambda - 2u)\Psi_x.$$

Recall: Darboux transformation for KdV

KdV:

$$u_{t_1} = -(6uu_x + u_{xxx}),$$

Lax pair for KdV:

$$\frac{\partial^2}{\partial x^2}\Psi + (\lambda + u)\Psi = 0,$$

$$\frac{\partial}{\partial t_1}\Psi = u_x\Psi + (4\lambda - 2u)\Psi_x.$$

Darboux transformation for KdV:
If Ψ and u satisfy the Lax pair for KdV,
f and u satisfy the Lax pair for KdV with $\lambda = \lambda_1$
$\Rightarrow$ $\widetilde{\Psi}$ and $\tilde{u}$ satisfy the Lax pair for KdV, where

$$\widetilde{\Psi} \equiv \Psi_x - \frac{f_x}{f}\Psi = \frac{1}{f}\begin{vmatrix} f & \Psi \\ f_x & \Psi_x \end{vmatrix}, \qquad \tilde{u} \equiv u + 2\partial_x^2 \ln f.$$

Wronskian determinant

Given functions $g_1(x)$, $g_1(x)$, ..., $g_m(x)$, define Wronskian determinant $W(g_1, ..., g_m)$ as

$$W(g_1, g_2, ..., g_m) = \begin{vmatrix} g_1 & g_2 & \cdots & g_m \\ g_{1,x} & g_{2,x} & \cdots & g_{m,x} \\ \cdots & \cdots & \cdots & \cdots \\ \partial_x^{m-1} g_1 & \partial_x^{m-1} g_2 & \cdots & \partial_x^{m-1} g_m \end{vmatrix}.$$

KdVES & its auxiliary linear problems

KdV equation with sources (KdVES):

$$u_{t_1} = -(6uu_x + u_{xxx}) - 2\frac{\partial}{\partial x}\sum_{j=1}^{N}\phi_j^2,$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \qquad j = 1, \cdots, N.$$

The auxiliary linear problems for KdVES:

$$\Psi_{xx} + (\lambda + u)\Psi = 0,$$

$$\Psi_{t_1} = u_x\Psi + (4\lambda - 2u)\Psi_x + \sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\phi_j(\phi_{j,x}\Psi - \phi_j\Psi_x).$$

Darboux Transformation for KdV with sources

If $u, \phi_1, ..., \phi_N$ is a solution of KdVES, Ψ satisfy

$$\Psi_{xx} + (\lambda + u)\Psi = 0,$$

$$\Psi_{t_1} = u_x\Psi + (4\lambda - 2u)\Psi_x + \sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j}\phi_j(\phi_{j,x}\Psi - \phi_j\Psi_x),$$

f and g are two solutions of the above linear problems with $\lambda = \lambda_{N+1}$, and $W(f, g) \neq 0$

$\Rightarrow$ Define $S \equiv f + g$,

$$\widetilde{\psi} = \frac{W(S,\psi)}{S}, \qquad \widetilde{u} = u + 2\partial_x^2 \ln S,$$

$$\widetilde{\phi}_j = \frac{1}{\sqrt{\lambda_j - \lambda_{N+1}}}\frac{W(S,\phi_j)}{S}, \quad j = 1, ..., N,$$

satisfy the auxiliary linear problems for KdVES

$$\widetilde{\Psi}_{xx} + (\lambda + \widetilde{u})\widetilde{\Psi} = 0,$$

$$\widetilde{\Psi}_{t_1} = \widetilde{u}_x\widetilde{\Psi} + (4\lambda - 2\widetilde{u})\widetilde{\Psi}_x + \sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j}\widetilde{\phi}_j(\widetilde{\phi}_{j,x}\widetilde{\Psi} - \widetilde{\phi}_j\widetilde{\Psi}_x).$$

Darboux Transf. (DT-I) for KdV with sources

If $u, \phi_1, ..., \phi_N$ is a solution of KdVES, Ψ satisfy

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f and g are two solutions of the above linear problems with $\lambda = \lambda_{N+1}$, and $W(f,g) \neq 0$

$\Rightarrow$ Define $S \equiv C(t)f + g$, ($C(t)$ is differentiable)

$$\tilde{\psi} = \frac{W(S,\psi)}{S}, \qquad \tilde{u} = u + 2\partial_x^2 \ln S,$$

$$\tilde{\phi}_j = \frac{1}{\sqrt{\lambda_j - \lambda_{N+1}}}\frac{W(S,\phi_j)}{S}, \quad j = 1, ..., N, \qquad \tilde{\phi}_{N+1} = \sqrt{\frac{C_t}{W(f,g)}}\frac{W(S,f)}{S},$$

satisfy the auxiliary linear problems for KdVES

$$\widetilde{\Psi}_{xx} + (\lambda + \tilde{u})\widetilde{\Psi} = 0,$$

$$\widetilde{\Psi}_{t_1} = \tilde{u}_x\widetilde{\Psi} + (4\lambda - 2\tilde{u})\widetilde{\Psi}_x + \sum_{j=1}^{N+1} \frac{1}{\lambda - \lambda_j}\tilde{\phi}_j(\tilde{\phi}_{j,x}\widetilde{\Psi} - \tilde{\phi}_j\widetilde{\Psi}_x).$$

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f and g are two solutions of the above linear problems with $\lambda = \lambda_{N+1}$, and $W(f,g) \neq 0$

$\Rightarrow$ Define $S \equiv C(t)f + g$, $\quad$ ($C(t)$ is differentiable)

$$\tilde{\psi} = \frac{W(S,\psi)}{S}, \qquad \tilde{u} = u + 2\partial_x^2 \ln S,$$

$$\tilde{\phi}_j = \frac{1}{\sqrt{\lambda_j - \lambda_{N+1}}}\frac{W(S,\phi_j)}{S}, \quad j = 1, ..., N, \qquad \tilde{\phi}_{N+1} = \sqrt{\frac{C_t}{W(f,g)}}\frac{W(S,f)}{S},$$

satisfy the auxiliary linear problems for KdVES

$$\widetilde{\Psi}_{xx} + (\lambda + \tilde{u})\widetilde{\Psi} = 0,$$

$$\widetilde{\Psi}_{t_1} = \tilde{u}_x\widetilde{\Psi} + (4\lambda - 2\tilde{u})\widetilde{\Psi}_x + \sum_{j=1}^{N+1} \frac{1}{\lambda-\lambda_j}\tilde{\phi}_j(\tilde{\phi}_{j,x}\widetilde{\Psi} - \tilde{\phi}_j\widetilde{\Psi}_x).$$

It's a non-auto-Bäcklund transformation between KdVES's.

(Lin, Zeng, 2006)

Soliton solution obtained by DT-I

The KdVES with $N = 1$ and $\lambda_1 = 0$ has the following solution

$$u = 0, \qquad \phi_1 = \eta(t).$$

With the above u and ϕ_1, we take two solutions of the auxiliary linear problems for KdVES with $\lambda = -k^2$ (where $k > 0$) as

$$f = \exp(kx - a(t)), \qquad g = \exp(-kx + a(t)), \qquad \frac{da}{dt} = 4k^3 - \frac{\eta(t)^2}{k}.$$

Then use the DT-I with $C(t) = \exp(-2z(t))$, where $z(t)$ is a differentiable function of t, we get a solution of the KdVES with $N = 2$, $\lambda_1 = 0$, $\lambda_2 = -k^2$,

$$\tilde{u} = 2k^2 \mathrm{sech}^2(kx - a(t) - z(t)), \qquad \tilde{\phi}_1 = -\eta(t) \tanh(kx - a(t) - z(t)),$$

$$\tilde{\phi}_2 = \sqrt{k\frac{dz}{dt}} \mathrm{sech}(kx - a(t) - z(t)),$$

Rational solution obtained by DT-I

The KdVES with $N = 0$ has a trivial solution

$$u = 0.$$

Take two solutions of the auxiliary linear problems for KdVES with $u = 0$ and $\lambda = 0$ as follows

$$f = 1, \qquad g = x,$$

then use the DT-I for KdVES, we get a rational solution of the KdVES with $N = 1$, $\lambda_1 = 0$,

$$\widetilde{u} = \frac{-2}{(x + C(t))^2}, \qquad \widetilde{\phi}_1 = \frac{-\sqrt{C_t}}{(x + C(t))}.$$

Darboux Transf. (DT-II) for KdV with sources

If $u, \phi_1, ..., \phi_N$ is a solution of KdVES, Ψ satisfy:

$$\Psi_{xx} + (\lambda + u)\Psi = 0,$$

$$\Psi_{t_1} = u_x\Psi + (4\lambda - 2u)\Psi_x + \sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j}\phi_j(\phi_{j,x}\Psi - \phi_j\Psi_x),$$

$f(x, t, \lambda_{N+1})$ and $g(x, t, \lambda_{N+1})$ are two solutions of the above linear problems with $\lambda = \lambda_{N+1}$, and $W(f, g) \neq 0$

$\Rightarrow$ Define $T \equiv C(t)f(x, t, \lambda_{N+1}) + \partial_{\lambda_{N+1}} g(x, t, \lambda_{N+1})$,

$$\widetilde{\Psi} = \frac{W(g,T,\Psi)}{W(g,T)}, \qquad \widetilde{u} = u + 2\partial_x^2 \ln W(g, T),$$

$$\widetilde{\phi}_j = \frac{1}{\lambda_j-\lambda_{N+1}}\frac{W(g,T,\phi_j)}{W(g,T)}, \qquad j = 1, ..., N, \qquad \widetilde{\phi}_{N+1} = \sqrt{\frac{C_t}{W(f,g)}}\frac{W(g,T,f)}{W(g,T)},$$

satisfy the auxiliary linear problems for KdVES

$$\widetilde{\Psi}_{xx} + (\lambda + \widetilde{u})\widetilde{\Psi} = 0,$$

$$\widetilde{\Psi}_{t_1} = \widetilde{u}_x\widetilde{\Psi} + (4\lambda - 2\widetilde{u})\widetilde{\Psi}_x + \sum_{j=1}^{N+1} \frac{1}{\lambda-\lambda_j}\widetilde{\phi}_j(\widetilde{\phi}_{j,x}\widetilde{\Psi} - \widetilde{\phi}_j\widetilde{\Psi}_x).$$

(Lin, Zeng, 2006)

Positon solution obtained by DT-II

The KdVES with $N = 1$ and $\lambda_1 = 0$ has a solution

$$u = 0, \qquad \phi_1 = \sqrt{\frac{d\eta(t)}{dt}}.$$

With the above u and ϕ_1, we take two solutions of the auxiliary linear problems for KdVES with $\lambda = k^2$ $(k > 0)$ as

$$f = \cos\Theta, \qquad g = \sin\Theta, \qquad \Theta = kx + 4k^3t - \frac{\eta(t)}{k} + b(k),$$

where $b(k)$ is a differentiable function of k. Using the DT-II, we get a solution of KdVES with $N = 2$, $\lambda_1 = 0$, $\lambda_2 = k^2$ $(k > 0)$,

$$\widetilde{u} = \frac{32k^2(2k^2\gamma\cos\Theta - \sin\Theta)\sin\Theta}{(4k^2\gamma - \sin(2\Theta))^2},$$

$$\widetilde{\phi}_1 = \frac{-\sqrt{\eta_t}(4k^2\gamma + \sin(2\Theta))}{4k^2\gamma - \sin(2\Theta)}, \qquad \widetilde{\phi}_2 = \frac{4k\sqrt{kC_t}\sin\Theta}{4k^2\gamma - \sin(2\Theta)},$$

where $\gamma = C(t) + \frac{1}{2k}\partial_k\Theta$.

Negaton solution obtained by DT-II

The KdVES with $N = 1$ and $\lambda_1 = 0$ has a solution

$$u = 0, \qquad \phi_1 = \sqrt{\tfrac{d\eta(t)}{dt}}.$$

With the above u and ϕ_1, we take two solutions of the auxiliary linear problems for KdVES with $\lambda = -k^2$ (where $k > 0$) as

$$f = \cosh\Theta, \qquad g = \sinh\Theta, \qquad \Theta = kx - 4k^3t + \frac{\eta(t)}{k} + b(k),$$

where $b(k)$ is a differentiable function of k. Using DT-II, we get a solution of KdVES with $N = 2$, $\lambda_1 = 0$, $\lambda_2 = -k^2$, $(k > 0)$,

$$\tilde{u} = \frac{8k^2(2k^2\gamma\cosh\Theta + \sinh\Theta)\sinh\Theta}{(2k^2\gamma + \sinh\Theta\cosh\Theta)^2},$$

$$\tilde{\phi}_1 = \frac{\sqrt{\eta_t}(-2k^2\gamma + \sinh\Theta\cosh\Theta)}{2k^2\gamma + \sinh\Theta\cosh\Theta}, \qquad \tilde{\phi}_2 = \frac{2k\sqrt{kC_t}\sinh\Theta}{2k^2\gamma + \sinh\Theta\cosh\Theta},$$

where $\gamma = C(t) - \frac{1}{2k}\partial_k\Theta$.

KP equation with self-consistent sources

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The 1st type: (Mel'nikov, Zeng, Hu, Zhang, Deng, ...)

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} + 4\sum_{i=1}^{N}(q_i r_i)_{xx} = 0, \qquad u := u_1$$

$$q_{i,y} = q_{i,xx} + 2uq_i, \quad r_{i,y} = -r_{i,xx} - 2ur_i, \quad i = 1, \ldots, N.$$

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$$q_{i,y} = q_{i,xx} + 2uq_i, \quad r_{i,y} = -r_{i,xx} - 2ur_i, \quad i = 1, \ldots, N.$$

The 2nd type: (Mel'nikov, Hu, Wang, ...)

$$4u_t - 12uu_x - u_{xxx} - 3D^{-1}u_{yy} = 3\sum_{i=1}^{N}[q_{i,xx}r_i - q_i r_{i,xx} + (q_i r_i)_y],$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2}q_i D^{-1}u_y + \frac{3}{2}q_i\sum_{j=1}^{N}q_j r_j + \frac{3}{2}u_x q_i,$$

$$r_{i,t} = r_{i,xxx} + 3ur_{i,x} - \frac{3}{2}r_i D^{-1}u_y - \frac{3}{2}r_i\sum_{j=1}^{N}q_j r_j + \frac{3}{2}u_x r_i,$$

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$$q_{i,y} = q_{i,xx} + 2uq_i, \quad r_{i,y} = -r_{i,xx} - 2ur_i, \quad i = 1, \ldots, N.$$

The 2nd type: (Mel'nikov, Hu, Wang, ...)

$$4u_t - 12uu_x - u_{xxx} - 3D^{-1}u_{yy} = 3\sum_{i=1}^{N}[q_{i,xx}r_i - q_i r_{i,xx} + (q_i r_i)_y],$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2}q_i D^{-1}u_y + \frac{3}{2}q_i \sum_{j=1}^{N} q_j r_j + \frac{3}{2}u_x q_i,$$

$$r_{i,t} = r_{i,xxx} + 3ur_{i,x} - \frac{3}{2}r_i D^{-1}u_y - \frac{3}{2}r_i \sum_{j=1}^{N} q_j r_j + \frac{3}{2}u_x r_i,$$

Problem: How to generate these two systems in a *systematical* way?

KP equation with self-consistent sources

The 1st type: (Mel'nikov, Zeng, Hu, Zhang, Deng, ...)

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} + 4\sum_{i=1}^{N}(q_i r_i)_{xx} = 0, \qquad u := u_1$$

$$q_{i,y} = q_{i,xx} + 2uq_i, \quad r_{i,y} = -r_{i,xx} - 2ur_i, \quad i = 1, \ldots, N.$$

The 2nd type: (Mel'nikov, Hu, Wang, ...)

$$4u_t - 12uu_x - u_{xxx} - 3D^{-1}u_{yy} = 3\sum_{i=1}^{N}[q_{i,xx}r_i - q_i r_{i,xx} + (q_i r_i)_y],$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2}q_i D^{-1}u_y + \frac{3}{2}q_i \sum_{j=1}^{N} q_j r_j + \frac{3}{2}u_x q_i,$$

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Problem: How to generate these two systems in a *systematical* way?

$\Longrightarrow$ constructing *a new extended KP hierarchy* (KP hierarchy with self-consistent sources, KPHWS) (Liu, Zeng, Lin, 2008)

The KP hierarchy with sources (KPHWS)

The KP hierarchy

The KP hierarchy

$$\partial_{t_n} L = [B_n, L], \qquad B_n = L_+^n,$$

where $L = \partial + \sum_{i=1}^{\infty} u_i \partial^{-i} = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \dots .$

The KP hierarchy

The KP hierarchy

$$\partial_{t_n} L = [B_n, L], \qquad B_n = L^n_+,$$

where $L = \partial + \sum\limits_{i=1}^{\infty} u_i \partial^{-i} = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \ldots.$

The commutativity of ∂_{t_n} flows gives the zero-curvature equations of KP hierarchy

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0.$$

The (adjoint) wave function

The wave function and the adjoint one satisfy

$$Lw = zw, \qquad \frac{\partial w}{\partial t_n} = B_n w,$$

$$L^* w^* = z w^*, \qquad \frac{\partial w^*}{\partial t_n} = -(B_n)^* w^*.$$

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It can be proved that (see, e.g., Dickey)

$$T(z)_- \equiv \sum_{i \in \mathbb{Z}} L^i_- z^{-i-1} = w \partial^{-1} w^*.$$

Introducing a new vector field

Define a new variable τ_k whose vector field is

$$\partial_{\tau_k} = \partial_{t_k} - \sum_{i=1}^{N} \sum_{s\geq 0} \zeta_i^{-s-1} \partial_{t_s},$$

where ζ_i's are arbitrary distinct non-zero parameters.

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where ζ_i's are arbitrary distinct non-zero parameters.

Then it can be proved that

$$L_{\tau_k} = [B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L],$$

where $q_i = w(x, \bar{t}; \zeta_i)$, $r_i = w^*(x, \bar{t}; \zeta_i)$, $\bar{t} = (t_1, t_2, t_3, \ldots)$ and

$$q_{i,t_n} = B_n(q_i), \qquad r_{i,t_n} = -B_n^*(r_i), \qquad i = 1, \cdots, N.$$

KP hierarchy with sources (KPHWS)

The Lax type equations

$$L_{t_n} = [B_n, L], \qquad L_{\tau_k} = [B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L], \quad (n \neq k),$$

give the KPHWS

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i)_{t_n} + [B_n, B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i] = 0,$$

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$$B_{n,\tau_k} - (B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i)_{t_n} + [B_n, B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i] = 0,$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = -B_n^*(r_i), \quad i = 1, \cdots, N.$$

The KPHWS admits a Lax representation

$$\Psi_{\tau_k} = (B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i)(\Psi), \qquad \Psi_{t_n} = B_n(\Psi).$$

(Liu, Lin, Zeng, 2008)

Example in the KPHWS: $(n = 2,\ k = 3)$

yields the 1st type of KP equation with self-consistent sources

$$u_{1,t_2} - u_{1,xx} - 2u_{2,x} = 0,$$

$$2u_{1,\tau_3} - 3u_{2,t_2} - 3u_{1,x,t_2} + u_{1,xxx} + 3u_{2,xx} - 6u_1u_{1,x} + 2\partial_x \sum_{i=1}^{N} q_i r_i = 0,$$

$$q_{i,t_2} - q_{i,xx} - 2u_1 q_i = 0,$$

$$r_{i,t_2} + r_{i,xx} + 2u_1 r_i = 0, \qquad i = 1, \ldots, N.$$

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$$q_{i,t_2} - q_{i,xx} - 2u_1 q_i = 0,$$
$$r_{i,t_2} + r_{i,xx} + 2u_1 r_i = 0, \qquad i = 1, \ldots, N.$$

The Lax representation is (where $u \equiv u_1$)

$$\Psi_{\tau_3} = (\partial^3 + 3u\partial + \tfrac{3}{2}D^{-1}u_{t_2} + \tfrac{3}{2}u_x + \sum_{i=1}^{N} q_i \partial^{-1} r_i)(\Psi),$$
$$\Psi_{t_2} = (\partial^2 + 2u)(\Psi).$$

Example in the KPHWS: $(n = 3,\ k = 2)$

yields the 2nd type of KP equation with sources

$$u_{1,\tau_2} - u_{1,xx} - 2u_{2,x} + \partial_x \sum_{i=1}^{N} q_i r_i = 0,$$

$$3u_{2,\tau_2} + 3u_{1,x,\tau_2} - 2u_{1,t_3} - u_{1,xxx} + 6u_1 u_{1,x} - 3u_{2,xx} + 3\partial_x \sum_{i=1}^{N} q_{i,x} r_i = 0,$$

$$q_{i,t_3} - q_{i,xxx} - 3u_1 q_{i,x} - 3(u_{1,x} + u_2) q_i = 0,$$

$$r_{i,t_3} - r_{i,xxx} - 3u_1 r_{i,x} + 3u_2 r_i = 0, \qquad i = 1, \ldots, N.$$

Example in the KPHWS: $(n = 3,\ k = 2)$

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$$q_{i,t_3} - q_{i,xxx} - 3u_1 q_{i,x} - 3(u_{1,x} + u_2) q_i = 0,$$

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The Lax representation is (where $u \equiv u_1$)

$$\Psi_{\tau_2} = (\partial^2 + 2u + \sum_{i=1}^{N} q_i \partial^{-1} r_i)(\Psi),$$

$$\Psi_{t_3} = (\partial^3 + 3u\partial + \tfrac{3}{2} D^{-1} u_{\tau_2} + \tfrac{3}{2} u_x + \tfrac{3}{2} \sum_{i=1}^{N} q_i r_i)(\Psi).$$

t_n-reduction of KPHWS:

The t_n-reduction is given by

$$L^n = B_n \qquad \text{or} \qquad L_-^n = 0,$$

t_n-reduction of KPHWS:

The t_n-reduction is given by

$$L^n = B_n \qquad \text{or} \qquad L^n_- = 0,$$

then the KPHWS reduces to the Gelfand-Dickey hierarchy with self-consistent sources

$$B_{n,\tau_k} = [(B_n)_+^{\frac{k}{n}} + \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_n],$$

$$B_n(q_i) = \zeta_i^n q_i, \qquad B_n^*(r_i) = \zeta_i^n r_i, \quad i = 1, \cdots, N.$$

t_n-reduction of KPHWS :

$n = 2,\ k = 3$ gives the 1st type of KdV equation with sources (Mel'nikov, ...)

$$
\begin{aligned}
& u_{1,\tau_3} - 3u_1 u_{1,x} - \frac{1}{4} u_{1,xxx} + \partial_x \sum_{i=1}^{N} q_i r_i = 0, \\
& q_{i,xx} + 2u_1 q_i - \zeta^2 q_i = 0, \\
& r_{i,xx} + 2u_1 r_i - \zeta^2 r_i = 0, \qquad i = 1, \cdots, N.
\end{aligned}
$$

t_n-reduction of KPHWS :

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& q_{i,xx} + 2u_1 q_i - \zeta^2 q_i = 0, \\
& r_{i,xx} + 2u_1 r_i - \zeta^2 r_i = 0, \qquad i = 1, \cdots, N.
\end{aligned}
$$

The Lax representation is (where $u \equiv u_1$)

$$
\begin{aligned}
& (\partial^2 + 2u)(\Psi) = \lambda \Psi, \\
& \Psi_t = (\partial^3 + 3u\partial + \frac{3}{2} u_x + \sum_{i=1}^{N} q_i \partial^{-1} r_i)(\Psi).
\end{aligned}
$$

t_n-reduction of KPHWS :

$n = 3$, $k = 2$ gives the 1st type of Boussinesq equation with self-consistent sources

$$
\begin{aligned}
& -2u_{2,x} - u_{1,xx} + u_{1,\tau_2} + \partial_x \sum_{i=1}^{N} q_i r_i = 0, \\
& 3u_{2,\tau_2} - 3u_{2,xx} + 3u_{1,x,\tau_2} + 6u_1 u_{1,x} - u_{1,xxx} + 3\partial_x \sum_{i=1}^{N} q_{i,x} r_i = 0, \\
& q_{i,xxx} + 3u_1 q_{i,x} + 3(u_{1,x} + u_2) q_i - \zeta^3 q_i = 0, \\
& r_{i,xxx} + 3u_1 r_{i,x} - 3u_2 r_i + \zeta^3 r_i = 0, \qquad i = 1, \cdots, N.
\end{aligned}
$$

t_n-reduction of KPHWS :

$n = 3$, $k = 2$ gives the 1st type of Boussinesq equation with self-consistent sources

$$-2u_{2,x} - u_{1,xx} + u_{1,\tau_2} + \partial_x \sum_{i=1}^{N} q_i r_i = 0,$$

$$3u_{2,\tau_2} - 3u_{2,xx} + 3u_{1,x,\tau_2} + 6u_1 u_{1,x} - u_{1,xxx} + 3\partial_x \sum_{i=1}^{N} q_{i,x} r_i = 0,$$

$$q_{i,xxx} + 3u_1 q_{i,x} + 3(u_{1,x} + u_2)q_i - \zeta^3 q_i = 0,$$

$$r_{i,xxx} + 3u_1 r_{i,x} - 3u_2 r_i + \zeta^3 r_i = 0, \qquad i = 1, \cdots, N.$$

The Lax representation is

$$(\partial^3 + 3u_1\partial + 3u_2 + 3u_{1,x})(\Psi) = \lambda\Psi,$$

$$\Psi_t = (\partial^2 + 2u_1 + \sum_{i=1}^{N} q_i \partial^{-1} r_i)(\Psi).$$

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then the KPHWS reduces to the k-constrained KP hierarchy

$$\left(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i\right)_{t_n} = \left[(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i)_+^{\frac{n}{k}}, B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i\right],$$

$$q_{i,t_n} = (B_k + \sum_{j=1}^{N} q_j \partial^{-1} r_j)_+^{\frac{n}{k}}(q_i),$$

$$r_{i,t_n} = -(B_k + \sum_{j=1}^{N} q_j \partial^{-1} r_j)_+^{\frac{n}{k}*}(r_i), \quad i = 1, \cdots, N,$$

τ_k-reduction of KPHWS:

$n = 3,\ k = 2$ gives the 2nd type of KdV equation with sources (or Yajima-Oikawa equation)

$$u_{1,t_3} = \frac{1}{4}u_{1,xxx} + 3u_1u_{1,x} + \frac{3}{4}\sum_{i=1}^{N}(q_{i,xx}r_i - q_ir_{i,xx}),$$

$$q_{i,t_3} = q_{i,xxx} + 3u_1q_{i,x} + \frac{3}{2}u_{1,x}q_i + \frac{3}{2}q_i\sum_{j=1}^{N}q_jr_j,$$

$$r_{i,t_3} = r_{i,xxx} + 3u_1r_{i,x} + \frac{3}{2}u_{1,x}r_i - \frac{3}{2}r_i\sum_{i=1}^{N}q_jr_j, \qquad i = 1, \cdots, N.$$

τ_k-reduction of KPHWS:

$n = 2$, $k = 3$ gives the 2nd type of Boussinesq equation with sources

$$
\begin{aligned}
&-2u_{2,x} - u_{1,xx} + u_{1,t_2} = 0,\\
&3u_{2,t_2} - 3u_{2,xx} + 3u_{1,x,t_2} + 6u_1u_{1,x} - u_{1,xxx} - 2\partial_x \sum_{i=1}^{N} q_i r_i = 0,\\
&q_{i,t_2} = q_{i,xx} + 2u_1 q_i,\\
&r_{i,t_2} = -r_{i,xx} - 2u_1 r_i, \qquad i = 1, \cdots, N.
\end{aligned}
$$

Generalized dressing approach for solving the KPHWS

Wronskian determinant:

For a set of functions $\{h_1, h_2, \ldots, h_N\}$, the Wronskian determinant is defined as

$$\mathrm{Wr}(h_1, \cdots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ h_1^{(1)} & h_2^{(1)} & \cdots & h_N^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ h_1^{(N-1)} & h_2^{(N-1)} & \cdots & h_N^{(N-1)} \end{vmatrix}, \qquad h_i^{(k)} \equiv \partial^k(h_i),$$

Dressing approach for KP hierarchy :

For the KP hierarchy

$$L_{t_n} = [B_n, L],$$

the following formula solves the KP hierarchy

$$L = S\partial S^{-1}, \qquad S = \frac{1}{\mathrm{Wr}(h_1, \cdots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ h_1^{(1)} & h_2^{(1)} & \cdots & h_N^{(1)} & \partial \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_1^{(N)} & h_2^{(N)} & \cdots & h_N^{(N)} & \partial^N \end{vmatrix},$$

with $h_i = f_i + \alpha_i g_i$, (α_i are constants)

$$\partial_{t_n}(f_i) = \partial^n f_i, \qquad \partial_{t_n}(g_i) = \partial^n g_i, \qquad i = 1, \dots, N.$$

Dressing approach for KP hierarchy with sources :

For the KP hierarchy with sources

$$L_{t_n} = [B_n, L], \qquad L_{\tau_k} = [B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L],$$

the following formula solves the KP hierarchy with sources

$$L = S\partial S^{-1}, \qquad S = \frac{1}{\mathrm{Wr}(h_1, \cdots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ h_1^{(1)} & h_2^{(1)} & \cdots & h_N^{(1)} & \partial \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_1^{(N)} & h_2^{(N)} & \cdots & h_N^{(N)} & \partial^N \end{vmatrix},$$

$$q_i = -\alpha_{i,\tau_k} S(g_i), \qquad r_i = (-1)^{N-i} \left(\frac{\mathrm{Wr}(h_1, \cdots, \widehat{h}_i, \cdots, h_N)}{\mathrm{Wr}(h_1, \cdots, h_N)} \right), \qquad i = 1, \ldots, N.$$

with $h_i = f_i + \alpha_i(\tau_k) g_i$, $\qquad$ ($\alpha_i(\tau_k)$ are differentiable functions)

$$\partial_{t_n}(f_i) = \partial^n f_i, \qquad \partial_{t_n}(g_i) = \partial^n g_i, \qquad i = 1, \ldots, N.$$

$$\partial_{\tau_k}(f_i) = \partial^k f_i, \qquad \partial_{\tau_k}(g_i) = \partial^k g_i, \qquad i = 1, \ldots, N.$$

Liu, Lin, *et al*, *J. Math. Phys.* 2009.

Lemmas for proving the dressing formula:

For the L, S, h_i, q_i, r_i given in the dressing formula for the KPH-WS, we have

Lemma 1. $S^{-1} = \sum_{i=1}^{N} h_i \partial^{-1} r_i$.

Lemma 2. $\partial^{-1} r_i S$ is a pure differential operator, and

$$(\partial^{-1} r_i S)(h_j) = \delta_{ij}, \qquad 1 \leq i, j \leq N.$$

Lemma 3.

$$S_{t_n} = -L_-^n S,$$

$$S_{\tau_k} = -L_-^k S + \sum_{i=1}^{N} q_i \partial^{-1} r_i S.$$

Soliton solution of 2nd type KP equation with sources:

$(n = 3, k = 2)$ Take

$$f_i = \exp(\lambda_i x + \lambda_i^2 y + \lambda_i^3 t) := e^{\xi_i},$$
$$g_i = \exp(\mu_i x + \mu_i^2 y + \mu_i^3 t) := e^{\eta_i},$$
$$h_i = f_i + \alpha_i(y) g_i = 2\sqrt{\alpha_i} e^{\frac{\xi_i+\eta_i}{2}} \cosh(\Omega_i)$$

where $\lambda_i \neq \mu_i$, $\Omega_i = \frac{\xi_i-\eta_i}{2} - \frac{1}{2}\ln(\alpha_i)$. then we get one-soliton solution by dressing method with $N = 1$

$$u = \frac{(\lambda_1 - \mu_1)^2}{4} \operatorname{sech}^2(\Omega),$$
$$q_1 = \sqrt{\alpha_1 y} (\lambda_1 - \mu_1) e^{\frac{\xi_1+\eta_1}{2}} \operatorname{sech}(\Omega_1),$$
$$r_1 = \frac{1}{2\sqrt{\alpha_1}} e^{-\frac{\xi_1+\eta_1}{2}} \operatorname{sech}(\Omega_1).$$

Liu, Lin, *et al*, *J. Math. Phys.* 2009.

Similar idea to other soliton hierarchies:

- Construction of the mKP hierarchy with sources (mKPHWS)
- Gauge transformation between the KPHWS and the mKPHWS
- Wronskian solutions of the KPHWS and the mKPHWS

(Liu, Lin, *et al*, *J. Math. Phys.* 2009)

- Construction of the q-deformed KP (mKP) hierarchy with sources (q-KPHWS & q-mKPHWS)
- Gauge transformation between the q-KPHWS and the q-mKPHWS
- q-Wronskian solutions of the q-KPHWS and the q-mKPHWS

(Lin, Liu, Zeng, *J. Nonl. Math. Phys.* 2008)

(Lin, Peng, M. Mañas, *J. Phys. A* 2010)

The mKP hierarchy with sources (mKPHWS)

The mKP hierarchy

The mKP hierarchy

$$\partial_{t_n}\widetilde{L} = [\widetilde{B}_n, \widetilde{L}], \qquad \widetilde{B}_n = (\widetilde{L}^n)_{\geq 1},$$

where $\widetilde{L} = \partial + \widetilde{u}_0 + \widetilde{u}_1\partial^{-1} + \widetilde{u}_2\partial^{-2} + \cdots$.

The mKP hierarchy

The mKP hierarchy

$$\partial_{t_n}\widetilde{L} = [\widetilde{B}_n, \widetilde{L}], \qquad \widetilde{B}_n = (\widetilde{L}^n)_{\geq 1},$$

where $\widetilde{L} = \partial + \widetilde{u}_0 + \widetilde{u}_1\partial^{-1} + \widetilde{u}_2\partial^{-2} + \cdots$.

The commutativity of ∂_{t_n} flows gives the zero-curvature equations of mKP hierarchy

$$\widetilde{B}_{n,t_m} - \widetilde{B}_{m,t_n} + [\widetilde{B}_n, \widetilde{B}_m] = 0.$$

The mKP hierarchy

The mKP hierarchy

$$\partial_{t_n}\widetilde{L} = [\widetilde{B}_n, \widetilde{L}], \qquad \widetilde{B}_n = (\widetilde{L}^n)_{\geq 1},$$

where $\widetilde{L} = \partial + \widetilde{u}_0 + \widetilde{u}_1\partial^{-1} + \widetilde{u}_2\partial^{-2} + \cdots$.

The commutativity of ∂_{t_n} flows gives the zero-curvature equations of mKP hierarchy

$$\widetilde{B}_{n,t_m} - \widetilde{B}_{m,t_n} + [\widetilde{B}_n, \widetilde{B}_m] = 0.$$

When $n = 2$, $m = 3$, $\Longrightarrow$ mKP equation:

$$4v_t - v_{xxx} + 6v^2 v_x - 3(D^{-1}v_{yy}) - 6v_x(D^{-1}v_y) = 0,$$

where $t := t_3$, $y := t_2$, $v := \widetilde{u}_0$.

mKP hierarchy with sources (mKPHWS)

The mKPHWS is constructed as

$$\widetilde{L}_{\tau_k} = [\widetilde{B}_k + \sum_{i=1}^{N} \widetilde{q}_i \partial^{-1} \widetilde{r}_i \partial, \widetilde{L}],$$

$$\widetilde{L}_{t_n} = [\widetilde{B}_n, \widetilde{L}], \quad \forall n \neq k,$$

$$\widetilde{q}_{i,t_n} = \widetilde{B}_n(\widetilde{q}_i), \quad \widetilde{r}_{i,t_n} = -(\partial \widetilde{B}_n \partial^{-1})^*(\widetilde{r}_i), \qquad i = 1, \cdots, N.$$

mKP hierarchy with sources (mKPHWS)

The mKPHWS is constructed as

$$\widetilde{L}_{\tau_k} = [\widetilde{B}_k + \sum_{i=1}^{N} \widetilde{q}_i \partial^{-1} \widetilde{r}_i \partial, \widetilde{L}],$$
$$\widetilde{L}_{t_n} = [\widetilde{B}_n, \widetilde{L}], \quad \forall n \neq k,$$
$$\widetilde{q}_{i,t_n} = \widetilde{B}_n(\widetilde{q}_i), \quad \widetilde{r}_{i,t_n} = -(\partial \widetilde{B}_n \partial^{-1})^*(\widetilde{r}_i), \qquad i = 1, \cdots, N.$$

The mKPHWS admits a Lax representation

$$\Psi_{t_n} = \widetilde{B}_n(\Psi), \qquad \Psi_{\tau_k} = (\widetilde{B}_k + \sum_{i=1}^{N} \widetilde{q}_i \partial^{-1} \widetilde{r}_i \partial)(\Psi).$$

(Liu, Lin, et al, *J. Math. Phys.* 2009)

Example in mKPHWS :

$n = 2$, $k = 3$ gives the 1st type of mKP equation with sources

$$4\widetilde{u}_{0,t} - \widetilde{u}_{0,xxx} + 6\widetilde{u}_0^2\widetilde{u}_{0,x} - 3D^{-1}\widetilde{u}_{0,yy} - 6\widetilde{u}_{0,x}D^{-1}\widetilde{u}_{0,y} + 4\sum_{i=1}^{N}(\widetilde{q}_i\widetilde{r}_i)_x = 0,$$

$$\widetilde{q}_{i,y} = \widetilde{q}_{i,xx} + 2\widetilde{u}_0\widetilde{q}_{i,x},$$

$$\widetilde{r}_{i,y} = -\widetilde{r}_{i,xx} + 2\widetilde{u}_0\widetilde{r}_{i,x}, \qquad i = 1, \ldots, N,$$

where $t := \tau_3, y := t_2$.

Example in mKPHWS :

$n = 3$, $k = 2$ gives the 2nd type of mKP equation with sources

$$4\widetilde{u}_{0,t} - \widetilde{u}_{0,xxx} + 6\widetilde{u}_0^2\widetilde{u}_{0,x} - 3D^{-1}\widetilde{u}_{0,yy} - 6\widetilde{u}_{0,x}D^{-1}\widetilde{u}_{0,y} + \sum_{i=1}^{N}[3(\widetilde{q}_i\widetilde{r}_{i,xx} - \widetilde{q}_{i,xx}\widetilde{r}_i) - 3(\widetilde{q}_i\widetilde{r}_i)_y - 6(\widetilde{u}_0\widetilde{q}_i\widetilde{r}_i)_x] = 0,$$

$$\widetilde{q}_{i,t} = \widetilde{q}_{i,xxx} + 3\widetilde{u}_0\widetilde{q}_{i,xx} + \frac{3}{2}(D^{-1}\widetilde{u}_{0,y})\widetilde{q}_{i,x} + \frac{3}{2}\widetilde{u}_{0,x}\widetilde{q}_{i,x} + \frac{3}{2}\widetilde{u}_0^2\widetilde{q}_{i,x} + \frac{3}{2}\widetilde{q}_{i,x}\sum_{j=1}^{N}(\widetilde{q}_j\widetilde{r}_j),$$

$$\widetilde{r}_{i,t} = \widetilde{r}_{i,xxx} - 3\widetilde{u}_0\widetilde{r}_{i,xx} + \frac{3}{2}(D^{-1}\widetilde{u}_{0,y})\widetilde{r}_{i,x} - \frac{3}{2}\widetilde{u}_{0,x}\widetilde{r}_{i,x} + \frac{3}{2}\widetilde{u}_0^2\widetilde{r}_{i,x} + \frac{3}{2}\widetilde{r}_{i,x}\sum_{j=1}^{N}(\widetilde{q}_j\widetilde{r}_j),$$

where $y := \tau_2, t := t_3$.

Gauge transformation between KPHWS and mKPHWS

Gauge transformation

Suppose L, q_i's, and r_i's satisfy the KPHWS, and f is a particular eigenfunction for the Lax pair of the KPHWS, i.e.,

$$f_{t_n} = B_n(f), \qquad f_{\tau_k} = (B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i)(f),$$

then

$$\widetilde{L} := f^{-1} L f, \quad \widetilde{q}_i := f^{-1} q_i, \quad \widetilde{r}_i := -\partial^{-1}(f r_i) = (\partial^{-1})^*(f r_i),$$

satisfy the mKPHWS.

(Liu, Lin, et al. *J. Math. Phys* 2009)

Wronskian solutions of mKPHWS

we choose

$$f = S(1) = (-1)^N \frac{\mathrm{Wr}(\partial(h_1), \partial(h_2), \cdots, \partial(h_N))}{\mathrm{Wr}(h_1, h_2, \cdots, h_N)}$$

as the particular eigenfunction for the Lax pair of the KPHWS, where S is the dressing operator defined in the dressing approach for KPHWS. Then the Wronskian solution for the mKPHWS is

$$\widetilde{L} = f^{-1} L f = \frac{\mathrm{Wr}(h_1, \cdots, h_N, \partial)}{\mathrm{Wr}(\partial(h_1), \cdots, \partial(h_N))} \partial \left[\frac{\mathrm{Wr}(h_1, \cdots, h_N, \partial)}{\mathrm{Wr}(\partial(h_1), \cdots, \partial(h_N))} \right]^{-1},$$

$$\widetilde{q}_i = f^{-1} q_i = -\dot{\alpha}_i \frac{\mathrm{Wr}(h_1, h_2, \cdots, h_N, g_i)}{\mathrm{Wr}(\partial(h_1), \partial(h_2), \cdots, \partial(h_N))}, \qquad i = 1, \ldots, N,$$

$$\widetilde{r}_i = -\partial^{-1}(f r_i) = \left(\frac{\mathrm{Wr}(\partial(h_1), \cdots, \partial(\widehat{h}_i), \cdots, \partial(h_N))}{\mathrm{Wr}(h_1, h_2, \cdots, h_N)} \right).$$

Soliton solution of 2nd type KP equation with sources:

$(n = 3, k = 2)$ Take

$$f_i = \exp(\lambda_i x + \lambda_i^2 y + \lambda_i^3 t) := e^{\xi_i},$$
$$g_i = \exp(\mu_i x + \mu_i^2 y + \mu_i^3 t) := e^{\eta_i},$$
$$h_i = f_i + \alpha_i(y) g_i = 2\sqrt{\alpha_i} e^{\frac{\xi_i+\eta_i}{2}} \cosh(\Omega_i)$$

where $\lambda_i \neq \mu_i$, $\Omega_i = \frac{\xi_i - \eta_i}{2} - \frac{1}{2}\ln(\alpha_i)$. then we get one-soliton solution by dressing method with $N = 1$

$$u = \frac{(\lambda_1 - \mu_1)^2}{4} \operatorname{sech}^2(\Omega),$$
$$q_1 = \sqrt{\alpha_1 y}(\lambda_1 - \mu_1) e^{\frac{\xi_1+\eta_1}{2}} \operatorname{sech}(\Omega_1),$$
$$r_1 = \frac{1}{2\sqrt{\alpha_1}} e^{-\frac{\xi_1+\eta_1}{2}} \operatorname{sech}(\Omega_1).$$

Soliton solution of 2nd type mKP equation with sources $(n = 3, k = 2)$

we get the one-soliton solution by the gauge transformation

$$\begin{aligned} v &= \frac{\lambda_1 - \mu_1}{2}[\tanh(\Omega_1 + \theta_1) - \tanh(\Omega_1)], \\ \tilde{q}_1 &= \partial_y(\sqrt{\alpha_1/(\lambda_1\mu_1)})(\mu_1 - \lambda_1)e^{\frac{\xi_1+\eta_1}{2}}\operatorname{sech}(\Omega_1 + \theta_1), \\ \tilde{r}_1 &= -\frac{1}{2\sqrt{\alpha_1}}e^{-\frac{\xi_1+\eta_1}{2}}\operatorname{sech}\Omega_1. \end{aligned}$$

KPHWS and q-KPHWS:

KPHWS
- 1st KP with sources
- 2nd KP with sources
- ...
- reductions
  - GD with sources: 1st KdV with sources ...
  - k-constrained KP: 2nd KdV with sources ...

$\uparrow \quad (q \to 1, u_0 \equiv 0)$

q-KPHWS
- 1st q-KP with sources
- 2nd q-KP with sources
- ...
- reductions
  - q-GD with sources: 1st q-KdV with sources ...
  - k-constrained q-KP: 2nd q-KdV with sources ...

- generalized dressing approach, gauge transformation (Liu, Lin, Zeng, 2009)
- soliton solutions for the KPHWS and MKPHWS (Liu, Lin, Zeng, 2009)
- q-KPHWS and qMKPHWS (Lin, Peng, M. Mañas, 2010)

Bilinear identity for KPHWS:

Theorem The bilinear identity for the KP hierarchy with self-consistent sources (KPHWS) (with new time flow denoted by $\bar{t}_k$) is given by the following sets of residue identities with auxiliary variable z:

$$\mathrm{Res}_\lambda\, w(z-\bar{t}_k,\mathbf{t},\lambda)\ \cdot w^*(z-\bar{t}'_k,\mathbf{t}',\lambda) = 0,$$
$$\mathrm{Res}_\lambda\, w_z(z-\bar{t}_k,\mathbf{t},\lambda)\ \cdot w^*(z-\bar{t}'_k,\mathbf{t}',\lambda) = q(z-\bar{t}_k,\mathbf{t})r(z-\bar{t}'_k,\mathbf{t}'),$$
$$\mathrm{Res}_\lambda\, w(z-\bar{t}_k,\mathbf{t},\lambda)\ \cdot \partial^{-1}\Big(q(z-\bar{t}'_k,\mathbf{t}')w^*(z-\bar{t}'_k,\mathbf{t}',\lambda)\Big) = -q(z-\bar{t}_k,\mathbf{t}),$$
$$\mathrm{Res}_\lambda\, \partial^{-1}\left(r(z-\bar{t}_k,\mathbf{t})w(z-\bar{t}_k,\mathbf{t},\lambda)\right)\ \cdot w^*(z-\bar{t}'_k,\mathbf{t}',\lambda) = r(z-\bar{t}'_k,\mathbf{t}'),$$

where $\mathbf{t} = (t_1, t_2, \cdots, t_{k-1}, \bar{t}_k, t_{k+1}, \cdots)$, $\mathbf{t}' = (t'_1, t'_2, \cdots, t'_{k-1}, \bar{t}'_k, t'_{k+1}, \cdots)$.

(Lin, Liu, Zeng, *J. Nonlinear Math. Phys.*, 2013)

Tau function for KPHWS:

Make the following ansatz:

$$w(z-\bar{t}_k,\mathbf{t},\lambda)=\frac{\tau(z-\bar{t}_k+\frac{1}{k\lambda^k},\mathbf{t}-[\lambda])}{\tau(z-\bar{t}_k,\mathbf{t})}\cdot\exp\xi(\mathbf{t},\lambda),$$

$$w^*(z-\bar{t}_k,\mathbf{t},\lambda)=\frac{\tau(z-\bar{t}_k-\frac{1}{k\lambda^k},\mathbf{t}+[\lambda])}{\tau(z-\bar{t}_k,\mathbf{t})}\cdot\exp(-\xi(\mathbf{t},\lambda)),$$

$$q(z,\mathbf{t})=\frac{\sigma(z,\mathbf{t})}{\tau(z,\mathbf{t})},\qquad r(z,\mathbf{t})=\frac{\rho(z,\mathbf{t})}{\tau(z,\mathbf{t})}.$$

where $\xi(\mathbf{t},\lambda)=\bar{t}_k\lambda^k+\sum_{i\neq k}t_i\lambda^i$, $[\lambda]=\left(\frac{1}{\lambda},\frac{1}{2\lambda^2},\frac{1}{3\lambda^3},\cdots\right)$

(Ref. Cheng and Zhang, 1994; Loris and Willox, 1997).

(Lin, Liu, Zeng, *J. Nonlinear Math. Phys.*, 2013)

Hirota bilinear equations for KPHWS:

Then we have

$$\mathrm{Res}_\lambda\, \bar\tau\Big(z, \mathbf{t}-[\lambda]\Big)\, \bar\tau\Big(z, \mathbf{t}'+[\lambda]\Big)\, e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} = 0,$$

$$\begin{aligned}\mathrm{Res}_\lambda\, \bar\tau_z\Big(z, \mathbf{t}-[\lambda]\Big)\, \bar\tau\Big(z, \mathbf{t}'+[\lambda]\Big)\, e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} \\ - \mathrm{Res}_\lambda\, \bar\tau\Big(z, \mathbf{t}-[\lambda]\Big)\, (\partial_z \log \bar\tau(z,\mathbf{t}))\, \bar\tau\Big(z, \mathbf{t}'+[\lambda]\Big)\, e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} \\ = \bar\sigma(z,\mathbf{t})\, \bar\rho(z,\mathbf{t}'),\end{aligned}$$

$$\mathrm{Res}_\lambda\, \lambda^{-1}\bar\tau\Big(z, \mathbf{t}-[\lambda]\Big)\, \bar\sigma\Big(z, \mathbf{t}'+[\lambda]\Big)\, e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} = \bar\sigma(z,\mathbf{t})\, \bar\tau(z,\mathbf{t}'),$$

$$\mathrm{Res}_\lambda\, \lambda^{-1}\bar\rho\Big(z, \mathbf{t}-[\lambda]\Big)\, \bar\tau\Big(z, \mathbf{t}'+[\lambda]\Big)\, e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} = \bar\rho(z,\mathbf{t}')\, \bar\tau(z,\mathbf{t}).$$

Here the bar $\bar{}$ over a function $f(z,\mathbf{t})$ is defined as $\bar f(z,\mathbf{t}) \equiv f(z-\bar t_k, \mathbf{t})$, e.g, $\bar\tau\Big(z, \mathbf{t}-[\lambda]\Big) \equiv \tau\Big(z-(\bar t_k - \frac{1}{k\lambda^k}), \mathbf{t}-[\lambda]\Big)$.

This gives the Hirota bilinear equations for the KPHWS.
(Lin, Liu, Zeng, *J. Nonlinear Math. Phys.*, 2013)

Example: for the 2nd type of KPWS

The Hirota bilinear equations for the KPWS-II can be obtained as

$$
\begin{aligned}
& D_x \tau_z \cdot \tau + \sigma\rho = 0, \\
& (D_x^4 + 3(D_{\bar{t}_2} - D_z)^2 - 4D_x D_{t_3})\tau \cdot \tau = 0, \\
& ((D_{\bar{t}_2} - D_z) + D_x^2)\tau \cdot \sigma = 0, \\
& ((D_{\bar{t}_2} - D_z) + D_x^2)\rho \cdot \tau = 0. \\
& (4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z))\tau \cdot \sigma = 0, \\
& (4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z))\rho \cdot \tau = 0.
\end{aligned}
$$

(Lin, Liu, Zeng, *J. Nonlinear Math. Phys.*, 2013)

Ref: Result by Hu and Wang (2007)

Another Hirota bilinear equations for the KPWS-II can be obtained by Pfaffian method (by Hu and Wang, 2007)

$$(D_x^4 - 4D_xD_t + 3D_y^2)f \cdot f = 6\sum_{i=1}^{M}(D_yk_i \cdot f - D_xg_i \cdot h_i),$$

$$D_xk_i \cdot f + g_ih_i = 0,$$

$$(D_y - D_x^2)g_i \cdot f = P_if - g_i\sum_{j=1}^{M}k_j,$$

$$(D_y - D_x^2)f \cdot h_i = h_i\sum_{j=1}^{M}k_j - fQ_i,$$

$$(D_x^3 + 3D_xD_y - 4D_t)g_i \cdot f = 3D_x\left[P_i \cdot f - g_i \cdot \left(\sum_{j=1}^{M}k_j\right)\right],$$

$$(D_x^3 + 3D_xD_y - 4D_t)f \cdot h_i = 3D_x\left[\left(\sum_{j=1}^{M}k_j\right) \cdot h_i - f \cdot Q_i\right].$$

Bilinear identity for B-type KPHWS (BKP with source):

Theorem The bilinear identity for the B-type KP hierarchy with self-consistent sources (BKPHWS) (with new time flow denoted by $\bar{t_k}$) is given by the following sets of residue identities with auxiliary variable z:

$$\text{Res}_\lambda \lambda^{-1} w(z-\bar{t_k}, t, \lambda) w(z-\bar{t_k}', t', -\lambda) = 1,$$

$$\text{Res}_\lambda \lambda^{-1} w_z(z-\bar{t_k}, t, \lambda) w(z-\bar{t_k}', t', -\lambda)$$
$$= q(z-\bar{t_k}, t) r(z-\bar{t_k}', t') - r(z-\bar{t_k}, t) q(z-\bar{t_k}', t'),$$

$$q(z-\bar{t_k}, t) = \text{Res}_\lambda \lambda^{-1} w(z-\bar{t_k}, t, \lambda) \cdot \partial_{x'}^{-1}(q(z-\bar{t_k}', t') w_{x'}(z-\bar{t_k}', t', -\lambda)),$$

$$r(z-\bar{t_k}, t) = \text{Res}_\lambda \lambda^{-1} w(z-\bar{t_k}, t, \lambda) \cdot \partial_{x'}^{-1}(r(z-\bar{t_k}', t') w_{x'}(z-\bar{t_k}', t', -\lambda)).$$

where $t = (t_1, t_2, \cdots, t_{k-1}, \bar{t}_k, t_{k+1}, \cdots)$, $t' = (t_1', t_2', \cdots, t_{k-1}', \bar{t}_k', t_{k+1}', \cdots)$.

(Lin, Cao, Liu, Zeng, *Theor. Math. Phys.* 2016)

Sources generated by Binary Darboux Transf.

Example: Discrete KP equation with self-consistent sources (dKPwS)

&

its linear problems

Discrete KP (dKP) equation:

Discrete KP (dKP) equation:

$$\tau_{(i)}\tau_{(jk)} - \tau_{(j)}\tau_{(ik)} + \tau_{(k)}\tau_{(ij)} = 0, \qquad 1 \le i < j < k,$$

where $\tau_{(\pm i)}(n_1, \dots, n_i, \dots) = \tau(n_1, \dots, n_i \pm 1, \dots)$.

Discrete KP (dKP) equation:

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Discrete KP with self-consistent sources (dKPwS) (by "source generation method", Hu, Wang, 2006):

$$\tau_{(1)}\tau_{(23)} - \tau_{(2)}\tau_{(13)} + \tau_{(3)}\tau_{(12)} = \boldsymbol{\rho}^*_{(13)}\boldsymbol{\sigma}_{(2)},$$

$$\tau_{(3)}\boldsymbol{\sigma}_{(1)} - \tau_{(1)}\boldsymbol{\sigma}_{(3)} = \boldsymbol{\sigma}\tau_{(13)}, \qquad \tau_{(1)}\boldsymbol{\rho}^*_{(3)} - \tau_{(3)}\boldsymbol{\rho}^*_{(1)} = \tau\boldsymbol{\rho}^*_{(13)},$$

where we denote the column-vector function $\boldsymbol{\sigma} = (\sigma_j)_{j=1,\dots,K}$, and the row-vector function $\boldsymbol{\rho}^* = (\rho^*_j)_{j=1,\dots,K}$.

Discrete KP (dKP) equation:

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where we denote the column-vector function $\boldsymbol{\sigma} = (\sigma_j)_{j=1,\dots,K}$, and the row-vector function $\boldsymbol{\rho}^* = (\rho^*_j)_{j=1,\dots,K}$.

Problem: How to generate the source in a geometric way?

Linear problems for the dKP equation

The linear problem for the dKP equation

$$\psi_{(i)} - \psi_{(j)} = \frac{\tau\tau_{(ij)}}{\tau_{(i)}\tau_{(j)}}\psi, \qquad 1 \leq i < j,$$

Linear problems for the dKP equation

The linear problem for the dKP equation

$$\boldsymbol{\psi}_{(i)} - \boldsymbol{\psi}_{(j)} = \frac{\tau\tau_{(ij)}}{\tau_{(i)}\tau_{(j)}}\boldsymbol{\psi}, \qquad 1 \leq i < j,$$

The adjoint linear problem for the dKP equation

$$\boldsymbol{\psi}^*_{(j)} - \boldsymbol{\psi}^*_{(i)} = \frac{\tau\tau_{(ij)}}{\tau_{(i)}\tau_{(j)}}\boldsymbol{\psi}^*_{(ij)}, \qquad 1 \leq i < j.$$

Binary Darboux transformation (BDT) for the dKP equation

Given the solution (column vector) $\boldsymbol{\omega} : \mathbb{Z}^{\widehat{N}} \to \mathbb{V}$, of the linear system, and given the solution (row vector) $\boldsymbol{\omega}^* : \mathbb{Z}^{\widehat{N}} \to (\mathbb{V})^*$, of the adjoint linear system. These allow to construct the linear operator valued potential $\boldsymbol{\Omega}[\boldsymbol{\omega}, \boldsymbol{\omega}^*] : \mathbb{Z}^{\widehat{N}} \to \mathsf{L}(\mathbb{V})$, defined by the system of compatible equations

$$\Delta_i \boldsymbol{\Omega}[\boldsymbol{\omega}, \boldsymbol{\omega}^*] = \boldsymbol{\omega} \otimes \boldsymbol{\omega}^*_{(i)}, \qquad i = 1, \ldots, \widehat{N},$$

where Δ_i is the standard partial difference operator in direction of n_i. Then (the binary Darboux transform of) the τ-function

$$\tilde{\tau} = \tau \det \boldsymbol{\Omega}[\boldsymbol{\omega}, \boldsymbol{\omega}^*]$$

satisfies the dKP equation again.

Sources generated by Binary Darboux transformation

Replace the n_i flow by a new $n_{\tilde{i}}$ flow by the binary Darboux tansformation as follows

$$\tau_{(\tilde{i})} = \tilde{\tau}_{(i)}, \qquad \text{etc.}$$

Then we get the dKP equation with self-consistent sources (d-KPwS)

$$\tau_{(\tilde{i})}\tau_{(jk)} - \tau_{(j)}\tau_{(\tilde{i}k)} + \tau_{(k)}\tau_{(\tilde{i}j)} = -(\tau\boldsymbol{\omega}^*)_{(jk)}(\tau\boldsymbol{\pi})_{(\tilde{i})}, \qquad i < j < k,$$

$$\boldsymbol{\pi}_{(j)} - \boldsymbol{\pi}_{(k)} = \boldsymbol{\pi}\frac{\tau\tau_{(jk)}}{\tau_{(j)}\tau_{(k)}}, \qquad \boldsymbol{\omega}^*_{(k)} - \boldsymbol{\omega}^*_{(j)} = \boldsymbol{\omega}^*_{(jk)}\frac{\tau\tau_{(jk)}}{\tau_{(j)}\tau_{(k)}},$$

where $\boldsymbol{\pi} = \boldsymbol{\Omega}[\boldsymbol{\omega}, \boldsymbol{\omega}^*]^{-1}\boldsymbol{\omega}$.

(Doliwa, Lin, *Phys. Lett. A*, 2014)

Sources generated by Binary Darboux transformation

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$$\tau_{(\tilde{i})}\tau_{(jk)} - \tau_{(j)}\tau_{(\tilde{i}k)} + \tau_{(k)}\tau_{(\tilde{i}j)} = -(\tau\boldsymbol{\omega}^*)_{(jk)}(\tau\boldsymbol{\pi})_{(\tilde{i})}, \qquad i < j < k,$$

$$\boldsymbol{\pi}_{(j)} - \boldsymbol{\pi}_{(k)} = \boldsymbol{\pi}\frac{\tau\tau_{(jk)}}{\tau_{(j)}\tau_{(k)}}, \qquad \boldsymbol{\omega}^*_{(k)} - \boldsymbol{\omega}^*_{(j)} = \boldsymbol{\omega}^*_{(jk)}\frac{\tau\tau_{(jk)}}{\tau_{(j)}\tau_{(k)}},$$

where $\boldsymbol{\pi} = \boldsymbol{\Omega}[\boldsymbol{\omega}, \boldsymbol{\omega}^*]^{-1}\boldsymbol{\omega}$.

(Doliwa, Lin, *Phys. Lett. A*, 2014)

Linear problems for the dKPwS

Linear problems for the dKPwS (we still assume $i < j < k$)

$$\begin{aligned}
\boldsymbol{\psi}_{(\tilde{i})} - \boldsymbol{\psi}_{(j)} &= \boldsymbol{\psi} \frac{\tau \tau_{(\tilde{i}j)}}{\tau_{(\tilde{i})} \tau_{(j)}} - \boldsymbol{\Omega}[\boldsymbol{\psi}, \boldsymbol{\omega}^*]_{(j)} \boldsymbol{\pi}_{(\tilde{i})}, \\
\boldsymbol{\psi}_{(j)} - \boldsymbol{\psi}_{(k)} &= \boldsymbol{\psi} \frac{\tau \tau_{(jk)}}{\tau_{(j)} \tau_{(k)}},
\end{aligned}$$

with $\boldsymbol{\Delta}_j \boldsymbol{\Omega}[\boldsymbol{\psi}, \boldsymbol{\omega}^*] = \boldsymbol{\psi} \otimes \boldsymbol{\omega}^*_{(j)}, \qquad j \neq i.$

Linear problems for the dKPwS

Linear problems for the dKPwS (we still assume $i < j < k$)

$$
\begin{aligned}
\boldsymbol{\psi}_{(\tilde{i})} - \boldsymbol{\psi}_{(j)} &= \boldsymbol{\psi} \frac{\tau \tau_{(\tilde{i}j)}}{\tau_{(\tilde{i})} \tau_{(j)}} - \boldsymbol{\Omega}[\boldsymbol{\psi}, \boldsymbol{\omega}^*]_{(j)} \boldsymbol{\pi}_{(\tilde{i})}, \\
\boldsymbol{\psi}_{(j)} - \boldsymbol{\psi}_{(k)} &= \boldsymbol{\psi} \frac{\tau \tau_{(jk)}}{\tau_{(j)} \tau_{(k)}},
\end{aligned}
$$

with $\Delta_j \boldsymbol{\Omega}[\boldsymbol{\psi}, \boldsymbol{\omega}^*] = \boldsymbol{\psi} \otimes \boldsymbol{\omega}^*_{(j)}, \qquad j \neq i.$

Adjoint linear prolblems for the dKPwS

$$
\begin{aligned}
\boldsymbol{\psi}^*_{(j)} - \boldsymbol{\psi}^*_{(\tilde{i})} &= \boldsymbol{\psi}^*_{(\tilde{i}j)} \frac{\tau \tau_{(\tilde{i}j)}}{\tau_{(\tilde{i})} \tau_{(j)}} + \boldsymbol{\omega}^*_{(j)} \boldsymbol{\Omega}[\boldsymbol{\pi}, \boldsymbol{\psi}^*]_{(\tilde{i})}, \\
\boldsymbol{\psi}^*_{(k)} - \boldsymbol{\psi}^*_{(j)} &= \boldsymbol{\psi}^*_{(jk)} \frac{\tau \tau_{(jk)}}{\tau_{(j)} \tau_{(k)}},
\end{aligned}
$$

with $\Delta_j \boldsymbol{\Omega}[\boldsymbol{\pi}, \boldsymbol{\psi}^*] = \boldsymbol{\pi} \otimes \boldsymbol{\psi}^*_{(j)}, \qquad j \neq i.$

(Doliwa, Lin, *Phys. Lett. A*, 2014)

Linear problems for the dKPwS

Linear problems for the dKPwS (we still assume $i < j < k$)

$$\boldsymbol{\psi}_{(\tilde{i})} - \boldsymbol{\psi}_{(j)} = \boldsymbol{\psi}\frac{\tau\tau_{(\tilde{i}j)}}{\tau_{(\tilde{i})}\tau_{(j)}} - \boldsymbol{\Omega}[\boldsymbol{\psi}, \boldsymbol{\omega}^*]_{(j)}\boldsymbol{\pi}_{(\tilde{i})},$$
$$\boldsymbol{\psi}_{(j)} - \boldsymbol{\psi}_{(k)} = \boldsymbol{\psi}\frac{\tau\tau_{(jk)}}{\tau_{(j)}\tau_{(k)}},$$

with $\Delta_j\boldsymbol{\Omega}[\boldsymbol{\psi}, \boldsymbol{\omega}^*] = \boldsymbol{\psi} \otimes \boldsymbol{\omega}^*_{(j)}, \qquad j \neq i.$

Adjoint linear prolblems for the dKPwS

$$\boldsymbol{\psi}^*_{(j)} - \boldsymbol{\psi}^*_{(\tilde{i})} = \boldsymbol{\psi}^*_{(\tilde{i}j)}\frac{\tau\tau_{(\tilde{i}j)}}{\tau_{(\tilde{i})}\tau_{(j)}} + \boldsymbol{\omega}^*_{(j)}\boldsymbol{\Omega}[\boldsymbol{\pi}, \boldsymbol{\psi}^*]_{(\tilde{i})},$$
$$\boldsymbol{\psi}^*_{(k)} - \boldsymbol{\psi}^*_{(j)} = \boldsymbol{\psi}^*_{(jk)}\frac{\tau\tau_{(jk)}}{\tau_{(j)}\tau_{(k)}},$$

with $\Delta_j\boldsymbol{\Omega}[\boldsymbol{\pi}, \boldsymbol{\psi}^*] = \boldsymbol{\pi} \otimes \boldsymbol{\psi}^*_{(j)}, \qquad j \neq i.$

(Doliwa, Lin, *Phys. Lett. A*, 2014)

Soliton solutions & Generalized Darboux Transf. for dKPWS

(Lin, Du, *Theor. Math. Phys.*, to be published)

Conclusion:

$$\text{KPHWS} \begin{cases} \text{1st KP with sources} \\ \text{2nd KP with sources} \\ \cdots \\ \text{reductions} \begin{cases} \text{GD with sources: 1st KdV with sources} \ldots \\ k\text{-constrained KP: 2nd KdV with sources} \ldots \end{cases} \end{cases}$$

- The KPHWS and mKPHWS are constructed
- a generalized dressing approach is introduced to solve the KPHWS;
- a gauge transformation is established between KPHWS and mKPHWS;
- the bilinear identity of KPHWS is derived.

Problems: How to construct other discrete integrable system with self-consistent sources?...

Thank you!