Normal forms of spectral surfaces of commuting partial differential operators

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Plan:

- Classification theory of commutative subrings in the ring $\hat{D}_n$.
- Examples of deformations
- Normal forms

The algebra $\hat{D}_n$ and its order function

Denote $\hat{R} := \mathbb{C}[[x_1, \ldots, x_n]]$.
Consider the $\mathbb{C}$–vector space:

$$\mathcal{M} := \hat{R}[[\partial_1, \ldots, \partial_n]] = \left\{ \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \;\middle|\; a_{\underline{k}} \in \hat{R} \text{ for all } \underline{k} \in \mathbb{N}_0^n \right\}.$$

Definition

For any $0 \neq P := \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \in \mathcal{M}$ we define its *order* to be

$$\mathbf{ord}(P) := \sup\{ |\underline{k}| - v(a_{\underline{k}}) \;\} \in \mathbb{Z} \cup \{\infty\},$$

where $v(a_{\underline{k}}) := \max\{n| \quad a_{\underline{k}} \in (x_1, \ldots, x_n)^n\}$.

Definition

$$\hat{D}_n := \{Q \in \mathcal{M} \mid \mathbf{ord}(Q) < \infty\}.$$

Properties of $\hat{D}_n$:

- $\hat{D}_n$ is an associative algebra (with natural operations $\cdot$, $+$ coming from D_n); $\hat{D}_n \supset D_n$.
- $\hat{R}$ has a natural structure of a left $\hat{D}_n$-module, which extends its natural structure of a left D_n-module.
- Operators from $\hat{D}_n$ can realize arbitrary $\mathbb{C}$-linear endomorphisms of $\hat{R}$, e.g. for $n = 1$ the operator

$$\exp(u * \partial) := \sum_{k=0}^{\infty} \frac{u^k}{k!} \partial^k, \quad u \in x\mathbb{C}[[x]]$$

acts as

$$\exp(u * \partial) \circ f(x) = f\big(u + x\big).$$

- There are Dirac delta functions: for $\delta_i := \exp((-x_i) * \partial_i)$

  $\delta_i \circ f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = f\big(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n\big);$

- Operators of integration:

$$\int_i := \big(1 - \exp((-x_i) * \partial_i)\big) \cdot \partial_i^{-1} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} (-\partial)^k,$$

$$\int_i \circ x_i^m = \frac{x_i^{m+1}}{m+1}$$

- Difference operators ($n = 1$):

$$\sum_{i=0}^{M} f_i(n) T^i \hookrightarrow \hat{D}_n \quad \text{via} \quad T \mapsto x, n \mapsto -x\partial,$$

etc.

Commutative subalgebras in $\hat{D}_n$ and their spectral module

Let $B \subset \hat{D}_n$ be a commutative subalgebra.

Definition

The B-module $F = \hat{D}_n/(x_1, \dots, x_n)\hat{D}_n \simeq \mathbb{C}[\partial_1, \dots, \partial_n]$ is called *spectral module* of the algebra B, with the action of B:
$\partial_i \diamond b := \partial_i \circ b \mod (x_1, \dots, x_n)$.

If F is finitely generated, then F parametrizes common eigenfunctions of operators from B:

$$\Lambda \longleftrightarrow B$$

$$\{\text{points of } \Lambda\} \longleftrightarrow \{\chi \in \mathrm{Hom}(B, \mathbb{C})\}$$

Definition

$$Sol(B, \chi) = \{f \in \mathbb{C}[[x_1, \dots x_n]] \Big| \quad Q \circ f = \chi(Q) f \quad \forall Q \in B\}$$

The notion of rank

Fact:

$$\mathcal{F}|_{\chi} = (B/\ker\chi)\otimes_B F \simeq Sol(B,\chi)^*$$
$$\partial^{\bar{i}} \mapsto (f \mapsto f^{(\bar{i})}(0))$$

Definition

The analytic rank of $B \subset \hat{D}_n$ is

$$An.rank(B) := \mathrm{rk}\, F = \dim(Q(B)\otimes_B F) =$$

$$\dim_{\mathbb{C}}\{\psi| \quad P\circ\psi = \chi(P)\psi \quad \forall P \in B, \chi - \text{generic point}\}.$$

The algebraic rank is

$$Alg.rank(B) = GCD\{\mathbf{ord}(P)| \quad P \in B\}.$$

Fact: $An.rank(B) \geq Alg.rank(B)$. We say that $B \subset \hat{D}_n$ is of rank r if $An.rank(B) = Alg.rank(B) = r$.

The notion of Γ-order

The Γ-order is defined on *some elements* of the algebra $\hat{D}_n$:
Let's denote by $\hat{D}_n^{i_1,\dots,i_q}$ the subalgebra in $\hat{D}_n$ consisting of operators *not depending on* $\partial_{i_1},\dots,\partial_{i_q}$. The Γ-order is defined recursively.

Definition

We say that a nonzero operator $P\in\hat{D}_n^{2,3,\dots n}$ *has* Γ*-order* k_1 if $P=\sum_{s=0}^{k_1} p_s\partial_1^s$, where $0\neq p_{k_1}\in\hat{R}$.
We say that a nonzero operator $P\in\hat{D}_n^{i+1,i+2,\dots,n}$ *has* Γ*-order* $(k_1,\dots,k_i)$ if $P=\sum_{s=0}^{k_i} p_s\partial_i^s$, where $p_s\in\hat{D}_n^{i,i+1,\dots,n}$, and the Γ-order of p_{k_i} is $(k_1,\dots,k_{i-1})$.
We say that a nonzero operator $P\in\hat{D}_n$ *has* Γ*-order*

$$\mathrm{ord}_\Gamma(P)=(k_1,\dots,k_n)$$

if $P=\sum_{s=0}^{k_n} p_s\partial_n^s$, where $p_s\in\hat{D}_n^{n}$, and the Γ-order of p_{k_n} is $(k_1,\dots,k_{n-1})$.
P is *monic* if the highest coefficient $p_{k_1,\dots,k_n}$ is 1.

Quasi-elliptic algebras

Now we define the algebras that admit an effective description in terms of its algebro-geometric spectral data (which will be defined below).

Definition

The subalgebra $B \subset \hat{D}_n$ of commuting operators is called 1-*quasi elliptic* if there are n operators $P_1, \ldots, P_n$ such that

- For $1 \leq i < n$

$$\mathrm{ord}_\Gamma(P_i) = (0, \ldots 0, 1, 0 \ldots 0, l_i),$$

  where 1 stands at the i-th place and $l_i \in \mathbb{Z}_+$;
- $\mathrm{ord}_\Gamma(P_n) = (0, \ldots, 0, l_n)$, where $l_n > 0$;
- For $1 \leq i \leq n$ $\mathbf{ord}(P_i) = |\mathrm{ord}_\Gamma(P_i)|$.
- P_i are monic.

B is *normalized* if P_i are of some special form.

Krichever's classification in $n = 1$ case.

Example: if $n = 1$, then quasi-elliptic algebras are well known *elliptic* subalgebras of *ordinary differential operators*.

Theorem

There is a one-to-one correspondence

$$[B \subset D_1 \quad \textit{of rank } r] \longleftrightarrow [(C, p, \mathcal{F}, z, \phi) \quad \textit{of rank } r]/ \simeq$$

$$[B \subset D_1 \quad \textit{of rank } 1]/ \sim \longleftrightarrow [(C, p, \mathcal{F}) \quad \textit{of rank } 1]/ \simeq$$

where

- B *is a commutative elliptic and normalized subalgebra,*
- $\sim$ *means "up to linear changes of variables"*
- $(C, p, \mathcal{F}, z, \phi)$ *are algebro-geometric spectral data of rank* r

Classification theorem

For $n > 1$ practically all known examples of commutative rings of *PDOs* can be made quasi-elliptic and normalized after a change of coordinates and conjugation by a unity in $\hat{R}$

Theorem (Z.)

There is a one-to-one correspondence

$$[B \subset \hat{D}_2 \quad \textit{of rank } r] \longleftrightarrow [(X, C, p, \mathcal{F}, \pi, \phi) \quad \textit{of rank } r]/\simeq$$

$$[B \subset \hat{D}_2 \quad \textit{of rank } 1]/\sim \longleftrightarrow [(X, C, \mathcal{F}) \quad \textit{of rank } 1]/\simeq$$

where

- B *is a commutative quasi-elliptic and normalized finitely generated subalgebra,*
- $\sim$ *means:* $B_1 \sim B_2$ *if there is a linear change of variables* φ *and a unity* $U \in \hat{D}_2$, $\mathrm{ord}(U) = 0$ *such that* $B_1 = U^{-1}\varphi(B_2)U$.
- $(X, C, p, \mathcal{F}, \pi, \phi)$ *are algebro-geometric spectral data of rank* r*:*

Spectral data

Definition

- X is an integral projective algebraic surface;
- C is an integral ample Cartier divisor on X. Moreover, $C^2 = r$.
- $p \in C$ is a closed point, which is regular on C and on X;
- $\mathcal{F}$ is a coherent torsion free sheaf of rank r on X, which is Cohen-Macaulay along C, *i.e. for each point $q \in C$ the $\mathcal{O}_{X,q}$-module $\mathcal{F}_q$ is a Cohen-Macaulay module (recall: there is a regular sequence of two elements)*, and for $n \geq 0$

$$h^0(X, \mathcal{F}(nC)) = \frac{(nr+1)(nr+2)}{2}$$

- $\pi : \hat{\mathcal{O}}_{X,p} \simeq \mathbb{C}[[u,t]]$ and $\phi : \hat{\mathcal{F}}_p \simeq \hat{\mathcal{O}}_{X,p}^{\oplus r}$ are some trivialisations of the local ring and module correspondingly.

Remark: We can additionally assume that X is *Cohen-Macaulay* because of the following result:

Proposition

If $B \subset \hat{D}_2$ is a commutative subring, then there exist a Cohen-Macaulay commutative subring $\tilde{B} \supset B$.
Moreover, if $B \subset D_2$, then $\tilde{B} \subset D_2$.

Cohen-Macaulay surfaces may have singularities: the singular locus is a union of codimension one curves.

Analogy with $n = 1$ case: Isospectral deformations of rank one commutative rings of ODOs determine the KP flows on the *Jacobian* of the spectral curve. Isospectral deformations of rank one commutative rings of PDOs determine some flows on the *moduli space of torsion free sheaves with fixed Hilbert polynomial* $\chi(n) = \frac{(n+1)(n+2)}{2}$.

A dense open subset of this moduli space parametrises *Cohen-Macaulay sheaves*. Cohen-Macaulay sheaves on Cohen-Macaulay surfaces can be effectively described with the help of *matrix-problem approach* due to Burban and Drozd. Then the higher-dimensional version of the *Sato theory* ("algebraic inverse scattering method") is used to obtain explicit examples or explicit deformations of known examples of commuting PDOs.

In the matrix problem approach it is important to know what are the Cohen-Macaulay sheaves with special properties on the *normalisation* of the spectral surface. So, it is important to know what are the possible *normal* surfaces X such that a pre-spectral data $(X, C, \mathcal{F})$ from classification theorem exists. I'll call such surfaces *normal forms*.

Question

What are the normal forms? Can they be smooth? Can they be classified?

Example: Quantum Calogero-Moser systems

There are many examples of commuting PDOs. The most interesting are *quantum Calogero-Moser systems* and their modifications studied in works of *Veselov, Chalykh, Feigin, Etingof, Ginzburg*, etc.
Quantum Calogero-Moser systems and their modifications are defined as families of PDOs commuting with the *Calogero-Moser operator*

$$H_{(x_1,x_2)} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + U(x)$$

for some special value of the potential $U(x)$.
Below we'll deal with *planar* (e.g. $n = 2$) Calogero-Moser systems with *rational* potentials. These systems are closely related with the so-called *rings of quasi-invariants*.

Example: Quantum Calogero-Moser systems

Consider the Calogero–Moser operator

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - 2 \left(\frac{1}{(x_1 - \xi_1)^2} + \frac{1}{(x_2 - \xi_2)^2} \right),$$

where $(\xi_1, \xi_2) \in \mathbb{C}^2$ is such that $\xi_1 \xi_2 \neq 0$. In this case we have due to Chalykh, Veselov, Styrkas:

- There is a commutative subring $B_H \subset D_2$, $B_H \simeq A = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$, the isomorphism is given with the help of the Berest BA-function:

$$\Psi_{Be} = z_1 z_2 + \frac{z_1}{\xi_2 - x_2} + \frac{z_2}{\xi_1 - x_1} + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

  s.t. for any $q \in A$ there exists a unique $L_q \in B_H$

$$L_q \Psi_{Be} = q \Psi_{Be}.$$

Example (Burban-Z.)

The spectral surface X is a Cohen-Macaulay singular surface with a locus of singularities = two lines. Its normalisation is $\mathbb{P}^2$ (so, X is an "umbrella").

The divisor "at infinity" C is a rational curve with two cuspidal singularities.

The spectral module F of the Calogero–Moser system B_H:

$$F \simeq \left\{ f \in \mathbb{C}[z_1, z_2] \;\middle|\; \begin{array}{l} \dfrac{\partial f}{\partial z_1}(0,\rho) = \xi_1 \rho f(0,\rho) \\ \dfrac{\partial f}{\partial z_2}(\rho,0) = \xi_2 \rho f(\rho,0) \end{array} \right\}.$$

Moduli space of sheaves has dimension 3: for any $\beta \in \mathbb{C}$, consider the vector space

$$W_\beta = \left\{ f \in \mathbb{C}[z_1, z_2] \;\middle|\; \begin{array}{l} \dfrac{\partial f}{\partial z_1}(0,\rho) = \dfrac{\xi_1^2 \rho}{\xi_1 + \beta\rho} f(0,\rho) \\ \dfrac{\partial f}{\partial z_2}(\rho,0) = \dfrac{\xi_2^2 \rho}{\xi_2 + \beta\rho} f(\rho,0) \end{array} \right\}.$$

Example

W_β is a Cohen–Macaulay module of rank one over $A = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$.

$W_\beta = \mathbb{C}\cdot w + (\xi_2 + \xi_2^2 z_2 + \beta z_1) z_1^2 \mathbb{C}[z_1] + (\xi_1 + \xi_1^2 z_1 + \beta z_2) z_2^2 \mathbb{C}[z_2] + z_1^2 z_2^2 \mathbb{C}[z_1, z_2]$,

where $w = 1 + \xi_1 z_1 + \xi_2 z_2 + (\xi_1\xi_2 - \beta)\cdot\left(z_1 z_2 + \left(\frac{z_1^2}{\xi_2^2} + \frac{z_2^2}{\xi_1^2}\right)\right)$.

Deformed BA-function:

$$\Psi(x_1, x_2, z_1, z_2) = \Psi_{Be} + \beta\overline{\Psi},$$

where Ψ_{Be} is the Berest function and

$$\overline{\Psi} = \frac{1 + \beta\left(\frac{z_1}{\xi_2} + \frac{z_2}{\xi_1}\right)}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} +$$

$$\frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_1 z_1) z_1 + (\xi_1 - x_1)\exp(x_1 z_1) z_1^2\right) +$$

$$\frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_2 z_2) z_2 + (\xi_2 - x_2)\exp(x_2 z_2) z_2^2\right).$$

Example

The simplest deformations of differential operators from the B_H:
for any $q \in z_1^2 z_2^2 A$ denote $q'(z_1, z_2) := q/(z_1^2 z_2^2)$.
Then the corresponding operator $L_q \in \hat{D}_2$ (where, as usual, $L_q \Psi = q\Psi$) is

$$L_q = Sq'(\partial_1, \partial_2)(\partial_1 - \frac{1}{1-x_1})(\partial_2 - \frac{1}{1-x_2}), \quad \text{where}$$

$$S = S_0 + \beta T,$$

$$S_0 = \partial_1 \partial_2 + \frac{1}{\xi_2 - x_2}\partial_1 + \frac{1}{\xi_1 - x_1}\partial_2 + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

$$T = \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)} \left(\frac{1}{\xi_2} \left(\delta_2 \partial_1 + (\xi_1 - x_1)\delta_2 \partial_1^2 \right) + \frac{1}{\xi_1} \left(\delta_1 \partial_2 + (\xi_2 - x_2)\delta_1 \partial_2^2 \right) \right) + \frac{1}{(\xi_1 \xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} \delta_1 \delta_2 \left(1 + \beta \left(\frac{\partial_1}{\xi_2} + \frac{\partial_2}{\xi_1} \right) \right)$$

Generalization of the **KP**-hierarchy in dimension 2: Definitions

Definition

Define

$$\hat{E} = \hat{D}_1((\partial_2^{-1})) \supset \hat{D} \supset D.$$

This is an associative ring of formal pseudo-differential operators. Denote by $\hat{E}^{(-1)}$ the algebra $\hat{D}_1[[\partial_2^{-1}]]\partial_2^{-1}$. We have a natural direct sum decomposition

$$\hat{E} = \hat{D} \oplus \hat{E}^{(-1)}$$

as a module.

Generalization of the KP-hierarchy in dimension 2

Definition

We consider $L, M \in \hat{E}[[\{t_k\}]]$, $k = (i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ such that

$$L = \partial_1 + u_1\partial_2^{-1} + \ldots, \qquad M = \partial_2 + v_1\partial_2^{-1} + \ldots,$$

$$\text{where} \quad u_i, v_i \in \hat{D}_1[[\{t_k\}]].$$

Let $N = (L, M)$ and $[L, M] = 0$, then **hierarchy** is

$$\frac{\partial N}{\partial t_{i,j}} = V_N^{i,j},$$

$$\text{where} \quad V_N^{i,j} = ([(L^i M^j)_+, L], [(L^i M^j)_+, M]),$$

$$(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+.$$

Sato-Wilson hierarchy

It can be shown that this new hierarchy is equivalent to the following analogue of the Sato-Wilson hierarchy.

Definition

$$\frac{\partial S}{\partial t_{i,j}} = -(S\partial_1^i\partial_2^j S^{-1})_- S,$$

where $S = 1 + s_1\partial_2^{-1} + \ldots \in \hat{E}[[\{t_k\}]]$, $s_i \in \hat{D}_1[[\{t_k\}]]$ and $L = S\partial_1 S^{-1}$, $M = S\partial_2 S^{-1}$ (such S is always defined, called also as Schur operator).

Example. Consider a commutative subring B generated by 3 operators:

$$P = \partial_2^2 - 2\frac{1}{(1-x_2)^2}\delta_1,$$

$$Q = \partial_1\partial_2 + \frac{1}{1-x_2}\delta_1\partial_1,$$

$$P' = \partial_2^3 - 3\frac{1}{(1-x_2)^2}\delta_1\partial_2 - 3\frac{1}{(1-x_2)^3}\delta_1.$$

Its spectral surface is again rational singular, with normalisation $\mathbb{P}^2$. If we derive equations of isospectral deformations of these operators (equations of the SW system with the initial condition = Schur operator of B), we obtain the following equations:

$$\frac{\partial s_1}{\partial t_1} = \frac{1}{4}(s_1)_{x_2x_2x_2} - \frac{3}{2}(s_1)_{x_2}^2, \quad \frac{\partial s_1}{\partial t_2} = -(s_1)_{x_2}(s_1)_{x_1} - \frac{1}{2}(s_1)_{x_2x_2}\partial_1,$$
$$\frac{\partial s_1}{\partial t_3} = -(s_1)_{x_1}^2 - (s_1)_{x_1x_2}\partial_1 - (s_1)_{x_2}\partial_1^2, \tag{1}$$

where $s_1(x_1, x_2, t_1, t_2, t_3) = s_1(t)$ is the first coefficient of the operator $S(t) = 1 + s_1(t)\partial_2^{-1} + \ldots$, and $S(0) = S$ is the Schur operator of B. Notably

$$s_1(0) = \frac{1}{1 - x_2}\delta_1$$

is a solution of the equations above. This corresponds to the following fact from one-dimensional KP theory: the function $u(x) = (x^{-1})_x$ is the rational solution of the KdV equation (and this function is the halved coefficient of the operator P in example below).

Geometric properties of commuting PDOs

Q: Which geometric data describe commutative subrings $B \subset D_2$ of *PDOs*?

Theorem (Kurke, Z.)

If $B \subset D_2$ is quasi-elliptic and normalized of rank 1, then

- *The sheaf $\mathcal{F}$ is Cohen-Macaulay of rank 1;*
- *The divisor C is a rational curve;*
- *If $n : \mathbb{P}^1 \to C$ is the normalisation map, then $\mathcal{F}|_C = (n_*(\mathcal{O}_{\mathbb{P}^1}))$.*

Corollary

If X is a smooth normal form of a commutative ring of PDOs, then $X \simeq \mathbb{P}^2$ (and then $C \simeq \mathbb{P}^1$, $\mathcal{F} \simeq \mathcal{O}_X$).

Conjecture

The conditions from theorem are sufficient.

Smooth normal forms

Q: Are there smooth normal forms of commutative subrings from $\hat{D}_2$?

Question

Find a smooth surface X such that there is a curve C and a divisor D with the following properties:

1. *C is ample (i.e. the sheaf $\mathcal{O}_X(C)$ is ample), $C^2 = 1$ and $h^0(X, \mathcal{O}_X(C)) = 1$;*
2. *$(D, C)_X = g(C) - 1$;*
3. *$h^i(X, \mathcal{O}_X(D)) = 0$, $i = 0, 1, 2$ and $h^0(X, \mathcal{O}_X(D + C)) = 1$.*

Remark: The condition $h^0(X, \mathcal{O}_X(C)) = 1$ means that we are looking for normal forms of "non-trivial" commutative subrings.

Definition

The subring $B \subset \hat{D}_2$ is "trivial", if it contains the operator ∂_1 or the operator ∂_2, i.e. B consists of operators not depending on x_1 or x_2.

Smooth normal forms

The examples of such algebras naturally arise from examples of commuting *ordinary differential operators* just by adding one extra derivation.

Proposition (Z.)

The subring $B \subset \hat{D}_2$ *is "trivial" iff* $h^0(X, \mathcal{O}_X(D+C)) \geq 2$.

Proposition (Z.)

Let $(X, C, \mathcal{F})$ *be a pre-spectral data of rank one with a smooth surface* X *and* $g(C) \leq 1$. *Then* $h^0(X, \mathcal{O}_X(C)) \geq 2$.

Conjecture

If X is a smooth normal form, then it is either rational (and corresponds to a "trivial" subring) or of general type.

Smooth normal forms

Theorem (Kulikov-Z.)

There is an eight-dimensional family of pairwise non-isomorphic Godeaux surfaces X such that on each X from this family there are at least 840 different divisors D_j and four curves C_i satisfying the conditions from Question.

Each of these Godeaux surfaces is a factor of a quintic in $\mathbb{P}^3$ by the group $\mathbb{Z}^5$.

Conjecture

All normal forms have the property $q = H^1(X, \mathcal{O}_X) = 0$. There are no other smooth normal forms of general type corresponding to "non-trivial" subrings.