NLS hierarchy, MRW solutions and P_n breathers: from α, β to t_k parametrisation. (joint work with A.O.Smirnov)

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Main Results

1. Multi-rogue wave solutions to the focusing NLS and Gross-Pitaevskii equation
2. MRW solutions or the first seven members of the NLS hierarchy
3. General Method for constructing the quasi-rational solutions
4. Rank-1 rational and quasi-rational solutions of the NLS hierarchy equations
5. Examples of higher order NLS equations

Abstract

The basic formulas, from the paper Ph. Dubard, V. B. Matveev. *Nonlinearity*, **26**:12, R93–R125 (2013), which we used for description of the Multiple Rogue Waves (MRW) solutions for NLS case, - can be extended to the whole AKNS and NLS hierarchies.
The reported results are included in our recent articles:

V.B. Matveev, A.O. Smirnov, "Solutions of the AKNS hierarchy equations of the “rogue wave” type"
Theor. Math. Phys. **186**(2), 156–182 (2016)

V.B. Matveev, A.O. Smirnov, "AKNS and NLS hierarchies, MRW solutions, P_n breathers, and beyond", -
Submitted to the JMP special issue dedicated to the memory of L.D. Faddeev.

Focusing NLS eq:

$$ip_t + 2|p|^2 p + p_{xx} = 0, \quad x, t \in \mathbb{R}.$$

Multi rogue waves solutions of the NLS equation are quasi rational solutions:

$$p = e^{2ia^2 t} R(x,t), \quad R(x,t) = \frac{N(x,t)}{D(x,t)}, \quad a > 0,$$

Here $N(x,t), D(x,t)$ are polynomials of x and t, and deg $N(x,t)$ = deg $R(x,t) = n(n+1)$,

$$|p^2| \to a^2, \quad x^2 + t^2 \to \infty$$

The rational function $R(x,t)$ satisfies the 1D Gross-Pitaevskii equation:

$$iR_t + 2R(|R|^2 - a^2) + R_{xx} = 0, \quad |R| = |p|.$$

Simplest covariance properties:

$$p(x,t) \to a p(ax, a^2 t) \quad a > 0,$$
$$p(x,t) \to e^{i\chi} p(x,t) \quad \chi \in \mathbb{R},$$
$$p(x,t) \to p(x - bt, t) \exp\left(ibx/2 - ib^2 t/4\right) \quad b \in \mathbb{R}$$
$$p(x,t) \to p(x - x_0, t - t_0)$$

(Normalised) Peregrine breather or P_1 breather:

$$P_1(x,t) := \left(1 - 4\frac{(1+iT)}{X^2+T^2+1}\right) e^{iT/2}, \quad X := 2x, \quad T := 4t. \tag{1}$$

$$P_1(0,0) = -3, \quad P(x,t) \to e^{2it}, \quad \text{when} \quad x^2+t^2 \to \infty$$

Normalised rank 2 MRW solution:

$$p_2(x,t,\alpha_1,\beta_1) := R_2 e^{iT/2},\ R_2(X,T) = \left(1 - 12\frac{G_2(X,T) + iH_2(X,T)}{Q_2(X,T)}\right)$$

$$\begin{aligned}
G_2(X,T) :=& X^4 + 6(T^2+1)X^2 + 4\alpha_1 X + 5T^4 + 18T^2 - 4\beta_1 T - 3,\\
H_2(X,T) :=& TX^4 + 2(T^3 - 3T + \beta_1)X^2 + 4\alpha_1 TX + T^5 + 2T^3\\
&- 2\beta_1 T^2 - 15T + 2\beta_1,\\
Q_2(X,T) :=& (1 + X^2 + T^2)^3 - 4\alpha_1 X^3 - 12(2T^2 - \beta_1 T - 2)X^2\\
&+ 4(3\alpha_1(T^2+1)X + 6T^4 - \beta_1 T^3 + 24T^2 - 9\beta_1 T\\
&+ \alpha_1^2 + \beta_1^2 + 2).
\end{aligned}$$

Canonical structure of the rank n MRW solution

$$p_n(x,t) = R_n(X,T)e^{iT/2}, \quad X := 2x, \quad T := 4t,$$

$$R_n(X,T) = \frac{N_n(X,T,\alpha,\beta)}{Q_n(X,T,\alpha,\beta)} \equiv \left(1 - 2n(n+1)\frac{G_n(X,T) + iH_n(X,T)}{Q_n(X,T)}\right),$$

$$(\alpha,\beta) := (\alpha_1, \beta_1, \dots \alpha_{n-1}, \beta_{n-1}), \quad \alpha_j, \beta_j \in \mathbb{R}$$

$$p_n(x,t) = e^{2it} + o(1), \quad \text{when} \quad x^2 + t^2 \to \infty$$

$$\frac{Q(n+1)}{Q(n)} = 1^2 3^2 \cdots (2n+1)^2 = \frac{2^{2(n+1)}}{\pi} \Gamma^2 \left(n + \frac{3}{2} \right) = [(2n+1)!!]^2$$

$$Q(2n) = -(2n+1)\, G(2n), \quad Q(2n-1) = (2n-1)\, G(2n-1),$$

$$G(2n-1) > 0, G(2n) < 0, \forall n \geq 1$$

From this structure it follows that

$$P_n(0,0) = (-1)^n (2n+1), \quad n \geq 1.$$

Normalised Rank 3 MRW solutions of the NLS equation

$$p_3(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2) = \left(1 - 24\frac{G_3(X,T) + iH_3(X,T)}{Q_3(X,T)}\right) e^{iT/2},$$

$$G_3(X,T) = X^{10} + 15(T^2+1)X^8 + \sum_{n=0}^{6} g_n(T)X^n$$

$$H_3(X,T) = TX^{10} + 5(T^3 - 3T + \beta_1)X^8 + \sum_{n=0}^{6} h_n(T)X^n$$

$$Q_3(X,T) = (1 + X^2 + T^2)^6 - 20\alpha_1 X^9 - 60(2T^2 - \beta_1 T - 2)X^8 + 4\sum_{n=0}^{7} q_n(T)X^n.$$

$$
\begin{aligned}
g_6 &= 50T^4 - 60T^2 + 80\beta_1 T + 210 \\
g_5 &= 120\alpha_1 T^2 - 18\alpha_2 + 300\alpha_1 \\
g_4 &= 70T^6 - 150T^4 + 200\beta_1 T^3 + 450T^2 + 30\beta_2 T - 450 + 150\alpha_1^2 - 50\beta_1^2 \\
g_3 &= 400\alpha_1 T^4 + (3000\alpha_1 - 60\alpha_2)T^2 - 800\alpha_1\beta_1 T - 600\alpha_1 - 60\alpha_2 \\
g_2 &= 45T^8 + 420T^6 + 6750T^4 - (6000\beta_1 - 180\beta_2)T^3 - (300\alpha_1^2 - 900\beta_1^2 + 13500)T^2 \\
&\quad +(3600\beta_1 + 180\beta_2)T - 675 - 300\alpha_1^2 - 300\beta_1^2 \\
g_1 &= 280\alpha_1 T^6 + (150\alpha_2 - 2100\alpha_1)T^4 + 800\alpha_1\beta_1 T^3 - (3600\alpha_1 - 540\alpha_2)T^2 \\
&\quad +(120\beta_2\alpha_1 + 1200\alpha_1\beta_1 - 120\alpha_2\beta_1)T - 200\alpha_1\beta_1^2 - 900\alpha_1 - 90\alpha_2 - 200\alpha_1^3 \\
g_0 &= 11T^{10} + 495T^8 - 120\beta_1 T^7 + 2190T^6 - (42\beta_2 + 1200\beta_1)T^5 \\
&\quad +(350\alpha_1^2 + 150\beta_1^2 - 7650)T^4 + (6600\beta_1 - 420\beta_2)T^3 \\
&\quad -(2100\beta_1^2 + 2025 - 120\beta_2\beta_1 - 120\alpha_2\alpha_1 + 900\alpha_1^2)T^2 + (200\alpha_1^2\beta_1 + 200\beta_1^3 - 90\beta_2)T \\
&\quad +675 + 150\alpha_1^2 + 6\alpha_2^2 + 150\beta_1^2 + 6\beta_2^2.
\end{aligned}
$$

$$
\begin{aligned}
h_6 &= 10T^5 - 140T^3 + 40\beta_1 T^2 - 150T + 60\beta_1 - 5\beta_2 \\
h_5 &= 40\alpha_1 T^3 + (60\alpha_1 - 18\alpha_2)T + 40\alpha_1\beta_1 \\
h_4 &= 10T^7 - 210T^5 + 50\beta_1 T^4 - 450T^3 + 15\beta_2 T^2 - (50\beta_1^2 + 1350 - 150\alpha_1^2)T \\
&\quad +150\beta_1 - 15\beta_2 \\
h_3 &= 80\alpha_1 T^5 + (1000\alpha_1 - 20\alpha_2)T^3 - 400\alpha_1\beta_1 T^2 - (1800\alpha_1 - 60\alpha_2)T \\
&\quad +200\alpha_1\beta_1 + 20\beta_2\alpha_1 - 20\alpha_2\beta_1 \\
h_2 &= 5T^9 - 60T^7 + 1710T^5 + (45\beta_2 - 2100\beta_1)T^4 + (300\beta_1^2 - 6300 - 100\alpha_1^2)T^3 \\
&\quad +(1800\beta_1 - 90\beta_2)T^2 + (4725 + 300\alpha_1^2 + 300\beta_1^2)T - 135\beta_2 - 100\beta_1^3 \\
&\quad -100\alpha_1^2\beta_1 - 900\beta_1 \\
h_1 &= 40\alpha_1 T^7 + (30\alpha_2 - 1140\alpha_1)T^5 + 200\alpha_1\beta_1 T^4 - (2400\alpha_1 - 60\alpha_2)T^3 \\
&\quad +(60\beta_2\alpha_1 - 60\alpha_2\beta_1 + 600\alpha_1\beta_1)T^2 - (900\alpha_1 + 450\alpha_2 + 200\alpha_1^3 + 200\alpha_1\beta_1^2)T \\
&\quad +60\alpha_2\beta_1 - 60\beta_2\alpha_1 \\
h_0 &= T^{11} + 25T^9 - 15\beta_1 T^8 - 870T^7 + (40\beta_1 - 7\beta_2)T^6 + (70\alpha_1^2 - 9630 + 30\beta_1^2)T^5 \\
&\quad +(5850\beta_1 - 75\beta_2)T^4 + (40\beta_2\beta_1 + 40\alpha_2\alpha_1 - 2475 - 900\alpha_1^2 - 1300\beta_1^2)T^3 \\
&\quad +(100\alpha_1^2\beta_1 + 495\beta_2 + 100\beta_1^3)T^2 + (6\alpha_2^2 + 4725 - 240\alpha_2\alpha_1 - 240\beta_2\beta_1 \\
&\quad +750\beta_1^2 + 6\beta_2^2 + 750\alpha_1^2)T - 20\alpha_1^2\beta_2 - 675\beta_1 - 45\beta_2 - 100\alpha_1^2\beta_1 - 100\beta_1^3 \\
&\quad +40\alpha_2\alpha_1\beta_1 + 20\beta_1^2\beta_2
\end{aligned}
$$

$$
\begin{aligned}
q_7 &= 3\alpha_2 - 30\alpha_1\\
q_6 &= -60T^4 + 40\beta_1 T^3 + 120T^2 - (15\beta_2 - 60\beta_1)T + 35\beta_1^2 + 15\alpha_1^2 + 580\\
q_5 &= 30\alpha_1 T^4 - (27\alpha_2 - 90\alpha_1)T^2 + 120\alpha_1\beta_1 T - 27\alpha_2 + 540\alpha_1\\
q_4 &= 30\beta_1 T^5 - 360T^4 + (15\beta_2 + 600\beta_1)T^3 + (3360 + 225\alpha_1^2 - 75\beta_1^2)T^2\\
&\quad +(135\beta_2 - 1350\beta_1)T + 225\beta_1^2 - 30\alpha_2\alpha_1 + 525\alpha_1^2 - 30\beta_2\beta_1 + 840\\
q_3 &= 40\alpha_1 T^6 + (1950\alpha_1 - 15\alpha_2)T^4 - 400\alpha_1\beta_1 T^3 + (90\alpha_2 + 4500\alpha_1)T^2\\
&\quad +(60\beta_2\alpha_1 - 1800\alpha_1\beta_1 - 60\alpha_2\beta_1)T - 450\alpha_1 + 100\alpha_1^3 + 100\alpha_1\beta_1^2 - 135\alpha_2\\
q_2 &= 60T^8 + 3360T^6 - (1620\beta_1 - 27\beta_2)T^5 + (225\beta_1^2 - 75\alpha_1^2 + 19560)T^4\\
&\quad -(16200\beta_1 - 270\beta_2)T^3 + (450\alpha_1^2 - 9120 + 4050\beta_1^2)T^2\\
&\quad +(675\beta_2 + 2700\beta_1 - 300\beta_1^3 - 300\alpha_1^2\beta_1)T + 3036 + 9\alpha_2^2 - 180\alpha_2\alpha_1\\
&\quad +225\beta_1^2 + 225\alpha_1^2 + 9\beta_2^2 - 180\beta_2\beta_1\\
q_1 &= 15\alpha_1 T^8 + (15\alpha_2 - 90\alpha_1)T^6 + 120\alpha_1\beta_1 T^5 + (405\alpha_2 - 5400\alpha_1)T^4\\
&\quad +(3000\alpha_1\beta_1 - 60\alpha_2\beta_1 + 60\beta_2\alpha_1)T^3 + (1485\alpha_2 - 300\alpha_1\beta_1^2 - 1350\alpha_1 - 300\alpha_1^3)T^2\\
&\quad +(540\beta_2\alpha_1 - 540\alpha_2\beta_1)T + 300\alpha_1^3 - 120\alpha_1\beta_1\beta_2 - 60\alpha_2\alpha_1^2 + 135\alpha_2 + 60\alpha_2\beta_1^2\\
&\quad +300\alpha_1\beta_1^2 + 2025\alpha_1\\
q_0 &= 30T^{10} - 5\beta_1 T^9 + 930T^8 - (240\beta_1 + 3\beta_2)T^7 + (15\beta_1^2 + 3820 + 35\alpha_1^2)T^6\\
&\quad +(1710\beta_1 - 153\beta_2)T^5 + (30\beta_2\beta_1 + 30\alpha_2\alpha_1 - 975\beta_1^2 + 35940 - 75\alpha_1^2)T^4\\
&\quad +(100\beta_1^3 + 100\alpha_1^2\beta_1 + 135\beta_2 - 23400\beta_1)T^3\\
&\quad +(9\beta_2^2 + 23286 + 9\alpha_2^2 - 360\beta_2\beta_1 - 360\alpha_2\alpha_1 + 4725\alpha_1^2 + 8325\beta_1^2)T^2\\
&\quad +(120\alpha_2\alpha_1\beta_1 - 60\alpha_1^2\beta_2 - 1500\alpha_1^2\beta_1 + 60\beta_1^2\beta_2 - 7425\beta_1 - 675\beta_2 - 1500\beta_1^3)T\\
&\quad +506 + 9\beta_2^2 + 100\beta_1^4 + 675\alpha_1^2 + 100\alpha_1^4 + 9\alpha_2^2 + 90\beta_2\beta_1 + 200\alpha_1^2\beta_1^2\\
&\quad +675\beta_1^2 + 90\alpha_2\alpha_1.
\end{aligned}
$$

Similar "polynomial " formulas for the rank 4 MRW solutions of the NLS equation are obtained and discussed in our article in Nonlinearity n.12 (2013) cited in the abstract.

Assume now that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are linked with first 7 times t_k of the NLS hierarchy by the formulas

$$\alpha_1 = -3\cdot 2^4(t_2 - 10t_4 + 70t_6) = -48(t_2 - 10t_4 + 70t_6),$$
$$\beta_1 = 3\cdot 2^5(t_3 - 10t_5 + 70t_7) = 96(t_3 - 10t_5 + 70t_7)$$
$$\alpha_2 = -2^5\cdot 3\cdot 5(t_2 - 42t_4 - 518t_6) = -480(t_2 - 42t_4 - 518t_6),$$
$$\beta_2 = 3\cdot 5\cdot 2^7(t_3 - 14t_5 + 126t_7) = 1920(t_3 - 14t_5 + 126t_7),$$

X, T, Φ in the formulas above are linked with t_k by the formulas:

$$\Phi := 2(t_1 - 3t_3 + 10t_5 - 35t_7),$$
$$X =:= 2(x - 6t_2 + 30t_4 - 140t_6),$$
$$T = T = 2^2(t_1 - 6t_3 + 30t_5 - 140t_7),$$

and the exponential in the first formula of the previous section is replaced by $e^{i\Phi}$ Then the formula above represents a joint rank-3 solution of the first 7 equations of the NLS hierarchy.

Definitions.

$$q_{2n}(k) := \prod_{j=1}^{n} \left(k^2 + i\cot\omega_j\right), \quad \omega_j := \frac{(2j-1)\pi}{4n+2},$$

$$\Phi(k) := \sum_{l\geq 1} i^l \phi_l k^{l+1}, \quad \phi_l,\ k \in \mathbb{R}.$$

$$f(k,x,t) := \frac{\exp(kx + ik^2t + \Phi(k))}{q_{2n}(k)}, \Rightarrow -if_t = f_{xx}.$$

$$D_k := \frac{k^2}{k^2+1}\frac{\partial}{\partial k}, \quad j = 1, \ldots, n,$$
$$f_j(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=1}\ ,$$
$$f_{n+j}(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=-1}\ .$$

$$W_1 := W(f_1, \dots, f_{2n}) \equiv \det A, \quad A_{lj} = \partial_x^{l-1} f_j, \quad W_2 := W(f_1, \dots, f_{2n}, f).$$

Theorem: The function

$$p_n(x,t) := (-1)^n q_{2n}(0) e^{2it} \frac{W_2 \,|_{k=0}}{W_1}$$

describes a 2n parametric familly of solutions depending on $2n$ real parameters $\phi_1, \dots, \phi_{2n}$, which describes rank n solutions of the focusing NLS equation.

$$P_1 = \left(1 - \frac{4(1 + iT_1)}{1 + X^2 + T_1^2}\right) e^{i\Phi}$$

$$X := 2\sum_{k\geq 0}(-1)^k(2k+1)\phi_{2k},$$

$$T_1 := 2\sum_{k\geq 0}(-1)^k(2k+2)\phi_{2k+1}.$$

The phases Φ and ϕ_k depend on $x \equiv t_0$ and on times $t_k, k \geq 1$ in a following way:

$$\Phi := \sum_{k\geq 0}(-1)^k \binom{2k+2}{k+1} t_{2k+1},$$

$$\phi_j := \sum_{k\geq 0}(-1)^k \binom{2k+j+1}{k+j+1} t_{2k+j}$$

For $j \leq 7$ and $2k + j \leq 7$ after setting $t_k = 0, \forall k > 7$ the we get the truncated systems allowing to express all $\phi_j,\ j = 0, 1, \dots 7$ by means of the first times $t_j, j = 0, \dots 7$:

$$\begin{aligned}
\Phi &:= 2(t_1 - 3t_3 + 10t_5 - 35t_7),\\
\phi_0 &:= t_0 - 3t_2 + 10t_4 - 35t_6,\\
\phi_1 &:= t_1 - 4t_3 + 15t_5 - 56t_7,\\
\phi_2 &:= t_2 - 5t_4 + 21t_6,\\
\phi_3 &:= t_3 - 6t_5 + 28t_7,\\
\phi_4 &:= t_4 - 7t_6,\\
\phi_5 &:= t_5 - 8t_7,\\
\phi_6 &:= t_6,\\
\phi_7 &:= t_7.
\end{aligned}$$

Reciprocally

$$
\begin{aligned}
t_0 &= \phi_0 + 3\phi_2 + 5\phi_4 + 7\phi_6,\\
t_1 &= \phi_1 + 4\phi_3 + 9\phi_5 + 16\phi_7,\\
t_2 &= \phi_2 + 5\phi_4 + 14\phi_6,\\
t_3 &= \phi_3 + 6\phi_5 + 20\phi_7\\
t_4 &= \phi_4 + 7\phi_6,\\
t_5 &= \phi_5 + 8\phi_7,\\
t_6 &= \phi_6,\\
t_7 &= \phi_7.
\end{aligned}
$$

Therefore, for construct the analogs of P_1 breather for the first 7 members of the NLS hierarchy we define X, T_1 and $\Phi = \Phi_7$: by the formulas

$$X := 2(x - 6t_2 + 30t_4 - 140t_6)$$
$$T_1 := 4(t_1 - 6t_3 + 30t_5 - 140t_7)$$
$$\Phi_7 := 2(t_1 - 3t_3 + 10t_5 - 35t_7)$$

$$P_1(x, t_1, t_2, \dots, t_7) := \left(1 - \frac{4(1 + iT_1)}{1 + X^2 + T_1^2}\right) e^{i\Phi_7}.$$

Formula above obviously reduces to the single $P_1^k(x,t)$ breather solution with $k \leq 7$ for NLS_k equation, deleting all t_j with $j \neq k, \; j \geq 1$.

$$P_1^1 = \left(1 - \frac{4(1+4it)}{4x^2+16t^2+1}\right) e^{2it},$$

$$P_1^2 = \left(1 - \frac{4(1+iT)}{4(x-6t)^2+T^2+1}\right),$$

$$P_1^3 = \left(1 - \frac{4(1+24it)}{4x^2+2^6 3^2 t^2+1}\right) e^{6it},$$

$$P_1^4 = \left(1 - \frac{4(1+iT)}{4(x-30t)^2+T^2+1}\right),$$

$$P_1^5 = \left(1 - \frac{4(1+120it)}{4x^2+2^6 3^2 5^2 t^2+1}\right) e^{20it},$$

$$P_1^6 = \left(1 - \frac{4(1+iT)}{4(x-140t)^2+T^2+1}\right)$$

$$P_1^7 = \left(1 - \frac{4(1+120it)}{4x^2+2^{12} 5^2 7^2 t^2+1}\right) e^{70it}$$

NLS_6 equation

$$
\begin{aligned}
p_t + p_{xxxxxxx} + 14\,[\,p^2 p_{xxxxx} + 3p^* p_x p_{xxxx} + 2pp_x^* p_{xxxx} \\
+ pp_x p_{xxxx}^* + 5p^4 p_{xxx} + 7{p_x}^2 p_{xxx} + 5p^* p_{xx} p_{xxx} \\
+ 3pp_{xx}^* p_{xxx} + 2p_x^2 p_{xxx}^* + 2pp_{xx} p_{xxx}^* + 20p^* p^2 p_x p_{xx} \\
+ 10p^2 pp_x^* p_{xx} + 10p^2 pp_x p_{xx}^* + 5p_x^* p_{xx}^2 + 8p_x {p_{xx}}^2 \\
+ 5(p^*)^2 p_x^3 + 20p^2 {p_x}^2 p_x + 5p^2 {p_x}^2 p_x^* + 10p^6 p_x\,] = 0.
\end{aligned}
$$

NLS_7 equation

$$
\begin{aligned}
&ip_t + p_{xxxxxxxx} + 16p^2 p_{xxxxxx} + 2p^2 p^*_{xxxxxx} + 56p^* p_x p_{xxxxx} + \\
&+ 40pp^*_x p_{xxxxx} + 12pp_x p^*_{xxxxx} + 98p^4 p_{xxxx} + 168|p_x|^2 p_{xxxx} + \\
&+ 112p^* p_{xx} p_{xxxx} + 72pp^*_{xx} p_{xxxx} + 28p^2 p^2 p^*_{xxxx} + 42p^2_x p^*_{xxxx} + \\
&+ 44pp_{xx} p^*_{xxxx} + 68pp_{xxx} p^*_{xxx} + 476p^2 p^* p_x p_{xxx} + 252p_x p^*_{xx} p_{xxx} + \\
&+ 308pp^2 p^*_x p_{xxx} + 308p^*_x p_{xx} p_{xxx} + 70p^* p^2_{xxx} + 196p_x p_{xx} p^*_{xxx} + \\
&+ 168pp^2 p_x p^*_{xxx} + 56p^3 p^*_x p^*_{xxx} + 280p^6 p_{xx} + 1456p^2 {p_x}^2 p_{xx} + \\
&+ 490(p^*)^2 p^2_x p_{xx} + 238p^2 (p^*_x)^2 p_{xx} + 588p^2 p^2_x p^*_{xx} + 336p^2 {p_x}^2 p^*_{xx} + \\
&+ 140p^4 p^2 p^*_{xx} + 42p^3 (p^*_{xx})^2 + 392p^2 p{p_{xx}}^2 + 322p^2 p^* p^2_{xx} + \\
&+ 182p^2_{xx} p^*_{xx} + 560p^4 p^* p^2_x + 560p^4 p{p_x}^2 + 420p^* p^2_x {p_x}^2 + \\
&+ 140p^3 p^2 (p^*_x)^2 + 378{p_x}^4 p + 70p^8 p = 0.
\end{aligned}
$$

Thank you for your attention