Lax pairs, recursion operators and new multi-parameter bi-Hamiltonian systems in (3+1) dimensions

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(based in part on the results obtained together with D. Yazıcı)

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Outline I

Outline II

Basic concepts

Evolutionary Hirota type 3+1-dimensional equations generalize the famous *heavenly equations* which describe self-dual gravity.

$$F = f - u_{tt}g = 0 \iff u_{tt} = \frac{f}{g} \tag{1}$$

where f and g depend on $u_{ij} = \frac{\partial^2 u}{\partial z_i \partial z_j}$, $\{z_i\} = \{t, z_1, z_2, z_3\}$. Here $u = u(t, z_1, z_2, z_3)$.

The main problems to be solved

We will describe a general method for obtaining Lax pairs and recursion operators for equations of the form (1) which possess a Lagrangian. We show that such equations have a general symplectic Monge–Ampère form and find their Lagrangians.

We study equations of the form (1) because of possible applications to self-dual gravity. In 1975, J. F. Plebański had shown that the Einstein equations with Euclidean or neutral signature with the constraint of Hodge self-duality reduce to a single scalar equation for the Kähler potential of the metric $u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = 1$ which he called the 1st heavenly equation. The metric is given by $ds^2 = u_{i\bar{j}}dz^i d\bar{z}^j$. He also derived second heavenly equation and the corresponding metric. Recently we have shown that some further equations of this type, which will be considered in this talk, also provide a description of self-dual gravity.

Interesting solutions of the first heavenly equation (complex Monge-Ampère equation with the reality condition) are gravitational instantons which yield a semi-classical description of the future theory of quantum gravity. There is one important gravitational instanton $K3$ whose metric is still unknown. It is named after three geometers: Kummer, Kähler, and Kodaira. $K3$ is a fundamental difficult problem similar to $K2$, a difficult mountain in the Karakorum region of Himalayas. One of possible approaches to $K3$ is to widen the class of scalar PDEs governing self-dual gravity with the hope that their solutions more readily will describe $K3$ in the corresponding new variables.

Second-order equations possessing a Lagrangian

The Fréchet derivative operator (linearization) of equation (1) reads

$$\begin{aligned} D_F = & -gD_t^2 + (f_{u_{t1}} - u_{tt}g_{u_{t1}})D_tD_1 + (f_{u_{t2}} - u_{tt}g_{u_{t2}})D_tD_2 \\ & + (f_{u_{t3}} - u_{tt}g_{u_{t3}})D_tD_3 + (f_{u_{11}} - u_{tt}g_{u_{11}})D_1^2 \\ & + (f_{u_{12}} - u_{tt}g_{u_{12}})D_1D_2 + (f_{u_{13}} - u_{tt}g_{u_{13}})D_1D_3 \\ & + (f_{u_{22}} - u_{tt}g_{u_{22}})D_2^2 + (f_{u_{23}} - u_{tt}g_{u_{23}})D_2D_3 + (f_{u_{33}} - u_{tt}g_{u_{33}})D_3^2 \end{aligned} \tag{2}$$

where D_i, D_t denote operators of total derivatives.

- The adjoint Fréchet derivative operator has the form

$$
\begin{aligned}
D_F^* = &-D_t^2 g + D_t D_1 (f_{u_{t1}} - u_{tt} g_{u_{t1}}) + D_t D_2 (f_{u_{t2}} - u_{tt} g_{u_{t2}}) \\
&+ D_t D_3 (f_{u_{t3}} - u_{tt} g_{u_{t3}}) + D_1^2 (f_{u_{11}} - u_{tt} g_{u_{11}}) \\
&+ D_1 D_2 (f_{u_{12}} - u_{tt} g_{u_{12}}) + D_1 D_3 (f_{u_{13}} - u_{tt} g_{u_{13}}) \\
&+ D_2^2 (f_{u_{22}} - u_{tt} g_{u_{22}}) + D_2 D_3 (f_{u_{23}} - u_{tt} g_{u_{23}}) + D_3^2 (f_{u_{33}} - u_{tt} g_{u_{33}})
\end{aligned}
$$

- **Helmholtz conditions**: equation (1) is an Euler-Lagrange equation for a variational problem iff its Fréchet derivative is self-adjoint, $D_F^* = D_F$.

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& + D_t D_3 (f_{u_{t3}} - u_{tt} g_{u_{t3}}) + D_1^2 (f_{u_{11}} - u_{tt} g_{u_{11}}) \\
& + D_1 D_2 (f_{u_{12}} - u_{tt} g_{u_{12}}) + D_1 D_3 (f_{u_{13}} - u_{tt} g_{u_{13}}) \\
& + D_2^2 (f_{u_{22}} - u_{tt} g_{u_{22}}) + D_2 D_3 (f_{u_{23}} - u_{tt} g_{u_{23}}) + D_3^2 (f_{u_{33}} - u_{tt} g_{u_{33}})
\end{aligned}
$$

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$$
\begin{aligned}
F = {} & a_1\{u_{tt}(u_{11}u_{22}-u_{12}^2)-u_{t1}(u_{t1}u_{22}-u_{t2}u_{12})+u_{t2}(u_{t1}u_{12}-u_{t2}u_{11})\}\\
&+a_2\{u_{tt}(u_{11}u_{33}-u_{13}^2)-u_{t1}(u_{t1}u_{33}-u_{t3}u_{13})+u_{t3}(u_{t1}u_{13}-u_{t3}u_{11})\}\\
&+a_3\{u_{tt}(u_{22}u_{33}-u_{23}^2)-u_{t2}(u_{t2}u_{33}-u_{t3}u_{23})+u_{t3}(u_{t2}u_{23}-u_{t3}u_{22})\}\\
&+a_4\{u_{tt}(u_{11}u_{23}-u_{12}u_{13})-u_{t1}(u_{t1}u_{23}-u_{t2}u_{13})+u_{t3}(u_{t1}u_{12}-u_{t2}u_{11})\}\\
&+a_5\{u_{tt}(u_{12}u_{23}-u_{13}u_{22})-u_{t1}(u_{t2}u_{23}-u_{t3}u_{22})+u_{t2}(u_{t2}u_{13}-u_{t3}u_{12})\}\\
&+a_6\{u_{tt}(u_{12}u_{33}-u_{13}u_{23})-u_{t1}(u_{t2}u_{33}-u_{t3}u_{23})+u_{t3}(u_{t2}u_{13}-u_{t3}u_{12})\}\\
&+b_1\{u_{t1}(u_{12}u_{23}-u_{13}u_{22})-u_{t2}(u_{11}u_{23}-u_{12}u_{13})+u_{t3}(u_{11}u_{22}-u_{12}^2)\}\\
&+b_2\{u_{t1}(u_{12}u_{33}-u_{13}u_{23})-u_{t2}(u_{11}u_{33}-u_{13}^2)+u_{t3}(u_{11}u_{23}-u_{12}u_{13})\}\\
&+b_3\{u_{t1}(u_{22}u_{33}-u_{23}^2)-u_{t2}(u_{12}u_{33}-u_{13}u_{23})+u_{t3}(u_{12}u_{23}-u_{13}u_{22})\}\\
&+b_4\{u_{11}(u_{22}u_{33}-u_{23}^2)-u_{12}(u_{12}u_{33}-u_{13}u_{23})+u_{13}(u_{12}u_{23}-u_{13}u_{22})\}
\end{aligned}
$$

$$
\begin{aligned}
&+ a_7(u_{tt}u_{11} - u_{t1}^2) + a_8(u_{tt}u_{12} - u_{t1}u_{t2}) + a_9(u_{tt}u_{13} - u_{t1}u_{t3}) \\
&+ a_{10}(u_{tt}u_{22} - u_{t2}^2) + a_{11}(u_{tt}u_{23} - u_{t2}u_{t3}) + a_{12}(u_{tt}u_{33} - u_{t3}^2) + a_{13}u_{tt} \\
&+ c_1(u_{t1}u_{12} - u_{t2}u_{11}) + c_2(u_{t1}u_{13} - u_{t3}u_{11}) + c_3(u_{t1}u_{22} - u_{t2}u_{12}) \\
&+ c_4(u_{t1}u_{23} - u_{t2}u_{13}) + c_5(u_{t2}u_{23} - u_{t3}u_{22}) + c_6(u_{t1}u_{33} - u_{t3}u_{13}) \\
&+ c_7(u_{t2}u_{33} - u_{t3}u_{23}) + c_8(u_{t2}u_{13} - u_{t3}u_{12}) + c_{8'}(u_{t1}u_{23} - u_{t3}u_{12}) \\
&+ c_9(u_{11}u_{23} - u_{12}u_{13}) + c_{10}(u_{12}u_{23} - u_{13}u_{22}) + c_{11}(u_{12}u_{33} - u_{13}u_{23}) \\
&+ c_{12}(u_{11}u_{22} - u_{12}^2) + c_{13}(u_{11}u_{33} - u_{13}^2) + c_{14}(u_{22}u_{33} - u_{23}^2) \\
&+ c_{15}u_{t1} + c_{16}u_{t2} + c_{17}u_{t3} + c_{18}u_{11} + c_{19}u_{12} + c_{20}u_{13} + c_{21}u_{22} + c_{22}u_{23} \\
&+ c_{23}u_{33} + c_{24} = 0 \qquad (3)
\end{aligned}
$$

where the quadratic terms have the Monge–Ampère form.

The *homotopy formula* (see P. Olver's book) yields the Lagrangian for $F = f - u_{tt}g$ in (3)

$$L[u] = \int_0^1 u \cdot F[\lambda u]\, d\lambda = \int_0^1 u \cdot f[\lambda u]\, d\lambda - \int_0^1 u \cdot (\lambda u_{tt}) g[\lambda u]\, d\lambda$$

with the result

$$
\begin{aligned}
L = \frac{u}{4} \Big\langle & a_1\{u_{tt}(u_{11}u_{22} - u_{12}^2) - u_{t1}(u_{t1}u_{22} - u_{t2}u_{12}) + u_{t2}(u_{t1}u_{12} - u_{t2}u_{11})\} \\
& + a_2\{u_{tt}(u_{11}u_{33} - u_{13}^2) - u_{t1}(u_{t1}u_{33} - u_{t3}u_{13}) + u_{t3}(u_{t1}u_{13} - u_{t3}u_{11})\} \\
& + a_3\{u_{tt}(u_{22}u_{33} - u_{23}^2) - u_{t2}(u_{t2}u_{33} - u_{t3}u_{23}) + u_{t3}(u_{t2}u_{23} - u_{t3}u_{22})\} \\
& + a_4\{u_{tt}(u_{11}u_{23} - u_{12}u_{13}) - u_{t1}(u_{t1}u_{23} - u_{t2}u_{13}) + u_{t3}(u_{t1}u_{12} - u_{t2}u_{11})\} \\
& + a_5\{u_{tt}(u_{12}u_{23} - u_{13}u_{22}) - u_{t1}(u_{t2}u_{23} - u_{t3}u_{22}) + u_{t2}(u_{t2}u_{13} - u_{t3}u_{12})\} \\
& + a_6\{u_{tt}(u_{12}u_{33} - u_{13}u_{23}) - u_{t1}(u_{t2}u_{33} - u_{t3}u_{23}) + u_{t3}(u_{t2}u_{13} - u_{t3}u_{12})\} \\
& + b_1\{u_{t1}(u_{12}u_{23} - u_{13}u_{22}) - u_{t2}(u_{11}u_{23} - u_{12}u_{13}) + u_{t3}(u_{11}u_{22} - u_{12}^2)\} \\
& + b_2\{u_{t1}(u_{12}u_{33} - u_{13}u_{23}) - u_{t2}(u_{11}u_{33} - u_{13}^2) + u_{t3}(u_{11}u_{23} - u_{12}u_{13})\} \\
& + b_3\{u_{t1}(u_{22}u_{33} - u_{23}^2) - u_{t2}(u_{12}u_{33} - u_{13}u_{23}) + u_{t3}(u_{12}u_{23} - u_{13}u_{22})\} \\
& + b_4\{u_{11}(u_{22}u_{33} - u_{23}^2) - u_{12}(u_{12}u_{33} - u_{13}u_{23}) + u_{13}(u_{12}u_{23} - u_{13}u_{22})\} \Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{u}{3}\big\{a_7(u_{tt}u_{11} - u_{t1}^2) + a_8(u_{tt}u_{12} - u_{t1}u_{t2}) + a_9(u_{tt}u_{13} - u_{t1}u_{t3}) \\
&+ a_{10}(u_{tt}u_{22} - u_{t2}^2) + a_{11}(u_{tt}u_{23} - u_{t2}u_{t3}) + a_{12}(u_{tt}u_{33} - u_{t3}^2) \\
&+ c_1(u_{t1}u_{12} - u_{t2}u_{11}) + c_2(u_{t1}u_{13} - u_{t3}u_{11}) + c_3(u_{t1}u_{22} - u_{t2}u_{12}) \\
&+ c_4(u_{t1}u_{23} - u_{t2}u_{13}) + c_5(u_{t2}u_{23} - u_{t3}u_{22}) + c_6(u_{t1}u_{33} - u_{t3}u_{13}) \\
&+ c_7(u_{t2}u_{33} - u_{t3}u_{23}) + c_8(u_{t2}u_{13} - u_{t3}u_{12}) + c_{8'}(u_{t1}u_{23} - u_{t3}u_{12}) \\
&+ c_9(u_{11}u_{23} - u_{12}u_{13}) + c_{10}(u_{12}u_{23} - u_{13}u_{22}) + c_{11}(u_{12}u_{33} - u_{13}u_{23}) \\
&+ c_{12}(u_{11}u_{22} - u_{12}^2) + c_{13}(u_{11}u_{33} - u_{13}^2) + c_{14}(u_{22}u_{33} - u_{23}^2)\big\} \\
&+ \frac{u}{2}(a_{13}u_{tt} + c_{15}u_{t1} + c_{16}u_{t2} + c_{17}u_{t3} + c_{18}u_{11} + c_{19}u_{12} + c_{20}u_{13} \\
&+ c_{21}u_{22} + c_{22}u_{23} + c_{23}u_{33}) + c_{24}u. \qquad (4)
\end{aligned}
$$

Operators $L_{ij(k)}$ and some of their properties

Symmetry condition is the differential compatibility condition of (3) and the Lie equation $u_\tau = \varphi$, where φ is the symmetry characteristic and τ is the group parameter. It has the form of Fréchet derivative (linearization) of equation (3). For a more compact form, we introduce linear differential operators

$$L_{ij(k)} = u_{jk}D_i - u_{ik}D_j = -L_{ji(k)} \quad \Longrightarrow L_{ii(k)} = 0, \tag{5}$$

$$L_{ij(k)} + L_{ki(j)} + L_{jk(i)} = 0, \quad D_l L_{ij(k)} - D_k L_{ij(l)} = L_{ij(k)} D_l - L_{ij(l)} D_k$$

$$L_{ij(l)} D_k + L_{jk(l)} D_i + L_{ki(l)} D_j = 0 \tag{6}$$

where $i, j, k = 1, 2, 3, t$. For example,

$$L_{12(3)} = u_{23}D_1 - u_{13}D_2, \quad L_{12(t)} = u_{2t}D_1 - u_{1t}D_2.$$

$$
\begin{aligned}
&\big\{a_7(L_{t1(1)}D_t - L_{t1(t)}D_1) + a_8(L_{t1(2)}D_t - L_{t1(t)}D_2)\\
&+ a_9(L_{t1(3)}D_t - L_{t1(t)}D_3) + a_{10}(L_{t2(2)}D_t - L_{t2(t)}D_2)\\
&+ a_{11}(L_{t2(3)}D_t - L_{t2(t)}D_3) + a_{12}(L_{t3(3)}D_t - L_{t3(t)}D_3)\\
&+ c_1(L_{12(1)}D_t - L_{12(t)}D_1) + c_2(L_{13(1)}D_t - L_{13(t)}D_1)\\
&+ c_3(L_{12(2)}D_t - L_{12(t)}D_2) + c_4(L_{12(3)}D_t - L_{12(t)}D_3)\\
&+ c_5(L_{23(2)}D_t - L_{23(t)}D_2) + c_6(L_{13(3)}D_t - L_{13(t)}D_3)\\
&+ c_7(L_{23(3)}D_t - L_{23(t)}D_3) + c_8(L_{23(1)}D_t - L_{23(t)}D_1)\\
&+ c_{8'}(L_{13(2)}D_t - L_{13(t)}D_2) + c_9(L_{12(3)}D_1 - L_{12(1)}D_3)\\
&+ c_{10}(L_{23(2)}D_1 - L_{23(1)}D_2) + c_{11}(L_{23(3)}D_1 - L_{23(1)}D_3)\\
&+ c_{12}(L_{12(2)}D_1 - L_{12(1)}D_2) + c_{13}(L_{13(3)}D_1 - L_{13(1)}D_3)\\
&+ c_{14}(L_{23(3)}D_2 - L_{23(2)}D_3)\big\}\varphi = 0 \qquad (7)
\end{aligned}
$$

in the particular case $b_i = 0$, $a_i = 0$ for $i = 1, \dots 6$, $a_{ij} = 0$. We have also skipped the terms which do not involve $L_{ij(k)}$

$$\{a_{13}D_t^2 + c_{15}D_tD_1 + c_{16}D_tD_2 + c_{17}D_tD_3 + c_{18}D_1^2 + c_{19}D_1D_2 + c_{20}D_1D_3 + c_{21}D_2^2 + c_{22}D_2D_3 + c_{23}D_3^2\}\varphi = 0.$$

Skew-factorized form of the symmetry condition

The linear operator of the symmetry condition for integrable equations of the form (3) should be converted to the "skew-factorized" form

$$(A_1 B_2 - A_2 B_1)\varphi = 0 \tag{8}$$

where A_i and B_i are first order linear differential operators. These operators should satisfy the commutator relations

$$[A_1, A_2] = 0, \quad [A_1, B_2] - [A_2, B_1] = 0, \quad [B_1, B_2] = 0 \tag{9}$$

on solutions of the equation (3).

Skew-factorized form of the symmetry condition

It immediately follows that the following two operators also commute on solutions

$$X_1 = \lambda A_1 + B_1, \quad X_2 = \lambda A_2 + B_2, \qquad [X_1, X_2] = 0 \tag{10}$$

and therefore constitute Lax representation for equation (3) with λ being a spectral parameter.

Symmetry condition in the form (8) not only provides the Lax pair for equation (3) but also leads directly to recursion relations for symmetries

$$A_1\tilde{\varphi} = B_1\varphi, \quad A_2\tilde{\varphi} = B_2\varphi \tag{11}$$

where $\tilde{\varphi}$ is a symmetry if φ is also a symmetry and vice versa.

Indeed, equations (11) together with (9) imply $(A_1B_2 - A_2B_1)\varphi = [A_1, A_2]\tilde{\varphi} = 0$, so φ is a symmetry characteristic. Moreover, due to (11)

$$(A_1B_2 - A_2B_1)\tilde{\varphi} = \big([A_1, B_2] - [A_2, B_1] + B_2A_1 - B_1A_2\big)\tilde{\varphi} = [B_2, B_1]\varphi = 0$$

which shows that $\tilde{\varphi}$ satisfies the symmetry condition (8) and hence is also a symmetry. The equations (11) define an auto-Bäcklund transformation between the symmetry conditions written for φ and $\tilde{\varphi}$. Hence, the auto-Bäcklund transformation of the symmetry condition is a recursion operator.

We note that the skew-factorized form (8) and the properties (9) of the operators A_i and B_i remain invariant under the simultaneous interchange $A_1 \leftrightarrow B_1$ and $A_2 \leftrightarrow B_2$.

Our procedure extends A. Sergyeyev's method for constructing recursion operators. Namely, we start with the skew-factorized form of the symmetry condition and extract from it a "special" Lax pair instead of building it from a previously known Lax pair. After that we construct a recursion operator from this newly found Lax pair.

Second heavenly equation

All known heavenly equations, describing self-dual gravity, can be treated in a unified way according to this approach.
The **second heavenly equation** $u_{tt}u_{11} - u_{t1}^2 + u_{t2} + u_{t3} = 0$ has the symmetry condition of the form

$$\{L_{t1(1)}D_t - L_{t1(t)}D_1 + D_2D_t + D_3D_1\}\varphi = 0. \qquad (12)$$

It has the skew-factorized form (8) with the operators $A_1 = D_t$, $A_2 = D_1$, $B_1 = L_{t1(t)} - D_3$, $B_2 = L_{t1(1)} + D_2$ satisfying conditions (9). According to (10) the Lax pair has the form
$X_1 = \lambda D_t + L_{t1(t)} - D_3$, $X_2 = \lambda D_1 + L_{t1(1)} + D_2$ and (11) yields the recursions for symmetries $D_t\tilde{\varphi} = (L_{t1(t)} - D_3)\varphi$, $D_1\tilde{\varphi} = (L_{t1(1)} + D_2)\varphi$.

First heavenly equation

The **first heavenly equation** in the evolutionary form $(u_{tt} - u_{11})u_{23} - (u_{t3} + u_{13})(u_{t2} - u_{12}) = 1$ has the symmetry condition

$$\{L_{t2(t)}D_3 - L_{t2(3)}D_t + L_{23(1)}D_t - L_{23(t)}D_1 + L_{12(3)}D_1 - L_{12(1)}D_3\}\varphi = 0$$

with the skew-factorized form composed from the operators $A_1 = D_t - D_1$, $A_2 = -D_3$, $B_1 = L_{t2(t)} - L_{12(1)} - L_{t1(2)}$, $B_2 = L_{t2(3)} + L_{12(3)}$ which satisfy conditions (9). The Lax pair (10) reads $X_1 = \lambda(D_t - D_1) + L_{t2(t)} - L_{12(1)} - L_{t1(2)}$, $X_2 = -\lambda D_3 + L_{t2(3)} + l_{12(3)}$ while the recursion relations (11) become $(D_t - D_1)\tilde{\varphi} = (L_{t2(t)} - L_{12(1)} - L_{t1(2)})\varphi$ and $-D_3\tilde{\varphi} = (L_{t2(3)} + L_{12(3)})\varphi$.

Modified heavenly equation

The **modified heavenly equation** $u_{1t}u_{2t} - u_{tt}u_{12} + u_{13} = 0$ has the symmetry condition $(L_{t2(1)}D_t - L_{t2(t)}D_1 - D_1D_3)\varphi = 0$. Its skew-factorized form is constructed from the operators $A_1 = D_t$, $A_2 = D_1$, $B_1 = L_{t2(t)} + D_3$, $B_2 = L_{t2(1)}$ obviously satisfying conditions (9). The Lax pair (10) is formed by $X_1 = \lambda D_t + L_{t2(t)} + D_3$ and $X_2 = \lambda D_1 + L_{t2(1)}$. Recursions (11) have the form $D_t\tilde{\varphi} = (L_{t2(t)} + D_3)\varphi$, $D_1\tilde{\varphi} = L_{t2(1)}\varphi$.

Husain equation

Husain equation in the evolutionary form $u_{tt} + u_{11} + u_{t2}u_{13} - u_{t3}u_{12} = 0$ has the symmetry condition $(L_{23(1)}D_t - L_{23(t)}D_1 + D_t^2 + D_1^2)\varphi = 0$. Its skew-factorized form is constituted by the operators $A_1 = D_t$, $A_2 = D_1$, $B_1 = L_{23(t)} - D_1$, $B_2 = L_{23(1)} + D_t$ satisfying all conditions (9). The Lax pair (10) becomes $X_1 = \lambda D_t + L_{23(t)} - D_1$, $X_2 = \lambda D_1 + L_{23(1)} + D_t$ while the recursions (11) read $D_t\tilde{\varphi} = (L_{23(t)} - D_1)\varphi$, $D_1\tilde{\varphi} = (L_{23(1)} + D_t)\varphi$.

General heavenly equation

General heavenly equation in the evolutionary form

$$(\beta+\gamma)(u_{t2}u_{t3}-u_{tt}u_{23}+u_{11}u_{23}-u_{12}u_{13})+(\gamma-\beta)(u_{t2}u_{13}-u_{t3}u_{12})=0 \tag{13}$$

has the symmetry condition

$$\begin{aligned}&\{(\beta+\gamma)(L_{t3(t)}D_2-L_{t3(2)}D_t+L_{12(3)}D_1-L_{12(1)}D_3)\\&+(\gamma-\beta)(L_{23(1)}D_t-L_{23(t)}D_1)\}\varphi=0.\end{aligned} \tag{14}$$

$$A_1=\frac{1}{u_{23}}L_{t2(3)},\quad A_2=\frac{1}{u_{23}}L_{12(3)}$$

$$B_1=\frac{1}{u_{23}}\{(\beta-\gamma)L_{t3(2)}+(\beta+\gamma)L_{13(2)}\},\ B_2=\frac{\beta+\gamma}{u_{23}}L_{t3(2)}.$$

General heavenly equation (continued)

The Lax pair (10) becomes
$X_1 = \frac{\lambda}{u_{23}} L_{t2(3)} + \frac{1}{u_{23}}\{(\beta-\gamma)L_{t3(2)} + (\beta+\gamma)L_{13(2)}\}$,
$X_2 = \frac{\lambda}{u_{23}} L_{12(3)} + \frac{\beta+\gamma}{u_{23}} L_{t3(2)}$. Recursion relations (11) have the form

$$\begin{aligned} \frac{1}{u_{23}} L_{t2(3)}\tilde{\varphi} &= \frac{1}{u_{23}}\{(\beta-\gamma)L_{t3(2)} + (\beta+\gamma)L_{13(2)}\}\varphi \\ \frac{1}{u_{23}} L_{12(3)}\tilde{\varphi} &= \frac{\beta+\gamma}{u_{23}} L_{t3(2)}\varphi. \end{aligned} \tag{15}$$

Further properties of the operators $L_{ij(k)}$

$$L_{ij(k)}D_l - L_{ij(l)}D_k = L_{ij(k)}\frac{1}{u_{jk}}L_{lk(j)} + D_j\frac{1}{u_{jk}}(u_{jk}u_{il} - u_{ik}u_{jl})D_k \quad (16)$$

$$L_{ij(k)}D_l - L_{ij(l)}D_k = L_{lk(j)}\frac{1}{u_{jk}}L_{ij(k)} + D_k\frac{1}{u_{jk}}(u_{jk}u_{il} - u_{ik}u_{jl})D_j \quad (17)$$

$$L_{ij(k)}D_l - L_{ij(l)}D_k = L_{ij(l)}\frac{1}{u_{jl}}L_{lk(j)} + D_j\frac{1}{u_{jl}}(u_{jk}u_{il} - u_{ik}u_{jl})D_l \quad (18)$$

$$L_{ij(k)}D_l - L_{ij(l)}D_k = L_{li(j)}\frac{1}{u_{ij}}L_{kj(i)} - L_{ki(j)}\frac{1}{u_{ij}}L_{lj(i)} + D_i\frac{1}{u_{ij}}(u_{jk}u_{il} - u_{ik}u_{jl})D_j. \quad (19)$$

Here the expression $(u_{jk}u_{il} - u_{ik}u_{jl})$ is precisely the group of terms in the equation (3) corresponding to the terms $(L_{ij(k)}D_l - L_{ij(l)}D_k)\varphi$ in the symmetry condition (7), so that the last terms in all these relations vanish on solutions of (3). Keeping different groups of terms in (3), we obtain skew-factorized forms of the symmetry condition (7) determined by the operators A_i, B_i listed below which satisfy all the conditions (9). Using (10) and (11) we immediately obtain the Lax pair and recursion relations, respectively.

First example

$$a_{11}(u_{tt}u_{23} - u_{t2}u_{t3}) + c_4(u_{t1}u_{23} - u_{t2}u_{13}) + c_5(u_{t2}u_{23} - u_{t3}u_{22}) + c_8(u_{t2}u_{13} - u_{t3}u_{12}) + c_9(u_{11}u_{23} - u_{12}u_{13}) + c_{10}(u_{12}u_{23} - u_{13}u_{22}) = 0 \quad (20)$$

$$A_1 = \frac{1}{u_{23}}L_{t2(3)}, \quad B_1 = \frac{1}{u_{23}}\{(c_4 - c_8)L_{t3(2)} + c_9L_{13(2)} + c_{10}L_{23(2)}\}$$
$$A_2 = -\frac{1}{u_{23}}L_{12(3)}, \quad B_2 = \frac{1}{u_{23}}(c_5L_{23(2)} + c_8L_{13(2)} + a_{11}L_{t3(2)}). \quad (21)$$

Second example

$$\begin{aligned} & a_{11}(u_{tt}u_{23} - u_{t2}u_{t3}) + c_4(u_{t1}u_{23} - u_{t2}u_{13}) + c_7(u_{t2}u_{33} - u_{t3}u_{23}) \\ & + c_8(u_{t2}u_{13} - u_{t3}u_{12}) + c_9(u_{11}u_{23} - u_{12}u_{13}) \\ & + c_{11}(u_{12}u_{33} - u_{13}u_{23}) = 0 \end{aligned} \tag{22}$$

$$A_1 = \frac{1}{u_{23}}L_{t3(2)}, \quad B_1 = \frac{1}{u_{23}}(c_8 L_{t2(3)} + c_9 L_{12(3)} + c_{11} L_{23(3)}) \tag{23}$$

$$A_2 = -\frac{1}{u_{23}}L_{13(2)}, \quad B_2 = \frac{1}{u_{23}}\{(c_4 - c_8)L_{12(3)} + c_7 L_{23(3)} + a_{11} L_{t2(3)}\}.$$

Third example

$$a_8(u_{tt}u_{12} - u_{t1}u_{t2}) + a_{10}(u_{tt}u_{22} - u_{t2}^2) + a_{11}(u_{tt}u_{23} - u_{t2}u_{t3}) \\ + c_7(u_{t2}u_{33} - u_{t3}u_{23}) + c_8(u_{t2}u_{13} - u_{t3}u_{12}) = 0 \qquad (24)$$

$$A_1 = \frac{1}{u_{t2}}L_{23(t)}, \quad B_1 = \frac{1}{u_{t2}}(a_8L_{t1(2)} + a_{10}L_{t2(2)} + a_{11}L_{t3(2)}) \\ A_2 = -\frac{1}{u_{t2}}L_{t2(t)}, \quad B_2 = \frac{1}{u_{t2}}(c_7L_{t3(2)} + c_8L_{t1(2)}). \qquad (25)$$

Fourth example

$$a_{12}(u_{tt}u_{33} - u_{t3}^2) + c_5(u_{t2}u_{23} - u_{t3}u_{22}) + c_6(u_{t1}u_{33} - u_{t3}u_{13}) + c_7(u_{t2}u_{33} - u_{t3}u_{23}) + c_8(u_{t2}u_{13} - u_{t3}u_{12}) = 0 \tag{26}$$

$$A_1 = \frac{1}{u_{t3}}L_{t3(3)}, \quad B_1 = -\frac{1}{u_{t3}}L_{23(t)}, \quad A_2 = \frac{1}{u_{t3}}(c_5L_{t2(3)} + c_8L_{t1(3)})$$
$$B_2 = \frac{1}{u_{t3}}(a_{12}L_{t3(t)} + c_6L_{13(t)} + c_7L_{23(t)}) \tag{27}$$

Fifth example

$$
\begin{aligned}
& a_7(u_{tt}u_{11} - u_{t1}^2) + a_8(u_{tt}u_{12} - u_{t1}u_{t2}) + a_9(u_{tt}u_{13} - u_{t1}u_{t3}) \\
& + c_1(u_{t1}u_{12} - u_{t2}u_{11}) + c_3(u_{t1}u_{22} - u_{t2}u_{12}) \\
& + c_4(u_{t1}u_{23} - u_{t2}u_{13}) = 0
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
A_1 &= \frac{1}{u_{t1}} L_{t1(t)}, \quad B_1 = \frac{1}{u_{t1}}(c_1 L_{t1(1)} + c_3 L_{t2(1)} + c_4 L_{t3(1)}) \\
A_2 &= -\frac{1}{u_{t1}} L_{12(t)}, \quad B_2 = \frac{1}{u_{t1}}(a_7 L_{t1(1)} + a_8 L_{t2(1)} + a_9 L_{t3(1)})
\end{aligned} \tag{29}
$$

.

Some of the equations listed above are not independent since they are related by a permutation of indices. For example, our second equation (22) and the corresponding operators A_i, B_i in (23), determining the Lax pair and recursion relations, can be obtained from the first equation (20) and its operators (21) by the transposition of indices $2 \leftrightarrow 3$ and the permutation of the coefficients $c_5 \leftrightarrow -c_7$, $c_8 \leftrightarrow -c_8$ and $c_{10} \leftrightarrow -c_{11}$.

We can obtain skew-factorized forms of symmetry conditions for many more equations of the type (3) by using permutations of indices $1, 2, 3, t$ with an appropriate permutation of coefficients which leave the equation (3) invariant. Such permutations will however do change the skew factorized forms of the symmetry conditions.

To see that conditions (9) are satisfied for any operators arising from the skew-factorized form of the symmetry condition (7), we note that this form should follow from a linear combination of such pairs of terms in the symmetry condition (7)

$$p(L_{ij(k)}D_l - L_{ij(l)}D_k) + q(L_{mj(k)}D_n - L_{mj(n)}D_k), \qquad (30)$$

with constant p, q, which are simultaneously factorized on solutions of the corresponding equations according the formula (16)

$$L_{ij(k)}D_l - L_{ij(l)}D_k = L_{ij(k)}\frac{1}{u_{jk}}L_{lk(j)} + D_j\frac{1}{u_{jk}}(u_{jk}u_{il} - u_{ik}u_{jl})D_k$$

$$L_{mj(k)}D_n - L_{mj(n)}D_k = L_{mj(k)}\frac{1}{u_{jk}}L_{nk(j)} + D_j\frac{1}{u_{jk}}(u_{jk}u_{mn} - u_{mk}u_{jn})D_k.$$

Here the factors $D_j(1/u_{jk})(E_{p,q})D_k$ are the same in both formulas with the exception of factors E_p, E_q, where

$$E_p = u_{jk}u_{il} - u_{ik}u_{jl}, \quad E_q = u_{jk}u_{mn} - u_{mk}u_{jn} \qquad (32)$$

constitute the parts of the equation $E_{pq} = pE_p + qE_q = 0$ which implies the symmetry condition (30).

Then on solutions of the equation $E_{pq} = 0$ we have

$$p(L_{ij(k)}D_l - L_{ij(l)}D_k) + q(L_{mj(k)}D_n - L_{mj(n)}D_k)$$
$$= pL_{ij(k)}\frac{1}{u_{jk}}L_{lk(j)} + qL_{mj(k)}\frac{1}{u_{jk}}L_{nk(j)} \tag{33}$$

$$A_1 = \frac{1}{u_{jk}}L_{ij(k)}, \quad B_2 = \frac{1}{u_{jk}}L_{lk(j)}, \quad A_2 = -\frac{1}{u_{jk}}L_{mj(k)}, \quad B_1 = \frac{1}{u_{jk}}L_{nk(j)}.$$

We have $[A_1, A_2] = 0$, $[B_1, B_2] = 0$ and $[A_1, B_2] - [A_2, B_1] = 0$ holds on solutions of $E_{pq} = 0$ due to the identity

$$[A_1, B_2] - [A_2, B_1] = \frac{1}{u_{jk}}\left\{D_k\left(\frac{E_{pq}}{u_{jk}}\right)D_j - D_j\left(\frac{E_{pq}}{u_{jk}}\right)D_k\right\}.$$

More general skew-factorized forms of the symmetry condition arise as suitable linear combinations of the equations (33).

$$
\begin{aligned}
&u_t = v,\\
&v_t = \frac{1}{\Delta}\Big\langle a_1(v_1^2u_{22} + v_2^2u_{11} - 2v_1v_2u_{12}) + a_2(v_1^2u_{33} + v_3^2u_{11} - 2v_1v_3u_{13})\\
&+ a_3(v_2^2u_{33} + v_3^2u_{22} - 2v_2v_3u_{23}) + a_4\{v_1(v_1u_{23} - v_2u_{13})\\
&- v_3(v_1u_{12} - v_2u_{11})\} + a_5\{v_1(v_2u_{23} - v_3u_{22}) - v_2(v_2u_{13} - v_3u_{12})\}\\
&+ a_6\{v_2(v_1u_{33} - v_3u_{13}) - v_3(v_1u_{23} - v_3u_{12})\}\\
&+ a_7v_1^2 + a_8v_1v_2 + a_9v_1v_3 + a_{10}v_2^2 + a_{11}v_2v_3 + a_{12}v_3^2\\
&- b_1\{v_1(u_{12}u_{23} - u_{13}u_{22}) - v_2(u_{11}u_{23} - u_{12}u_{13}) + v_3(u_{11}u_{22} - u_{12}^2)\}\\
&- b_2\{v_1(u_{12}u_{33} - u_{13}u_{23}) - v_2(u_{11}u_{33} - u_{13}^2) + v_3(u_{11}u_{23} - u_{12}u_{13})\}\\
&- b_3\{v_1(u_{22}u_{33} - u_{23}^2) - v_2(u_{12}u_{33} - u_{13}u_{23}) + v_3(u_{12}u_{23} - u_{13}u_{22})\}\\
&- b_4\{u_{11}(u_{22}u_{33} - u_{23}^2) - u_{12}(u_{12}u_{33} - u_{13}u_{23}) + u_{13}(u_{12}u_{23} - u_{13}u_{22})\}
\end{aligned}
$$

$$
\begin{aligned}
&- c_1(v_1u_{12} - v_2u_{11}) - c_2(v_1u_{13} - v_3u_{11}) - c_3(v_1u_{22} - v_2u_{12}) \\
&- c_4(v_1u_{23} - v_2u_{13}) - c_5(v_2u_{23} - v_3u_{22}) - c_6(v_1u_{33} - v_3u_{13}) \\
&- c_7(v_2u_{33} - v_3u_{23}) - c_8(v_2u_{13} - v_3u_{12}) - c_{8'}(v_1u_{23} - v_3u_{12}) \\
&- c_9(u_{11}u_{23} - u_{12}u_{13}) - c_{10}(u_{12}u_{23} - u_{13}u_{22}) - c_{11}(u_{12}u_{33} - u_{13}u_{23}) \\
&- c_{12}(u_{11}u_{22} - u_{12}^2) - c_{13}(u_{11}u_{33} - u_{13}^2) - c_{14}(u_{22}u_{33} - u_{23}^2) \\
&- c_{15}v_1 - c_{16}v_2 - c_{17}v_3 - c_{18}u_{11} - c_{19}u_{12} - c_{20}u_{13} \\
&- c_{21}u_{22} - c_{22}u_{23} - c_{23}u_{33} - c_{24}\Big\rangle \\
&\equiv \frac{1}{\Delta}\left(\sum_{i=1}^{12} a_i q^{(ai)} + \sum_{i=1}^{4} b_i q^{(bi)} + {\sum_{i=1}^{24}}' c_i q^{(i)}\right) \equiv \frac{q}{\Delta} \qquad (34)
\end{aligned}
$$

$$\Delta = a_1(u_{11}u_{22} - u_{12}^2) + a_2(u_{11}u_{33} - u_{13}^2) + a_3(u_{22}u_{33} - u_{23}^2) + a_4(u_{11}u_{23} - u_{12}u_{13}) + a_5(u_{12}u_{23} - u_{13}u_{22}) + a_6(u_{12}u_{33} - u_{13}u_{23}) + a_7u_{11} + a_8u_{12} + a_9u_{13} + a_{10}u_{22} + a_{11}u_{23} + a_{12}u_{33} + a_{13}. \quad (35)$$

$$
\begin{aligned}
L = {} & \left(u_t v - \frac{1}{2}v^2\right)\{a_1(u_{11}u_{22} - u_{12}^2) + a_2(u_{11}u_{33} - u_{13}^2) + a_3(u_{22}u_{33} - u_{23}^2) \\
& + a_4(u_{11}u_{23} - u_{12}u_{13}) + a_5(u_{12}u_{23} - u_{13}u_{22}) + a_6(u_{12}u_{33} - u_{13}u_{23}) \\
& + a_7u_{11} + a_8u_{12} + a_9u_{13} + a_{10}u_{22} + a_{11}u_{23} + a_{12}u_{33} + a_{13}\} \\
& + \frac{u_t}{4}\Big\langle b_1\{u_1(u_{12}u_{23} - u_{13}u_{22}) - u_2(u_{11}u_{23} - u_{12}u_{13}) + u_3(u_{11}u_{22} - u_{12}^2)\} \\
& + b_2\{u_1(u_{12}u_{33} - u_{13}u_{23}) - u_2(u_{11}u_{33} - u_{13}^2) + u_3(u_{11}u_{23} - u_{12}u_{13})\} \\
& + b_3\{u_1(u_{22}u_{33} - u_{23}^2) - u_2(u_{12}u_{33} - u_{13}u_{23}) + u_3(u_{12}u_{23} - u_{13}u_{22})\}\Big\rangle \\
& - b_4\frac{u}{4}\{u_{11}(u_{22}u_{33} - u_{23}^2) - u_{12}(u_{12}u_{33} - u_{13}u_{23}) + u_{13}(u_{12}u_{23} - u_{13}u_{22})\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{u_t}{3}\big\{c_1(u_1u_{12}-u_2u_{11})+c_2(u_1u_{13}-u_3u_{11})+c_3(u_1u_{22}-u_2u_{12})\\
&+c_4(u_1u_{23}-u_2u_{13})+c_5(u_2u_{23}-u_3u_{22})+c_6(u_1u_{33}-u_3u_{13})\\
&+c_7(u_2u_{33}-u_3u_{23})+c_8(u_2u_{13}-u_3u_{12})+c_{8'}(u_1u_{23}-u_3u_{12})\big\}\\
-\frac{u}{3}&\big\{c_9(u_{11}u_{23}-u_{12}u_{13})+c_{10}(u_{12}u_{23}-u_{13}u_{22})+c_{11}(u_{12}u_{33}-u_{13}u_{23})\\
&+c_{12}(u_{11}u_{22}-u_{12}^2)+c_{13}(u_{11}u_{33}-u_{13}^2)+c_{14}(u_{22}u_{33}-u_{23}^2)\big\}\\
&+\frac{u_t}{2}(c_{15}u_1+c_{16}u_2+c_{17}u_3)\\
&-\frac{u}{2}(c_{18}u_{11}+c_{19}u_{12}+c_{20}u_{13}+c_{21}u_{22}+c_{22}u_{23}+c_{23}u_{33})-c_{24}u
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\pi_u = \frac{\partial L}{\partial u_t} = v\{& a_1(u_{11}u_{22} - u_{12}^2) + a_2(u_{11}u_{33} - u_{13}^2) + a_3(u_{22}u_{33} - u_{23}^2) \\
&+ a_4(u_{11}u_{23} - u_{12}u_{13}) + a_5(u_{12}u_{23} - u_{13}u_{22}) + a_6(u_{12}u_{33} - u_{13}u_{23}) \\
&+ a_7u_{11} + a_8u_{12} + a_9u_{13} + a_{10}u_{22} + a_{11}u_{23} + a_{12}u_{33} + a_{13}\} \\
&+ \frac{1}{4}\Big\langle b_1\{u_1(u_{12}u_{23} - u_{13}u_{22}) - u_2(u_{11}u_{23} - u_{12}u_{13}) + u_3(u_{11}u_{22} - u_{12}^2)\} \\
&+ b_2\{u_1(u_{12}u_{33} - u_{13}u_{23}) - u_2(u_{11}u_{33} - u_{13}^2) + u_3(u_{11}u_{23} - u_{12}u_{13})\} \\
&+ b_3\{u_1(u_{22}u_{33} - u_{23}^2) - u_2(u_{12}u_{33} - u_{13}u_{23}) + u_3(u_{12}u_{23} - u_{13}u_{22})\}\Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\big\{c_1(u_1u_{12}-u_2u_{11})+c_2(u_1u_{13}-u_3u_{11})+c_3(u_1u_{22}-u_2u_{12})\\
&+c_4(u_1u_{23}-u_2u_{13})+c_5(u_2u_{23}-u_3u_{22})+c_6(u_1u_{33}-u_3u_{13})\\
&+c_7(u_2u_{33}-u_3u_{23})+c_8(u_2u_{13}-u_3u_{12})+c_{8'}(u_1u_{23}-u_3u_{12})\big\}\\
+&\frac{1}{2}(c_{15}u_1+c_{16}u_2+c_{17}u_3),\quad \pi_v=\frac{\partial L}{\partial v_t}=0 \qquad (37)
\end{aligned}
$$

The canonical momenta satisfy canonical Poisson brackets $[u^i(z), \pi^k(z')] = \delta^{ik}\delta(z - z')$, where $u^1 = u$, $u^2 = v$, $\pi^1 = \pi_u$, $\pi^2 = \pi_v$ and $z = (z_1, z_2, z_3)$. The Lagrangian (36) is degenerate because the momenta cannot be inverted for the velocities. We impose (37) as constraints $\Phi_u = 0$, $\Phi_v = 0$ where

$$\Phi_u = \pi_u - v\{a_1(u_{11}u_{22} - u_{12}^2) + a_2(u_{11}u_{33} - u_{13}^2) + a_3(u_{22}u_{33} - u_{23}^2)$$
$$+ a_4(u_{11}u_{23} - u_{12}u_{13}) + a_5(u_{12}u_{23} - u_{13}u_{22}) + a_6(u_{12}u_{33} - u_{13}u_{23})$$
$$+ a_7u_{11} + a_8u_{12} + a_9u_{13} + a_{10}u_{22} + a_{11}u_{23} + a_{12}u_{33} + a_{13}\}$$
$$- \frac{1}{4}\Big\langle b_1\{u_1(u_{12}u_{23} - u_{13}u_{22}) - u_2(u_{11}u_{23} - u_{12}u_{13}) + u_3(u_{11}u_{22} - u_{12}^2)\}$$
$$+ b_2\{u_1(u_{12}u_{33} - u_{13}u_{23}) - u_2(u_{11}u_{33} - u_{13}^2) + u_3(u_{11}u_{23} - u_{12}u_{13})\}$$
$$+ b_3\{u_1(u_{22}u_{33} - u_{23}^2) - u_2(u_{12}u_{33} - u_{13}u_{23}) + u_3(u_{12}u_{23} - u_{13}u_{22})\}\Big\rangle$$

$$
\begin{aligned}
&-\frac{1}{3}\big\{c_1(u_1u_{12}-u_2u_{11})+c_2(u_1u_{13}-u_3u_{11})+c_3(u_1u_{22}-u_2u_{12})\\
&+c_4(u_1u_{23}-u_2u_{13})+c_5(u_2u_{23}-u_3u_{22})+c_6(u_1u_{33}-u_3u_{13})\\
&+c_7(u_2u_{33}-u_3u_{23})+c_8(u_2u_{13}-u_3u_{12})+c_{8'}(u_1u_{23}-u_3u_{12})\big\}\\
&-\frac{1}{2}(c_{15}u_1+c_{16}u_2+c_{17}u_3) \qquad (38)\\
&\Phi_v=\pi_v \qquad (39)
\end{aligned}
$$

and calculate Poisson brackets for the constraints

$$
\begin{aligned}
K_{11}&=[\Phi_u(z),\Phi_{u'}(z')], \quad K_{12}=[\Phi_u(z),\Phi_{v'}(z')]\\
K_{21}&=[\Phi_v(z),\Phi_{u'}(z')], \quad K_{22}=[\Phi_v(z),\Phi_{v'}(z')] \qquad (40)
\end{aligned}
$$

following the Dirac's theory of constraints.

We obtain the following matrix of Poisson brackets

$$K = \begin{pmatrix} K_{11} & K_{12} \\ -K_{12} & 0 \end{pmatrix} \tag{41}$$

$$K_{11} = \sum_{i=1}^{13} a_i K_{11}^{(ai)} + \sum_{i=1}^{3} b_i K_{11}^{(bi)} + \sum_{i=1}^{8'} c_i K_{11}^{(i)} - \sum_{i=1}^{3} c_{i+14} D_i,$$
$$K_{12} = \sum_{i=1}^{13} a_i K_{12}^{(i)} \tag{42}$$

with the following definitions

$$K_{11}^{(a1)} = 2(v_1u_{22} - v_2u_{12})D_1 + 2(v_2u_{11} - v_1u_{12})D_2 + v_{11}u_{22} + v_{22}u_{11} - 2v_{12}u_{12}, \quad K_{12}^{(1)} = -(u_{11}u_{22} - u_{12}^2), \quad K_{11}^{(a2)} = 2(v_1u_{33} - v_3u_{13})D_1 + 2(v_3u_{11} - v_1u_{13})D_3 + v_{11}u_{33} + v_{33}u_{11} - 2v_{13}u_{13},$$

$$K_{12}^{(2)} = -(u_{11}u_{33} - u_{13}^2), \quad K_{11}^{(a3)} = 2(v_2u_{33} - v_3u_{23})D_2 + 2(v_3u_{22} - v_2u_{23})D_3 + v_{22}u_{33} + v_{33}u_{22} - 2v_{23}u_{23},$$

$$K_{12}^{(3)} = -(u_{22}u_{33} - u_{23}^2),$$

$$K_{11}^{(a4)} = (2v_1u_{23} - v_2u_{13} - v_3u_{12})D_1 + (v_3u_{11} - v_1u_{13})D_2 + (v_2u_{11} - v_1u_{12})D_3 + v_{11}u_{23} + v_{23}u_{11} - v_{12}u_{13} - v_{13}u_{12}$$

$$K_{12}^{(4)} = -(u_{11}u_{23} - u_{12}u_{13})$$

$$
\begin{aligned}
K_{11}^{(a5)} &= (v_2 u_{23} - v_3 u_{22}) D_1 + (v_1 u_{23} - 2 v_2 u_{13} + v_3 u_{12}) D_2 \\
&\quad + (v_2 u_{12} - v_1 u_{22}) D_3 + v_{12} u_{23} + v_{23} u_{12} - v_{13} u_{22} - v_{22} u_{13} \\
K_{12}^{(5)} &= -(u_{12} u_{23} - u_{13} u_{22}), \quad K_{12}^{(6)} = -(u_{12} u_{33} - u_{13} u_{23}) \\
K_{11}^{(a6)} &= (v_2 u_{33} - v_3 u_{23}) D_1 + (v_1 u_{33} - v_3 u_{13}) D_2 \\
&+ (2 v_3 u_{12} - v_1 u_{23} - v_2 u_{13}) D_3 + v_{12} u_{33} + v_{33} u_{12} - v_{13} u_{23} - v_{23} u_{13} \\
K_{11}^{(a7)} &= 2 v_1 D_1 + v_{11},\ K_{12}^{(7)} = -u_{11}, \quad K_{11}^{(a8)} = v_2 D_1 + v_1 D_2 + v_{12} \\
K_{12}^{(8)} &= -u_{12}, \quad K_{11}^{(a9)} = v_3 D_1 + v_1 D_3 + v_{13},\ K_{12}^{(9)} = -u_{13} \\
K_{11}^{(a10)} &= 2 v_2 D_2 + v_{22},\ K_{12}^{(10)} = -u_{22}, \quad K_{11}^{(a11)} = v_3 D_2 + v_2 D_3 + v_{23} \\
K_{12}^{(11)} &= -u_{23}, \quad K_{11}^{(a12)} = 2 v_3 D_3 + v_{33},\ K_{12}^{(12)} = -u_{33}, \quad K_{11}^{(a13)} = 0 \\
K_{12}^{(13)} &= -1. \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
K_{11}^{(b1)} &= (u_{13}u_{22} - u_{12}u_{23})D_1 + (u_{11}u_{23} - u_{12}u_{13})D_2 - (u_{11}u_{22} - u_{12}^2)D_3\\
K_{11}^{(b2)} &= (u_{13}u_{23} - u_{12}u_{33})D_1 + (u_{11}u_{33} - u_{13}^2)D_2 - (u_{11}u_{23} - u_{12}u_{13})D_3\\
K_{11}^{(b3)} &= -(u_{22}u_{33} - u_{23}^2)D_1 + (u_{12}u_{33} - u_{13}u_{23})D_2 - (u_{12}u_{23} - u_{13}u_{22})D_3\\
K_{11}^{(1)} &= u_{11}D_2 - u_{12}D_1,\ K_{11}^{(2)} = u_{11}D_3 - u_{13}D_1,\ K_{11}^{(3)} = u_{12}D_2 - u_{22}D_1\\
K_{11}^{(4)} &= u_{13}D_2 - u_{23}D_1,\ K_{11}^{(5)} = u_{22}D_3 - u_{23}D_2,\ K_{11}^{(6)} = u_{13}D_3 - u_{33}D_1\\
K_{11}^{(7)} &= u_{23}D_3 - u_{33}D_2,\ K_{11}^{(8)} = u_{12}D_3 - u_{13}D_2,\ K_{11}^{(8')} = u_{12}D_3 - u_{23}D_1\\
K_{11}^{(15)} &= -D_1,\ K_{11}^{(16)} = -D_2,\ K_{11}^{(17)} = -D_3 \qquad (44)
\end{aligned}
$$

The components of K_{11} can be presented in a manifestly skew symmetric form, so that K is skew symmetric.

$$J_0 = K^{-1} = \begin{pmatrix} 0 & -K_{12}^{-1} \\ K_{12}^{-1} & K_{12}^{-1}K_{11}K_{12}^{-1} \end{pmatrix}. \tag{45}$$

Operator J_0 is Hamiltonian if and only if its inverse K is symplectic: the volume integral of $\omega = (1/2)du^i \wedge K_{ij}du^j$ should be a symplectic form, i.e. $d\omega = 0$ modulo total divergence. Here $u^1 = u,\ u^2 = v$, so that

$$\omega = \sum_{i=1}^{13} a_i\omega_i^a + \sum_{i=1}^{3} b_i\omega_i^b + \sum_{i=1}^{8'} c_i\omega_i + \sum_{i=1}^{3} c_{i+14}\,\omega_{i+14},$$
$$\omega_i^a = \frac{1}{2}du \wedge K_{11}^{(ai)}du + du \wedge K_{12}^{(i)}dv, \quad \omega_i^b = \frac{1}{2}du \wedge K_{11}^{(bi)}du$$
$$\omega_i = \frac{1}{2}du \wedge K_{11}^{(i)}du, \quad K_{12}^{(bi)} = 0, \quad K_{12}^{(i)} = 0. \tag{46}$$

Using (43) and (44) for $K_{11}^{(ai)}$, $K_{11}^{(bi)}$, $K_{11}^{(i)}$ and $K_{12}^{(i)}$ in (46), we get

$$\omega_1^a = (v_1 u_{22} - v_2 u_{12}) du \wedge du_1 + (v_2 u_{11} - v_1 u_{12}) du \wedge du_2 - (u_{11} u_{22} - u_{12}^2) du \wedge dv$$

$$\omega_2^a = (v_1 u_{33} - v_3 u_{13}) du \wedge du_1 + (v_3 u_{11} - v_1 u_{13}) du \wedge du_3 - (u_{11} u_{33} - u_{13}^2) du \wedge dv$$

$$\omega_3^a = (v_2 u_{33} - v_3 u_{23}) du \wedge du_2 + (v_3 u_{22} - v_2 u_{23}) du \wedge du_3 - (u_{22} u_{33} - u_{23}^2) du \wedge dv$$

$$\omega_4^a = \frac{1}{2}\{(2v_1 u_{23} - v_2 u_{13} - v_3 u_{12}) du \wedge du_1 + (v_3 u_{11} - v_1 u_{13}) du \wedge du_2 + (v_2 u_{11} - v_1 u_{12}) du \wedge du_3\} - (u_{11} u_{23} - u_{12} u_{13}) du \wedge dv$$

$$\omega_5^a = \frac{1}{2}\{(-2v_2u_{13} + v_1u_{23} + v_3u_{12})du \wedge du_2 + (v_2u_{12} - v_1u_{22})du \wedge du_3 + (v_2u_{23} - v_3u_{22})du \wedge du_1\} - (u_{12}u_{23} - u_{13}u_{22})du \wedge dv$$

$$\omega_6^a = \frac{1}{2}\{(2v_3u_{12} - v_1u_{23} - v_2u_{13})du \wedge du_3 + (v_1u_{33} - v_3u_{13})du \wedge du_2 + (v_2u_{33} - v_3u_{23})du \wedge du_1\} - (u_{12}u_{33} - u_{13}u_{23})du \wedge dv$$

$$\omega_7^a = v_1du \wedge du_1 - u_{11}du \wedge dv, \quad \omega_8^a = \frac{1}{2}(v_1du \wedge du_2 + v_2du \wedge du_1) - u_{12}du \wedge dv, \quad \omega_9^a = \frac{1}{2}(v_1du \wedge du_3 + v_3du \wedge du_1) - u_{13}du \wedge dv$$

$$\omega_{10}^a = v_2du \wedge du_2 - u_{22}du \wedge dv, \quad \omega_{11}^a = \frac{1}{2}(v_3du \wedge du_2 + v_2du \wedge du_3) - u_{23}du \wedge dv, \quad \omega_{12}^a = v_3du \wedge du_3 - u_{33}du \wedge dv, \quad \omega_{13}^a = du \wedge dv.$$

$$\omega_1^b = \frac{1}{2}\{(u_{13}u_{22} - u_{12}u_{23})du \wedge du_1 + (u_{11}u_{23} - u_{12}u_{13})du \wedge du_2$$
$$- (u_{11}u_{22} - u_{12}^2)du \wedge du_3\}, \quad \omega_2^b = \frac{1}{2}\{(u_{13}u_{23} - u_{12}u_{33})du \wedge du_1$$
$$+ (u_{11}u_{33} - u_{13}^2)du \wedge du_2 - (u_{11}u_{23} - u_{12}u_{13})du \wedge du_3\}$$
$$\omega_3^b = \frac{1}{2}\{(u_{23}^2 - u_{22}u_{33})du \wedge du_1 + (u_{12}u_{33} - u_{13}u_{23})du \wedge du_2$$
$$- (u_{12}u_{23} - u_{13}u_{22})du \wedge du_3\}, \quad \omega_1 = \frac{1}{2}(u_{11}du \wedge du_2 - u_{12}du \wedge du_1)$$
$$\omega_2 = \frac{1}{2}(u_{11}du \wedge du_3 - u_{13}du \wedge du_1), \ \omega_3 = \frac{1}{2}(u_{12}du \wedge du_2 - u_{22}du \wedge du_1)$$

$$\omega_4 = \frac{1}{2}(u_{13} du \wedge du_2 - u_{23} du \wedge du_1),\ \omega_5 = \frac{1}{2}(u_{22} du \wedge du_3 - u_{23} du \wedge du_2)$$
$$\omega_6 = \frac{1}{2}(u_{13} du \wedge du_3 - u_{33} du \wedge du_1),\ \omega_7 = \frac{1}{2}(u_{23} du \wedge du_3 - u_{33} du \wedge du_2)$$
$$\omega_8 = \frac{1}{2}(u_{12} du \wedge du_3 - u_{13} du \wedge du_2),\ \omega_{8'} = \frac{1}{2}(u_{12} du \wedge du_3 - u_{23} du \wedge du_1)$$
$$\omega_{15} = -\frac{1}{2} du \wedge du_1, \quad \omega_{16} = -\frac{1}{2} du \wedge du_2, \quad \omega_{17} = -\frac{1}{2} du \wedge du_3. \qquad (47)$$

Taking exterior derivatives of (47) and skipping total divergence terms, we have checked that $d\omega = 0$ modulo total divergence which proves that operator K is symplectic because the closedness condition for ω is equivalent to the Jacobi identity for J_0. Hence, J_0 defined in (45) is indeed a Hamiltonian operator. Hamiltonian form of this system is

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \tag{48}$$

where we still need to determine the corresponding Hamiltonian density H_1 by the formula $H_1 = \pi_u u_t + \pi_v v_t - L$, where $\pi_v = 0$, with the following final result

$$
\begin{aligned}
H_1 &= -\frac{v^2}{2}\sum_{i=1}^{13} a_i K_{12}^{(i)} \\
&+ b_4 \frac{u}{4}\{u_{11}(u_{22}u_{33} - u_{23}^2) - u_{12}(u_{12}u_{33} - u_{13}u_{23}) + u_{13}(u_{12}u_{23} - u_{13}u_{22})\} \\
&+ \frac{u}{3}\{c_9(u_{11}u_{23} - u_{12}u_{13}) + c_{10}(u_{12}u_{23} - u_{13}u_{22}) + c_{11}(u_{12}u_{33} - u_{13}u_{23}) \\
&+ c_{12}(u_{11}u_{22} - u_{12}^2) + c_{13}(u_{11}u_{33} - u_{13}^2) + c_{14}(u_{22}u_{33} - u_{23}^2)\} \\
&+ \frac{u}{2}(c_{18}u_{11} + c_{19}u_{12} + c_{20}u_{13} + c_{21}u_{22} + c_{22}u_{23} + c_{23}u_{33}) + c_{24}u. \qquad (49)
\end{aligned}
$$

We can write the Hamiltonian density in (49) in the following short-hand notation

$$H_1 = \sum_{i=1}^{13} a_i H_1^{(ai)} + \sum_{i=1}^{4} b_i H_1^{(bi)} + \sum_{i=1}^{24}{}' c_i H_1^{(i)} \qquad (50)$$

where the sum $\sum_{i=1}^{24}{}'$ includes $i = 8'$ and individual terms of the sums in (50) are defined by

$$H_1^{(ai)} = -\frac{v^2}{2} K_{12}^{(i)}, \quad H_1^{(b1)} = H_1^{(b2)} = H_1^{(b3)} = 0 \qquad (51)$$

$$H_1^{(1)} = H_1^{(2)} = \cdots = H_1^{(8)} = H_1^{(8')} = 0, \quad H_1^{(15)} = H_1^{(16)} = H_1^{(17)} = 0$$

and the remaining terms $H_1^{(b4)}, H_1^{(9)}, \cdots, H_1^{(14)}, H_1^{(18)}, \cdots, H_1^{(24)}$ are given in (49).

The formula (48) provides a Hamiltonian form of our two-component system (34)

$$
\begin{aligned}
u_t &= v \\
v_t &= \frac{1}{\Delta}\left(\sum_{i=1}^{13} a_i q^{(ai)} + \sum_{i=1}^{4} b_i q^{(bi)} + \sum_{i=1}^{24}{}' c_i q^{(i)}\right) \equiv \frac{q}{\Delta} \quad (52)
\end{aligned}
$$

For equation (20) recursions due to operators (21) become

$$u_{23}\tilde{\varphi}_t - u_{t3}\tilde{\varphi}_2 = (c_4 - c_8)(u_{23}\varphi_t - u_{t2}\varphi_3) + (c_9 L_{13(2)} + c_{10} L_{23(2)})\varphi$$
$$-L_{12(3)}\tilde{\varphi} = (c_5 L_{23(2)} + c_8 L_{13(2)})\varphi + a_{11}(u_{23}\varphi_t - u_{t2}\varphi_3). \qquad (53)$$

Lax pair for the equation (20) reads

$$X_1 = \frac{\lambda}{u_{23}} L_{t2(3)} + \frac{1}{u_{23}}\{(c_4 - c_8)L_{t3(2)} + c_9 L_{13(2)} + c_{10} L_{23(2)}\}$$
$$X_2 = -\frac{\lambda}{u_{23}} L_{12(3)} + \frac{1}{u_{23}}(c_5 L_{23(2)} + c_8 L_{13(2)} + a_{11} L_{t3(2)}). \; (54)$$

In a two-component form the equation (20) becomes

$$u_t = v, \quad v_t = \frac{q}{\Delta} = \frac{1}{a_{11}u_{23}}\Big(a_{11}q^{(a11)} + c_4 q^{(4)} + c_5 q^{(5)} + c_8 q^{(8)}$$
$$+ c_9 q^{(9)} + c_{10} q^{(10)}\Big), \quad q^{(a11)} = v_2 v_3, \quad q^{(4)} = -(v_1 u_{23} - v_2 u_{13})$$
$$q^{(5)} = -(v_2 u_{23} - v_3 u_{22}), \quad q^{(8)} = -(v_2 u_{13} - v_3 u_{12})$$
$$q^{(9)} = -(u_{11} u_{23} - u_{12} u_{13}), \quad q^{(10)} = -(u_{12} u_{23} - u_{13} u_{22}). \qquad (55)$$

Lie equations in a two-component form become $u_\tau = \varphi$, $v_\tau = \psi$, so that $u_t = v$ implies $\varphi_t = \psi$. We define two-component symmetry characteristic $(\varphi, \psi)^T$ with $\psi = \varphi_t$ and $(\tilde{\varphi}, \tilde{\psi})^T$ with $\tilde{\psi} = \tilde{\varphi}_t$.

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & -a_{11} L_{12(3)}^{-1} u_{23} \\ R_{21} & -a_{11} \frac{v_3}{u_{23}} D_2 L_{12(3)}^{-1} u_{23} + c_4 - c_8 \end{pmatrix} \tag{56}$$

$$R_{11} = -L_{12(3)}^{-1}(c_5 L_{23(2)} + c_8 L_{13(2)} - a_{11} v_2 D_3)$$

$$R_{21} = \frac{1}{u_{23}}\{(c_8 - c_4) v_2 D_3 + c_9 L_{13(2)} + c_{10} L_{23(2)}\} - \frac{v_3}{u_{23}} D_2 L_{12(3)}^{-1}(c_5 L_{23(2)} + c_8 L_{13(2)} - a_{11} v_2 D_3).$$

Here $L_{12(3)}^{-1} L_{12(3)} = 1$. Operator $L_{12(3)}^{-1}$ can make sense merely as a *formal* inverse of $L_{12(3)}$. Thus, the recursion relations above are formal as well. The proper interpretation of $L_{12(3)}^{-1}$ requires the language of differential coverings.

Composing the recursion operator (56) with the Hamiltonian operator J_0 defined in (45) we obtain the second Hamiltonian operator $J_1 = RJ_0$. For equation (55) we have $K_{12} = -a_{11}u_{23}$, $K_{11} = a_{11}(v_3D_2 + D_3v_2) - c_4L_{12(3)} - c_5L_{23(2)} - c_8L_{23(1)}$.

$$J_0 = \frac{1}{a_{11}u_{23}}\begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{a_{11}}K_{11}\frac{1}{u_{23}} \end{pmatrix}. \tag{57}$$

The corresponding Hamiltonian density according to (49), (50) reads

$$\begin{aligned} H_1 &= a_{11}H_1^{(a11)} + c_9H_1^{(9)} + c_{10}H_1^{(10)} \\ &= a_{11}\frac{v^2}{2}u_{23} + \frac{u}{3}\{c_9(u_{11}u_{23} - u_{12}u_{13}) + c_{10}(u_{12}u_{23} - u_{13}u_{22})\}. \end{aligned} \tag{58}$$

The equation (20) taken in the two-component form (55) can be written now as the Hamiltonian system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}. \tag{59}$$

For bi-Hamiltonian system we need a second Hamiltonian operator and corresponding Hamiltonian density. Performing matrix multiplication RJ_0 of the expressions (56) and (57) we obtain the second Hamiltonian operator

$$J_1 = \begin{pmatrix} L_{12(3)}^{-1} & -\left(L_{12(3)}^{-1} D_2 v_3 + \dfrac{c_8 - c_4}{a_{11}}\right) \dfrac{1}{u_{23}} \\ \dfrac{1}{u_{23}} \left(v_3 D_2 L_{12(3)}^{-1} + \dfrac{c_8 - c_4}{a_{11}}\right) & J_1^{22} \end{pmatrix} \tag{60}$$

where the entry J_1^{22} is defined by

$$J_1^{22} = \frac{1}{a_{11}u_{23}}(c_9 L_{13(2)} + c_{10}L_{23(2)})\frac{1}{u_{23}} - \frac{v_3}{u_{23}}D_2 L_{12(3)}^{-1} D_2 \frac{v_3}{u_{23}} \qquad (61)$$
$$+ \frac{c_4 - c_8}{a_{11}u_{23}}\left\{D_2 v_3 + v_3 D_2 - \frac{1}{a_{11}}(c_4 L_{12(3)} + c_5 L_{23(2)} + c_8 L_{23(1)})\right\}\frac{1}{u_{23}}.$$

The operator J_1 is manifestly skew symmetric. A check of the Jacobi identities and compatibility of the two Hamiltonian structures J_0 and J_1 has been made by P. Olver's method of the functional multi-vectors under the well-founded conjecture that this method is applicable for nonlocal Hamiltonian operators.

The next problem is to derive the Hamiltonian density H_0 corresponding to the second Hamiltonian operator J_1 such that implies the bi-Hamiltonian representation of the system (55)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{q}{\Delta} \end{pmatrix} \tag{62}$$

where q/Δ is the right-hand side of the second equation in (55). Then we may conclude that our system is integrable in the sense of Magri.

We assume quadratic dependence of the Hamiltonian H_0 on v

$$H_0 = a[u]v^2 + b[u]v + c[u] \tag{63}$$

the coefficients depending only on u and its partial derivatives.

Proposition

Bi-Hamiltonian representation (62) *of the system* (55) *with the assumption* (63) *is valid under the constraint*

$$c_8 c_{10} = c_5 c_9 \tag{64}$$

with the following Hamiltonian density

$$H_0 = -\frac{\{a_{11}c_8 v^2 + (a_{11}c_9 u_1 + b_0)v - c_9(c_8 - c_4)u_1^2\}u_{23}}{2\{a_{11}c_9 + c_8(c_8 - c_4)\}} \tag{65}$$

Thus, we have shown that our first integrable equation (20) in the two-component form (55) under the constraint (64) admits bi-Hamiltonian representation (62) with the second Hamiltonian operator J_1 defined in (60), (61) and the corresponding Hamiltonian density H_0 given in (65). In the next section, we construct bi-Hamiltonian systems corresponding to other four equations admitting skew-factorized form of the symmetry condition.

Results for our **2nd example** (22) can be obtained from those for equation (20) by interchanging the indices 2 and 3 together with the simultaneous interchange of the coefficients $c_5 \leftrightarrow -c_7$, $c_8 \leftrightarrow (c_4 - c_8)$ and $c_{10} \leftrightarrow -c_{11}$ with all other coefficients (including c_4) unchanged. The Lax pair for equation (22) reads

$$X_1 = \frac{\lambda}{u_{23}} L_{t3(2)} + \frac{1}{u_{23}} \{c_8 L_{t2(3)} + c_9 L_{12(3)} + c_{11} L_{23(3)}\} \qquad (66)$$
$$X_2 = -\frac{\lambda}{u_{23}} L_{13(2)} + \frac{1}{u_{23}} \{(c_4 - c_8) L_{12(3)} + c_7 L_{23(3)} + a_{11} L_{t2(3)}\}.$$

The equation (22) in the two-component form becomes

$$
\begin{aligned}
& u_t = v \qquad (67)\\
& v_t = \frac{q}{\Delta} = \frac{1}{a_{11}u_{23}}\{a_{11}v_2v_3 + c_4(v_2u_{13} - v_1u_{23}) - c_7(v_2u_{33} - v_3u_{23}) \\
& - c_8(v_2u_{13} - v_3u_{12}) + c_9(u_{12}u_{13} - u_{11}u_{23}) - c_{11}(u_{12}u_{33} - u_{13}u_{23})\}.
\end{aligned}
$$

The recursion operator is obtained from (56) by the same combined permutation

$$R = \begin{pmatrix} R_{11} & -a_{11} L_{13(2)}^{-1} u_{23} \\ R_{21} & -a_{11} \dfrac{v_2}{u_{23}} D_3 L_{13(2)}^{-1} u_{23} + c_8 \end{pmatrix} \tag{68}$$

with the matrix elements

$$R_{11} = -L_{13(2)}^{-1}(c_7 L_{23(3)} + (c_4 - c_8) L_{12(3)} - a_{11} v_3 D_2)$$

$$R_{21} = \frac{1}{u_{23}}(-c_8 v_3 D_2 + c_9 L_{12(3)} + c_{11} L_{23(3)})$$

$$- \frac{v_2}{u_{23}} D_3 L_{13(2)}^{-1}\{c_7 L_{23(3)} + (c_4 - c_8) L_{12(3)} - a_{11} v_3 D_2\}.$$

The first Hamiltonian operator has the form

$$J_0 = \frac{1}{a_{11}u_{23}} \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{a_{11}} K_{11} \frac{1}{u_{23}} \end{pmatrix} \tag{69}$$

where $K_{12} = -a_{11}u_{23}$,
$K_{11} = a_{11}(v_2D_3 + D_2v_3) - c_4L_{13(2)} - c_7L_{23(3)} + (c_4 - c_8)L_{23(1)}$.
The corresponding Hamiltonian density reads

$$H_1 = a_{11}\frac{v^2}{2}u_{23} + \frac{u}{3}\{c_9(u_{11}u_{23} - u_{12}u_{13}) + c_{11}(u_{12}u_{33} - u_{13}u_{23})\}. \tag{70}$$

The second Hamiltonian operator is obtained by composing R and J_0 as $J_1 = RJ_0$

$$J_1 = \begin{pmatrix} L^{-1}_{13(2)} & -\left(L^{-1}_{13(2)} D_3 v_2 - \dfrac{c_8}{a_{11}}\right)\dfrac{1}{u_{23}} \\ \dfrac{1}{u_{23}}\left(v_2 D_3 L^{-1}_{13(2)} - \dfrac{c_8}{a_{11}}\right) & J_1^{22} \end{pmatrix} \tag{71}$$

$$J_1^{22} = \frac{1}{a_{11}u_{23}}(c_9 L_{12(3)} + c_{11} L_{23(3)})\frac{1}{u_{23}} - \frac{v_2}{u_{23}} D_3 L^{-1}_{13(2)} D_3 \frac{v_2}{u_{23}}$$
$$+ \frac{c_8}{a_{11}u_{23}}\left\{D_3 v_2 + v_2 D_3 - \frac{1}{a_{11}}(c_4 L_{13(2)} + c_7 L_{23(3)} - (c_4 - c_8) L_{23(1)})\right\}$$

We see that J_1 is manifestly skew-symmetric.

The constraint (64) for the existence of the Hamiltonian density H_0 corresponding to J_1 becomes $c_{11}(c_4 - c_8) = c_7 c_9$. Then H_0 reads

$$H_0 = -\frac{\{a_{11}(c_4 - c_8)v^2 + (a_{11}c_9 u_1 + b_0)v + c_9 c_8 u_1^2\}u_{23}}{2\{a_{11}c_9 + c_8(c_8 - c_4)\}}. \quad (73)$$

We show here the recursion operator and bi-Hamiltonian representation for our **3rd example** (24). The Lax pair for this equation due to (25) reads

$$X_1 = \frac{\lambda}{u_{t2}} L_{23(t)} + \frac{1}{u_{t2}} (a_8 L_{t1(2)} + a_{10} L_{t2(2)} + a_{11} L_{t3(2)})$$
$$X_2 = -\frac{\lambda}{u_{t2}} L_{t2(t)} + \frac{1}{u_{t2}} (c_7 L_{t3(2)} + c_8 L_{t1(2)}). \tag{74}$$

In the following it is convenient to introduce the following notation

$$\hat{\Delta} = a_8 D_1 + a_{10} D_2 + a_{11} D_3$$
$$\Delta = \hat{\Delta}[u_2] = a_8 u_{12} + a_{10} u_{22} + a_{11} u_{23}$$
$$\hat{c} = c_7 D_3 + c_8 D_1. \tag{75}$$

In the two component form the equation (24) becomes

$$u_t = v, \quad v_t = \frac{q}{\Delta}, \quad q = v_2(\hat{\Delta}[v] - \hat{c}[u_3]) + v_3\hat{c}[u_2]. \qquad (76)$$

From now on, square brackets denote the value of an operator. Formulas (25) also imply the recursion relations for symmetry characteristics.

$$\begin{aligned} L_{23(t)}\tilde{\varphi} &= (a_8 L_{t1(2)} + a_{10} L_{t2(2)} + a_{11} L_{t3(2)})\varphi \\ -L_{t2(t)}\tilde{\varphi} &= (c_7 L_{t3(2)} + c_8 L_{t1(2)})\varphi. \end{aligned} \qquad (77)$$

In a two-component form $u_t = v$, $\varphi_t = \psi$, $\tilde{\varphi}_t = \tilde{\psi}$ equations (77) become

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

where the recursion operator is defined by

$$R = \begin{pmatrix} -L_{23(t)}^{-1} v_2 \hat{\Delta} & L_{23(t)}^{-1} \Delta \\ -\frac{q}{v_2 \Delta} D_2 L_{23(t)}^{-1} v_2 \hat{\Delta} + \hat{c} & \frac{1}{v_2} \left\{ \frac{q}{\Delta} D_2 L_{23(t)}^{-1} \Delta - \hat{c}[u_2] \right\} \end{pmatrix} \quad (78)$$

The first Hamiltonian operator has the form

$$J_0 = \begin{pmatrix} 0 & \Delta^{-1} \\ -\Delta^{-1} & \Delta^{-1} K_{11} \Delta^{-1} \end{pmatrix} \tag{79}$$

where $K_{11} = v_2\hat{\Delta} + D_2\hat{\Delta}[v] - c_7 L_{23(3)} - c_8 L_{23(1)}$. With the corresponding Hamiltonian density

$$H_1 = \frac{v^2}{2}\Delta \tag{80}$$

the system (76) takes the Hamiltonian form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}. \tag{81}$$

Composing the recursion operator (78) with the first Hamiltonian operator (79) we obtain the second Hamiltonian operator

$$J_1 = RJ_0 = \begin{pmatrix} -L_{23(t)}^{-1} & (L_{23(t)}^{-1} D_2 q - \hat{c}[u_2])\dfrac{1}{v_2\Delta} \\ -\dfrac{1}{v_2\Delta}(q D_2 L_{23(t)}^{-1} - \hat{c}[u_2]) & J_1^{22} \end{pmatrix} \tag{82}$$

$$J_1^{22} = \hat{c}\frac{1}{\Delta} - \hat{c}[u_2]\frac{1}{\Delta}\hat{\Delta}\frac{1}{\Delta} + \frac{q}{v_2\Delta}D_2 L_{23(t)}^{-1} D_2 \frac{q}{v_2\Delta} - \frac{q}{v_2\Delta}D_2\frac{\hat{c}[u_2]}{v_2\Delta} - \frac{\hat{c}[u_2]}{v_2\Delta}D_2\frac{q}{v_2\Delta} + \frac{\hat{c}[u_2]}{v_2\Delta}L_{23(t)}\frac{\hat{c}[u_2]}{v_2\Delta} \tag{83}$$

which shows that J_1 is manifestly skew-symmetric.

The Hamiltonian density H_0 corresponding to the second Hamiltonian operator J_1 has the form

$$H_0 = kv\Delta = kv(a_8 u_{12} + a_{10} u_{22} + a_{11} u_{23}) \tag{84}$$

with a constant k. Thus, we obtain a bi-Hamiltonian representation for the system (76), which is a two-component form of the equation (24)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix}. \tag{85}$$

Lax pair for our **4th example** (26) due to (27) reads

$$X_1 = \frac{\lambda}{u_{t3}} L_{t3(3)} - \frac{1}{u_{t3}} L_{23(t)}$$

$$X_2 = \frac{\lambda}{u_{t3}}(c_5 L_{t2(3)} + c_8 L_{t1(3)}) + \frac{1}{u_{t3}}(a_{12} L_{t3(t)} + c_6 L_{13(t)} + c_7 L_{23(t)}). \tag{86}$$

In a two-component form, equation (26) becomes

$$u_t = v, \quad v_t = \frac{q}{\Delta} \tag{87}$$

$$q = a_{12} v_3^2 (c_5 v_2 u_{23} - v_3 u_{22}) - c_6 (v_1 u_{33} - v_3 u_{13}) - c_7 (v_2 u_{33} - v_3 u_{23}) - c_8 (v_2 u_{13} - v_3 u_{12}), \quad \Delta = a_{12} u_{33}.$$

First Hamiltonian operator has the form

$$J_0 = \begin{pmatrix} 0 & \Delta^{-1} \\ -\Delta^{-1} & \Delta^{-1} K_{11} \Delta^{-1} \end{pmatrix} \tag{88}$$

where

$K_{11} = a_{12}(v_3 D_3 + D_3 v_3) - c_5 L_{23(2)} - c_6 L_{13(3)} - c_7 L_{23(3)} - c_8 L_{23(1)}$
and $K_{12} = -a_{12} u_{33} = -\Delta$. With the corresponding Hamiltonian density

$$H_1 = \frac{a_{12}}{2} v^2 u_{33} \tag{89}$$

the system (87) takes the Hamiltonian form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}. \tag{90}$$

The recursion operator in 2×2 matrix form is

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \; R = \begin{pmatrix} R^{11} & -a_{12} L^{-1}_{[12]3(3)} u_{33} \\ R^{21} & -a_{12} \dfrac{v_3}{u_{33}} D_3 L^{-1}_{[12]3(3)} u_{33} \end{pmatrix} \quad (91)$$

where we introduce the notation $L_{[12]3(3)} = c_8 L_{13(3)} + c_5 L_{23(3)}$

$$R^{11} = L^{-1}_{[12]3(3)} \frac{1}{v_3} \{ q D_3 - c_6 u_{33} L_{13(t)} - (c_5 u_{23} + c_8 u_{13} + c_7 u_{33}) L_{23(t)} \}$$

$$R^{21} = \frac{v_3}{u_{33}} D_3 L^{-1}_{[12]3(3)} \frac{1}{v_3} \{ q D_3 - c_6 u_{33} L_{13(t)}$$

$$- (c_5 u_{23} + c_8 u_{13} + c_7 u_{33}) L_{23(t)} \} - \frac{1}{u_{33}} L_{23(t)}.$$

Composing recursion operator (91) with J_0 in (88) we obtain the second Hamiltonian operator $J_1 = RJ_0$

$$J_1 = \begin{pmatrix} -L^{-1}_{[12]3(3)} & -L^{-1}_{[12]3(3)} D_3 \frac{v_3}{u_{33}} \\ \frac{v_3}{u_{33}} D_3 L^{-1}_{[12]3(3)} & -\left(\frac{v_3}{u_{33}} D_3 L^{-1}_{[12]3(3)} D_3 \frac{v_3}{u_{33}} + \frac{1}{a_{12} u_{33}} L_{23(t)} \frac{1}{u_{33}} \right) \end{pmatrix} \tag{92}$$

J_1 is manifestly skew-symmetric. With the Hamiltonian density

$$H_0 = \{k(t, z_1) v^2 - (c_8 u_1 + c_5 u_2 + c_7 u_3) v)\} u_{33} \tag{93}$$

system (87) takes bi-Hamiltonian form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix}. \tag{94}$$

Lax pair for our **5th example** (28) due to (29) reads

$$\begin{aligned} X_1 &= \frac{\lambda}{u_{t1}} L_{t1(t)} + \frac{1}{u_{t1}}(c_1 L_{t1(1)} + c_3 L_{t2(1)} + c_4 L_{t3(1)}) \\ X_2 &= -\frac{\lambda}{u_{t1}} L_{12(t)} + \frac{1}{u_{t1}}(a_7 L_{t1(1)} + a_8 L_{t2(1)} + a_9 L_{t3(1)}). \end{aligned} \tag{95}$$

In the following it is convenient to introduce the following notation

$$\hat{\Delta} = a_7 D_1 + a_8 D_2 + a_9 D_3, \quad \Delta = \hat{\Delta}[u_1] = a_7 u_{11} + a_8 u_{12} + a_9 u_{13}$$
$$\hat{c} = c_1 D_1 + c_3 D_2 + c_4 D_3. \tag{96}$$

In a two component form the equation (28) becomes

$$u_t = v, \quad v_t = \frac{q}{\Delta}, \quad q = v_1(\hat{\Delta}[v] - \hat{c}[u_2]) + v_2 \hat{c}[u_1]. \tag{97}$$

First Hamiltonian operator has the form

$$J_0 = \begin{pmatrix} 0 & \Delta^{-1} \\ -\Delta^{-1} & \Delta^{-1} K_{11} \Delta^{-1} \end{pmatrix} \tag{98}$$

where $K_{11} = v_1 \hat{\Delta} + D_1 \hat{\Delta}[v] - c_1 L_{12(1)} - c_3 L_{12(2)} - c_4 L_{12(3)}$ and $K_{12} = -\Delta$. With the corresponding Hamiltonian density

$$H_1 = \frac{v^2}{2} \Delta \tag{99}$$

the system (97) takes the Hamiltonian form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}. \tag{100}$$

Recursion operator in 2×2 matrix form is

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

$$R = \begin{pmatrix} L_{12(t)}^{-1} v_1 \hat{\Delta} & -L_{12(t)}^{-1} \Delta \\ \dfrac{q}{\Delta v_1} D_1 L_{12(t)}^{-1} v_1 \hat{\Delta} - \hat{c} & \dfrac{1}{v_1} \hat{c}[u_1] - \dfrac{q}{\Delta v_1} D_1 L_{12(t)}^{-1} \Delta \end{pmatrix}. \quad (101)$$

Composing the recursion operator (101) with J_0 in (98) we obtain the second Hamiltonian operator

$$J_1 = RJ_0 = \begin{pmatrix} L_{12(t)}^{-1} & -(L_{12(t)}^{-1} D_1 q - \hat{c}[u_1]) \dfrac{1}{v_1 \Delta} \\ \dfrac{1}{v_1 \Delta}(q D_1 L_{12(t)}^{-1} - \hat{c}[u_1]) & J_1^{22} \end{pmatrix} \tag{102}$$

$$J_1^{22} = -\hat{c}\frac{1}{\Delta} + \hat{c}[u_1]\frac{1}{\Delta}\hat{\Delta}\frac{1}{\Delta} - \frac{q}{\Delta v_1} D_1 L_{12(t)}^{-1} D_1 \frac{q}{v_1 \Delta} + \frac{q}{\Delta v_1} D_1 \frac{\hat{c}[u_1]}{v_1 \Delta} + \frac{\hat{c}[u_1]}{\Delta v_1} D_1 \frac{q}{v_1 \Delta} - \frac{\hat{c}[u_1]}{\Delta v_1} L_{12(t)} \frac{\hat{c}[u_1]}{v_1 \Delta} \tag{103}$$

which shows that J_1 is manifestly skew-symmetric on account of $\Delta = \hat{\Delta}[u_1]$.

The Hamiltonian density H_0 corresponding to the second Hamiltonian operator J_1 is

$$H_0 = kv\Delta = kv(a_8 u_{12} + a_{10} u_{22} + a_{11} u_{23}) \qquad (104)$$

with a constant k. Thus, we obtain a bi-Hamiltonian representation for the system (76), which is a two-component form of the equation (24)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix}. \qquad (105)$$

Summary

- All equations of the evolutionary Hirota type in $(3+1)$ dimensions possessing a Lagrangian have the symplectic Monge–Ampère form.
- In a two-component evolutionary form, all our equations have Hamiltonian form.
- We have developed a regular way for converting the symmetry condition to a skew-factorized form. Recursion relations and Lax pairs are obtained as immediate consequences of this representation.
- We have obtained new bi-Hamiltonian systems as an illustration of the general method.

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For Further Reading

M. B. Sheftel and D. Yazıcı.
Lax pairs, recursion operators and bi-Hamiltonian representations
of (3+1)-dimensional Hirota type equations, 2018, arXiv:1804.10620v2.

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M. B. Sheftel, D. Yazıcı and A. A. Malykh.
Recursion operators and bi-Hamiltonian structure of the general heavenly equation, *J. Geom. Phys.* **116**:124–139,

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Thank you very much for your attention.