On a class of solutions of KP-I equation

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Outline

Compatibility conditions

The equations from AKNS hierarchy have the form

$$p_{t_k} = i^k H_k(p,q) = 0, \quad q_{t_k} = (-i)^k H_k(q,p) = 0,$$

and they can be obtained from the following equations

$$\Psi_x = \mathfrak{U}\Psi, \quad \Psi_{t_k} = \mathfrak{V}_k\Psi,$$

where

$$\mathfrak{U} = \lambda J + \mathfrak{U}^0, \quad \mathfrak{V}_1 = 2\lambda\mathfrak{U} + \mathfrak{V}_1^0, \quad \mathfrak{V}_{k+1} = 2\lambda\mathfrak{V}_k + \mathfrak{V}_{k+1}^0,$$
$$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \mathfrak{U}^0 = \begin{pmatrix} 0 & ip \\ -iq & 0 \end{pmatrix}.$$

The functions H_k

In particular

$$
\begin{aligned}
H_1(p,q) &= p_{xx} - 2p^2 q,\\
H_2(p,q) &= p_{xxx} - 6pqp_x,\\
H_3(p,q) &= p_{xxxx} - 8pqp_{xx} - 2p^2 q_{xx} - 6p_x^2 q - 4pp_x q_x + 6p^3 q^2,\\
H_4(p,q) &= p_{xxxxx} - 10pqp_{xxx} - 20p_{xx}p_x q - 10(p_x q_x p)_x + 30p^2 q^2 p_x,\\
H_5(p,q) &= p_{xxxxxx} - 12pqp_{xxxx} - 2p^2 q_{xxxx} - 30p_{xxx}p_x q-\\
&\quad - 18p_{xxx}pq_x - 8p_x pq_{xxx} - 50p_{xx}p - xq_x + 50p_x xp^2 q^2-\\
&\quad - 20p_{xx}^2 q - 22p_{xx}q_{xx}p - 20p_x^2 q_{xx} + 20p^3 qq_{xx}+\\
&\quad + 10p^3 q_x^2 + 70p_x^2 pq^2 + 60p^2 p_x qq_x + 20p^4 q^3.
\end{aligned}
$$

The link between KP-I equation and AKNS hierarchy

The KP-I equation has the form

$$3u_{yy} = (4u_t + u_{xxx} + 6uu_x)_x.$$

The function

$$u(x, y, t) = -2p(x, y, t)q(x, y, t)$$

where $p(x, t_1, t_2)$ and $q(x, t_1, t_2)$ are solutions of equations from AKNS hierarchy, is a solution of KP-I equation.
In cases of NLS ($q = -p^*$) and NLS$^-$ ($q = p^*$) hierarchies the solution of KP-I equation is real

$$u(x, y, t) = \pm 2 \left| p(x, y, t) \right|^2 .$$

The finite-gap solutions of the KdV equation

The finite-gap solutions of the KdV equation

$$4v_t + v_{xxx} + 6vv_x = 0$$

have the following form

$$v(x,t) = -2\partial_x^2 \ln \Theta\left(\mathbf{W}^1 x + \mathbf{W}^2 t - \mathbf{X}|B\right) + 2s_1.$$

Theta function is defined by following equation

$$\Theta(\mathbf{p}|B) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\{\pi i \mathbf{m}^t B\mathbf{m} + 2\pi i \mathbf{m}^t \mathbf{p}\}.$$

Parameters s_1, $\mathbf{W}^k$ and B of finite-gap solutions depend of these spectral curves.

Parameters of finite-gap solutions

The spectral curve of real finite-gap solution of KdV equation has the form

$$\Gamma_{kdv}: \quad w^2 = \prod_{j=1}^{2g_0+1} (E - E_j), \quad \mathrm{Im}\,(E_j) = 0, \quad E_1 < \ldots < E_{2g_0+1}.$$

Vectors $2\pi i \mathbf{W}^j$ are vectors of b-periods of normalized Abelian integrals with asymptotic

$$\widetilde{\Omega}_j = (ik)^{2j-1} + \frac{s_j}{ik} + o(1/(ik)), \quad E = k^2, \quad E \to \infty.$$

Matrix B is a matrix of b-periods of normalized holomorphic integrals.

The finite-gap solutions of equations from NLS hierarchy

The finite-gap solutions of equations from NLS hierarchy $(q = -p^*)$ have the form

$$p(x, t_1, \ldots) = \rho_1 \frac{\Theta(\mathbf{U}(x, t_1, \ldots) + \mathbf{Z} - \boldsymbol{\Delta})}{\Theta(\mathbf{U}(x, t_1, \ldots) + \mathbf{Z})} \exp\{2i\Phi(x, t_1, \ldots)\},$$
$$q(x, t_1, \ldots) = \rho_2 \frac{\Theta(\mathbf{U}(x, t_1, \ldots) + \mathbf{Z} + \boldsymbol{\Delta})}{\Theta(\mathbf{U}(x, t_1, \ldots) + \mathbf{Z})} \exp\{-2i\Phi(x, t_1, \ldots)\},$$

where

$$\mathbf{U}(x, t_1, \ldots) = \mathbf{V}^1 x + \sum_{j>1} \mathbf{V}^j t_{j-1},$$
$$\Phi(x, t_1, \ldots) = -K_1 x - \sum_{j>1} K_j t_{j-1}.$$

Parameters of finite-gap solutions

The spectral curve in the case of NLS hierarchy has the form

$$\Gamma: \quad \chi^2 = \prod_{j=1}^{g+1} [(\lambda-\lambda_j)(\lambda-\lambda_j^*)], \quad \mathrm{Im}\,(\lambda_j) > 0.$$

Vectors $2\pi i \mathbf{V}^j$ are vectors of b-periods of Abelian integrals with asymptotics

$$\begin{aligned} \Omega_j(\mathcal{P}) &= \mp i \left(2^{j-1}\lambda^j - K_j + O\left(\lambda^{-1}\right)\right), && \mathcal{P} \to \mathcal{P}_\infty^\pm, \\ \chi &= \pm\left(\lambda^{g+1} + O\left(\lambda^g\right)\right), && \mathcal{P} \to \mathcal{P}_\infty^\pm. \end{aligned}$$

The link between spectral curves

Let us take

$$E = \lambda^2 + \varkappa, \quad w = \chi.$$

Then from spectral curve Γ_{kdv} we get following spectral curves

$$\Gamma_{akns} : \chi^2 = \prod_{j=1}^{2g_0+1} \left(\lambda^2 + \varkappa - E_j\right), \quad \varkappa > \max E_j, \quad g = 2g_0,$$

$$\Gamma'_{akns} : \chi^2 = \prod_{j=1}^{2g_0} \left(\lambda^2 + E_{2g_0+1} - E_j\right), \quad \varkappa = E_{2g_0+1}, \quad g = 2g_0 - 1$$

for the NLS hierarchy equations.

Covering

Spectral curves Γ_{akns} and Γ'_{akns} cover Γ_{kdv} and second curves

$$\Gamma_{kdv+}: \quad w^2 = (E - \varkappa) \prod_{j=1}^{2g_0+1} (E - E_j), \quad \varkappa > E_{2g_0+1},$$

$$\Gamma_{kdv-}: \quad w^2 = \prod_{j=1}^{2g_0} (E - E_j), \quad \varkappa = E_{2g_0+1}.$$

The curves Γ_{kdv+} $(g_+ = g_0)$ and Γ_{kdv-} $(g_- = g_0 - 1)$ are spectral curves for finite-gap solutions of equations from NLS$^-$ hierarchy $(q = p^*)$.

Reduction

Let us define

1. Vectors $2\pi i\widehat{\mathbf{V}}^j$ are vectors of b-periods of corresponding normalized Abelian integrals on $\Gamma_{kdv\pm}$;
2. Matrix $\widehat{B}$ is matrix of b-periods of holomorphic normolized integrals on $\Gamma_{kdv\pm}$.

Then from theorem of reduction of multi-dimensional Riemann theta functions we get the following relation

$$\Theta(U(x,t_1,\ldots)) = \sum_{k_j\in\{0;1\}} \varepsilon(k_1,\ldots,k_g)\Theta(4\widehat{\mathbf{V}}^1 t_1+\ldots|2\widehat{B}) \times \Theta(2\mathbf{W}^1(x+6\varkappa t_2)+8\mathbf{W}^2 t_2+\ldots|2B),$$

The number of terms in sum equals 2^g.

Elliptic solutions

$$\begin{pmatrix} \Gamma_{kdv} \\ v(x,t) \end{pmatrix} \to \begin{pmatrix} \Gamma_{akns} \\ p(x,y,t) \end{pmatrix} \to u(x,y,t)$$

Therefore, if Γ_{kdv} is a spectral curve

1. of elliptic in x finite-gap solution of KdV equation, the solution $u = 2\left|p\right|^2$ of KP-I equation is also elliptic in x;
2. of elliptic in x and t two-pase finite-gap solution of KdV equation, the four-phase solution $u = 2\left|p\right|^2$ of KP-I equation is elliptic in x and not elliptic in t and y;
3. of elliptic in x and t two-pase finite-gap solution of KdV equation and $\varkappa = E_5$, the three-pase solution $u = 2\left|p\right|^2$ of KP-I equation is elliptic in x and y and not elliptic in t.