Emergent Geometry of KP Hierarchy

Russian-Chinese Conference on Integrable Systems

and Geometry

Euler International Mathematical Institute
St. Petersburg, 8/20-8/25, 2018

Euler, symmetric functions, and quantum groups

The theory of symmetric functions, initiated by Euler, plays a crucial role in the theory of KP hierarchy via boson-fermion correspondence.

Representation theory of quantum groups, first developed in St. Petersburg in quantum integrable systems also play important roles in some examples to be presented.

Plan of the Talk

1. What does "emergent geometry" mean?

2. Witten Conjecture/Kontsevich Theorem as example of emergent geometry: The Virasoro constraints point of view

3. Witten Conjecture/Kontsevich Theorem as example of emergent geometry: The KP hierarchy point of view

4. Emergent geometry of KP hierarchy

5. More examples

What does "emergent geometry" mean?

In string theory, there are following approaches to compute partition function, free energy or n-point correlation functions:

1. Recursion relations

2. Integrable hierarchies

3. Mirror symmetry

Recursion Relations

These include:

1. Topological recursion relations

2. Virasoro constraints

3. Eynard-Orantin topological recursions

From lower genera, few points to higher genera, more points.

Integrable hierarchies

These include:

1. Matrix models and integrable hierarchies

2. Topological gravity and KdV hierarchy (Witten Conjecture/ Kontsevich Theorem)

3. FJRW theory and Drinfeld-Sokolov hierarchies

4. Dubrovin-Zhang theory (from Frobenius manifolds to integrable hierarchies)

Mirror symmetry

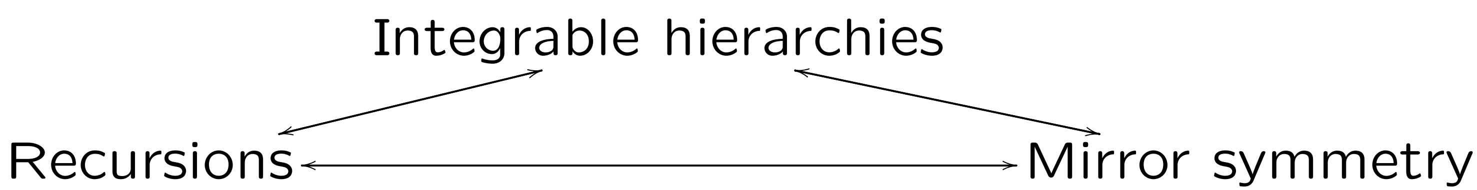
1. In genus zero: deformation theory of mirror manifold and variation of Hodge structures –special Kähler geometry on moduli space of complex structures

2. In genus one: analytic torsions and tt^*-geometry

3. In higher general: holomorphic anomaly equation (quantization and recursion)

Emergent geometry

The idea behind what we call emergent geometry is a synthesis of ideas from the above three different aspects of string theory.



It is borrowed from statistical physics, it means the geometric structures that emerge when there is an infinite degree of freedom.

The idea of emergence was advocated by Nobel Laureate P. Anderson in a 1972 paper entitled "More is different" published in Science.

An example of emergent phenomenon: Witten Conjecture/Kontsevich Theorem

For topological 2D gravity, the n-point correlators are defined by

$$\langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1} \cdots \psi_n^{m_n}.$$

$$F_g(\mathbf{t}) = \sum_{n \geq 0} \frac{1}{n!} \sum_{a_1, \ldots, a_n \geq 0} t_{a_1} \cdots t_{a_n} \langle \tau_{a_1}, \cdots, \tau_{a_n} \rangle_g.$$

$$F(\mathbf{t}; \lambda) = \sum_{g \geq 0} \lambda^{2g-2} F_g(\mathbf{t}).$$

$$Z_{WK} = \exp F(\mathbf{t}; \lambda).$$

This is called the Witten-Kontsevich tau-function .

Witten Conjecture/Kontsevich Theorem

Z_{WK} is a tau-function of the KdV hierarchy.

This means that $u = \lambda^2 \frac{\partial F}{\partial^2 t_0^2}$ satisfies a sequence of equations:

$$\partial_{t_n} u = \partial_{t_0} R_{n+1},$$

where $t_0 = x$.

Here R_n is a sequence of differential polynomials (Gelfand-Dickey polynomials).

Virasoro Constraints

Dijkgraaf-Verlinde-Verlinde:

$$\frac{\partial}{\partial u_{n+1}} Z_{WK} = \widehat{L}_n Z_{WK}, \quad n \geq -1,$$

where the operators $\widehat{L}_n$ are defined by:

$$\widehat{L}_n = \sum_{k=0}^{\infty} (2k+1) u_k \frac{\partial}{\partial u_{k+n}} + \frac{\lambda^2}{2} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial u_k \partial u_{n-k-1}} + \frac{u_0^2}{2\lambda^2} \delta_{n,-1} + \frac{\delta_{n,0}}{8}.$$

Here we have made the following change of coordinates:

$$t_k = (2k+1)!! u_k.$$

Virasoro Constraints for free energy

In terms of the free energy,

$$\frac{\partial F}{\partial u_0} = \sum_{k=1}^{\infty} (2k+1) u_k \frac{\partial F}{\partial u_{k-1}} + \frac{u_0^2}{2\lambda^2},$$

$$\frac{\partial F}{\partial u_1} = \sum_{k=0}^{\infty} (2k+1) u_k \frac{\partial F}{\partial u_k} + \frac{1}{8},$$

$$\frac{\partial F}{\partial u_n} = \sum_{k=0}^{\infty} (2k+1) u_k \frac{\partial F}{\partial u_{k+n-1}} + \frac{\lambda^2}{2} \sum_{k=0}^{n-2} \left(\frac{\partial^2 F}{\partial u_k \partial u_{n-k-2}} + \frac{\partial F}{\partial u_k} \frac{\partial F}{\partial u_{n-k-2}} \right), \quad n \geq 2.$$

Virasoro Constraints for free energy in genus zero

$$\frac{\partial F_0}{\partial u_0} = \sum_{k=1}^{\infty}(2k+1)u_k\frac{\partial F_0}{\partial u_{k-1}} + \frac{u_0^2}{2},$$

$$\frac{\partial F_0}{\partial u_1} = \sum_{k=0}^{\infty}(2k+1)u_k\frac{\partial F_0}{\partial u_k},$$

$$\frac{\partial F_0}{\partial u_n} = \sum_{k=0}^{\infty}(2k+1)u_k\frac{\partial F_0}{\partial u_{k+n-1}} + \frac{1}{2}\sum_{k=0}^{n-2}\frac{\partial F_0}{\partial u_k}\frac{\partial F_0}{\partial u_{n-k-2}}, \quad n \geq 2.$$

Derivatives of F_0 on the small phase space

Now we take $u_i = 0$ for $i \geq 1$ in these equations, and set $f_n = \frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots)$, we get:

$$f_0 = \frac{u_0^2}{2},$$
$$f_1 = u_0 f_0,$$
$$f_n = u_0 f_{n-1} + \frac{1}{2} \sum_{k=0}^{n-2} f_k f_{n-k-2}, \quad n \geq 2.$$

Derivatives of F_0 on the small phase space

One can easily find the following explicit solution:

$$\frac{\partial F_0}{\partial u_n}(u_0, 0, \dots) = \frac{(2n+1)!!}{(n+2)!} u_0^{n+2}.$$

We note the above result can be reformulated as follows:

$$z\left(1 - \frac{2u_0}{z^2}\right)^{1/2} = z - \frac{u_0}{z} - \sum_{n=0}^{\infty} \frac{\partial F_0}{\partial u_n}(u_0, 0, \dots) \cdot z^{-(2n+3)}.$$

Emergence of the spectral curve for topological 2D gravity

Consider the Puiseux series:

$$x = f - \frac{u_0}{f} - \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n}(u_0, 0, \dots) \cdot f^{-2n-3},$$

where $f^2 = 2y$, then one has

$$y = \frac{1}{2}x^2 + u_0.$$

When $u_0 = 0$, this gives us the Airy curve:

$$y = \frac{1}{2}x^2.$$

It is the spectral curve of the topological 2D gravity.

This is an emergent phenomenon : You have to go through the infinite-dimensional big phase space to see the spectral curve.

Emergence of the special deformation of spectral curve for topological 2D gravity

Theorem (Z.) Consider the following series:

$$x = f - \sum_{n\geq 0}(2n+1)u_n f^{2n-1} - \sum_{n\geq 0}\frac{\partial F_0}{\partial u_n}(\mathbf{u})\cdot f^{-2n-3}.$$

Then one has

$$\begin{aligned}
x^2 =& 2y\Big(1-\sum_{n\geq 1}(2n+1)u_n(2y)^{n-1}\Big)^2 \\
& -2u_0\Big(1-\sum_{n\geq 1}(2n+1)u_n(2y)^{n-1}\Big) \\
& +2\sum_{n\geq 0}\sum_{k\geq n+2}(2k+1)u_k\cdot\frac{\partial F_0}{\partial u_n}\cdot(2y)^{k-n-2}.
\end{aligned}$$

In particular,

$$(x^2)_- = 0.$$

Here for a formal series $\sum_{n\in\mathbb{Z}} a_n f^n$,

$$(\sum_{n\in\mathbb{Z}} a_n f^n)_+ = \sum_{n\geq 0} a_n f^n, \quad (\sum_{n\in\mathbb{Z}} a_n f^n)_- = \sum_{n<0} a_n f^n.$$

Proof. Equivalent to Virasoro constraints for genus zero free energy.

Examples of the special deformation

When $t_j = 0$ for $j \geq 3$, the curve is deformed to:

$$\begin{aligned} x^2 &= -2u_0(1-3u_1) + 10u_2\frac{\partial F_0}{\partial u_0}(u_0,u_1,u_2) \\ &+ ((1-3u_1)^2 + 10u_0u_2)(2y) \\ &- 10u_2(1-3u_1)(2y)^2 + 25u_2^2(2y)^3. \end{aligned}$$

When $t_1 = 1$ i.e., $u_1 = 1/3$,

$$x^2 = 10u_2\frac{\partial F_0}{\partial u_0}(u_0,u_1,u_2) + 10u_0u_2(2y) + 25u_2^2(2y)^3.$$

The spectral curve undergoes a phase change from a rational curve to an elliptic curve! In general, the spectral curve can undergo a phase change from $p^2 = q$ to $p^2 = q^{2n+1}$ in this way. This is again an emergent phenomenon.

Quantum deformation theory

Q. How to get mirror symmetry for topological 2D gravity in all genera?

A. Quantum deformation theory.

We quantize the special deformation of the Airy curve.

Rewrite the special deformation of the Airy curve as follows:

$$x(z) = -\sum_{n\geq 0}(2n+1)\tilde{u}_n z^{\frac{2n-1}{2}} - \sum_{n\geq 0}\frac{\partial F_0}{\partial \tilde{u}_n}(\mathbf{u})\cdot z^{-\frac{2n+3}{2}},$$

where $z = 2y = f^2$, and

$$\tilde{u}_n = u_n - \frac{1}{3}\delta_{n,1}.$$

Consider the space of

$$V = z^{1/2}\mathbb{C}[[z, z^{-1}]] = \{ \sum_{n=0}^{\infty} (2n+1)\tilde{u}_n z^{(2n-1)/2} + \sum_{n=0}^{\infty} \tilde{v}_n z^{-(2n+3)/2} \}.$$

We regard $\{\tilde{u}_n, \tilde{v}_n\}$ as linear coordinates on V, and introduce the following symplectic structure on V:

$$\omega = \sum_{n=0}^{\infty} d\tilde{u}_n \wedge d\tilde{v}_n.$$

It follows that

$$\tilde{v}_n = \frac{\partial F_0}{\partial u_n}(\mathbf{u})$$

defines a Lagrangian submanifold in V.

Canonical quantization of the special deformation of Airy curve

Take the natural polarization that $\{q_n = \tilde{u}_n\}$ and $\{p_n = \tilde{v}_n\}$, one can consider the canonical quantization:

$$\hat{\tilde{u}}_n = \tilde{u}_n \cdot, \quad \hat{\tilde{v}}_n = \frac{\partial}{\partial \tilde{u}_n}.$$

Corresponding to the field x, consider the following field of operators on the Airy curve:

$$\hat{x}(z) = -\sum_{m\in\mathbb{Z}} \beta_{-(2m+1)} z^{m-1/2} = -\sum_{m\in\mathbb{Z}} \beta_{2m+1} z^{-m-3/2}.$$

where the operators β_{2k+1} are defined by:

$$\beta_{-(2k+1)} = \lambda^{-1}(2k+1)\tilde{u}_k \cdot, \quad \beta_{2k+1} = \lambda \frac{\partial}{\partial \tilde{u}_k}.$$

Regularized products of two fields

We define the regularized product of $\widehat{x}(z)$ with itself by

$$\begin{aligned} \widehat{x}(z) \odot \widehat{x}(z) &= \widehat{x}(z)^{\odot 2} := \lim_{\epsilon \to 0} (\widehat{x}(z+\epsilon)\widehat{x}(z) - \frac{2}{\epsilon^2}) \\ &= \, :\widehat{x}(z)\widehat{x}(z): + \frac{1}{4z^2}. \end{aligned}$$

Virasoro constraints and mirror symmetry for 2D topological gravity

Recall the special deformation of the Airy curve constructed using the genus zero free energy of the 2D topological gravity, is characterized by :

$$(x^2)_- = 0.$$

The quantized version of this equation is

Theorem (Z.) The Witten-Kontsevich tau-function Z_{WK} satisfies the following equation:

$$(\widehat{x}(z)^{\odot 2})_- Z_{WK} = 0.$$

This result establishes the mirror symmetry of the theory of 2D topological gravity and the quantum deformation theory of the Airy curve.

Fermionic expression for Z_{WK} from Virasoro constraints

Theorem (Z.) Under a change of coordinates

$$t_n = (-1)^n \sqrt{-2} T_{2n+1} \cdot \prod_{j=0}^{n} (j + \frac{1}{2}), \;\; T_n = \frac{1}{n} p_n,$$

and the boson-fermion correspondence, the Witten-Kontsevich tau-function is given by a Bogoliubov transformation:

$$Z_{WK} = e^A |0\rangle,$$

where the operator

$$A = \sum_{m,n \geq 0} A_{m,n} \psi_{-m-\frac{1}{2}} \psi^*_{-n-\frac{1}{2}}$$

with the coefficients $A_{m,n}$ as given on the next slide

$A_{m,n} = 0$ if $m + n \not\equiv -1 \pmod 3$ and

$$A_{3m-1,3n} = A_{3m-3,3n+2} = (-1)^n \left(-\frac{\sqrt{-2}}{144}\right)^{m+n} \frac{(6m+1)!!}{(2(m+n))!}$$
$$\cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^{n} (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m+1}),$$

$$A_{3m-2,3n+1} = (-1)^{n+1} \left(-\frac{\sqrt{-2}}{144}\right)^{m+n} \frac{(6m+1)!!}{(2(m+n))!}$$
$$\cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^{n} (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m-1}),$$

where $B_n(m)$ is a polynomial in m of degree $n-1$ defined by:

$$B_n(x) = \frac{1}{6} \sum_{j=1}^{n} 108^j b_{n-j} \cdot (x+n)_{[j-1]},$$

where

$$(a)_{[j]} = \begin{cases} 1, & j = 0, \\ a(a-1)\cdots(a-j+1), & j > 0, \end{cases}$$

and b_n is a constant depending on n defined by:

$$b_n = \frac{2^n \cdot (6n+1)!!}{(2n)!}.$$

The following are some examples:

$$B_0 = 0,$$
$$B_1 = 18,$$
$$B_2 = 1944x + 5778,$$
$$B_3 = 209952x^2 + 1253880x + 2277477,$$
$$B_4 = 22674816x^3 + 226118304x^2 + 787643676x + 1114815879,$$
$$B_5 = 2448880128x^4 + 36665177472x^3 + 207169401168x^2$$
$$+545727699972x + 2633883829515/4,$$
$$B_6 = 264479053824x^5 + 5546713489920x^4 + 46133330328000x^3$$
$$+193184363553840x^2 + 424746412978761x$$
$$+1828597219279695/4,$$

and

$$b_0 = 1,$$
$$b_1 = 105,$$
$$b_2 = 45045/2,$$
$$b_3 = 14549535/2,$$
$$b_4 = 25097947875/8,$$
$$b_5 = 13537833083775/8,$$
$$b_6 = 17531493843488625/16.$$

Examples of $A_{3m-n-1,n}$

$$A_{3m-1,0} = \left(-\frac{\sqrt{-2}}{144}\right)^m \frac{(6m+1)!!}{(2m)!} \cdot \frac{1}{6m+1},$$
$$A_{3m-2,1} = -\left(-\frac{\sqrt{-2}}{144}\right)^m \frac{(6m+1)!!}{(2m)!} \cdot \frac{1}{6m-1},$$

Up to the factor $(-\frac{\sqrt{-2}}{144})^m$, our $A_{3m-1,0}$ and $A_{3m-2,1}$ are exactly the coefficients of the two seires in the Faber-Zagier tautological relations.

Their generating series are the so-called Airy functions.

Why do Airy functions appear?

Recall the mirror geometry of a point is the curve:

$$y = \frac{1}{2}x^2.$$

We will quantize this curve in the following way:

$$x \mapsto \widehat{x} = \frac{d}{dy}, \quad y \mapsto \widehat{y} = y \cdot .$$

After quantization the above equation leads to a differential e-quation:

$$\frac{1}{2}\frac{d^2}{dy^2}\psi(y) = y \cdot \psi(y).$$

This is the Airy equation.

Summary

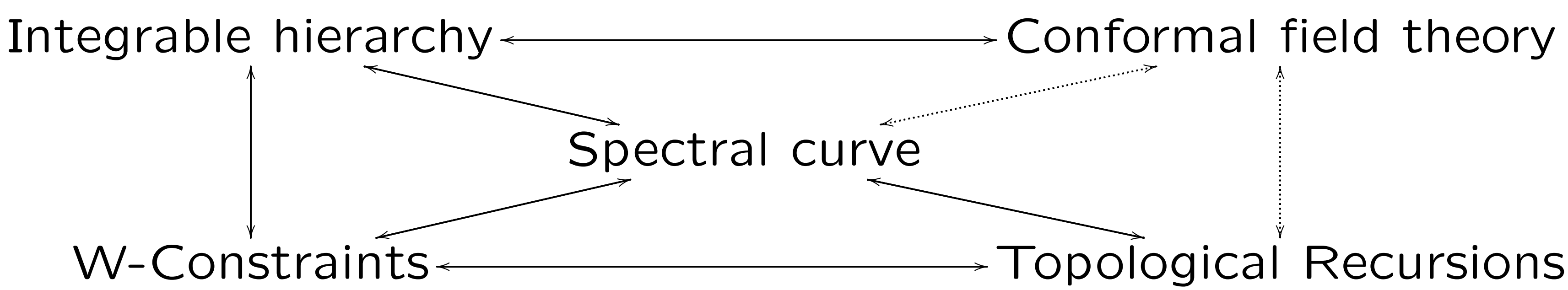
In the above results we have actually used Virasoro constraints only.

The first result is in bosonic picture: Witten-Kontsevich tau-function is annihilated by some bosonic fields on the Airy curve.

The second result is in the fermionic picture: The Witten-Kontsevich function is the Bogoliubov transform of the vacuum, it is related to the two solutions of the Airy equation.

These results suggest to study them from the point of view of KdV hierarchy, or more generally, KP hierarchy, and from the point of view of conformal field theory of free bosons and free fermions on Riemann surfaces.

Emergent Picture for Gromov-Witten theory of a point



For Witten-Kontsevich tau-function, the relevant integrable hierarchy is the KdV hierarchy which is a reduction of the KP hierarchy.

It is important to work with the KP hierarchy first, then consider the reduction.

KP hierarchy and its reductions

Let $x = t_1$, $\partial_x = \frac{\partial}{\partial t_1}$,

$$L = \partial_x + \sum_{n=1}^{\infty} u_{n+1}(\mathbf{t})(\hbar\partial_x)^{-n},$$

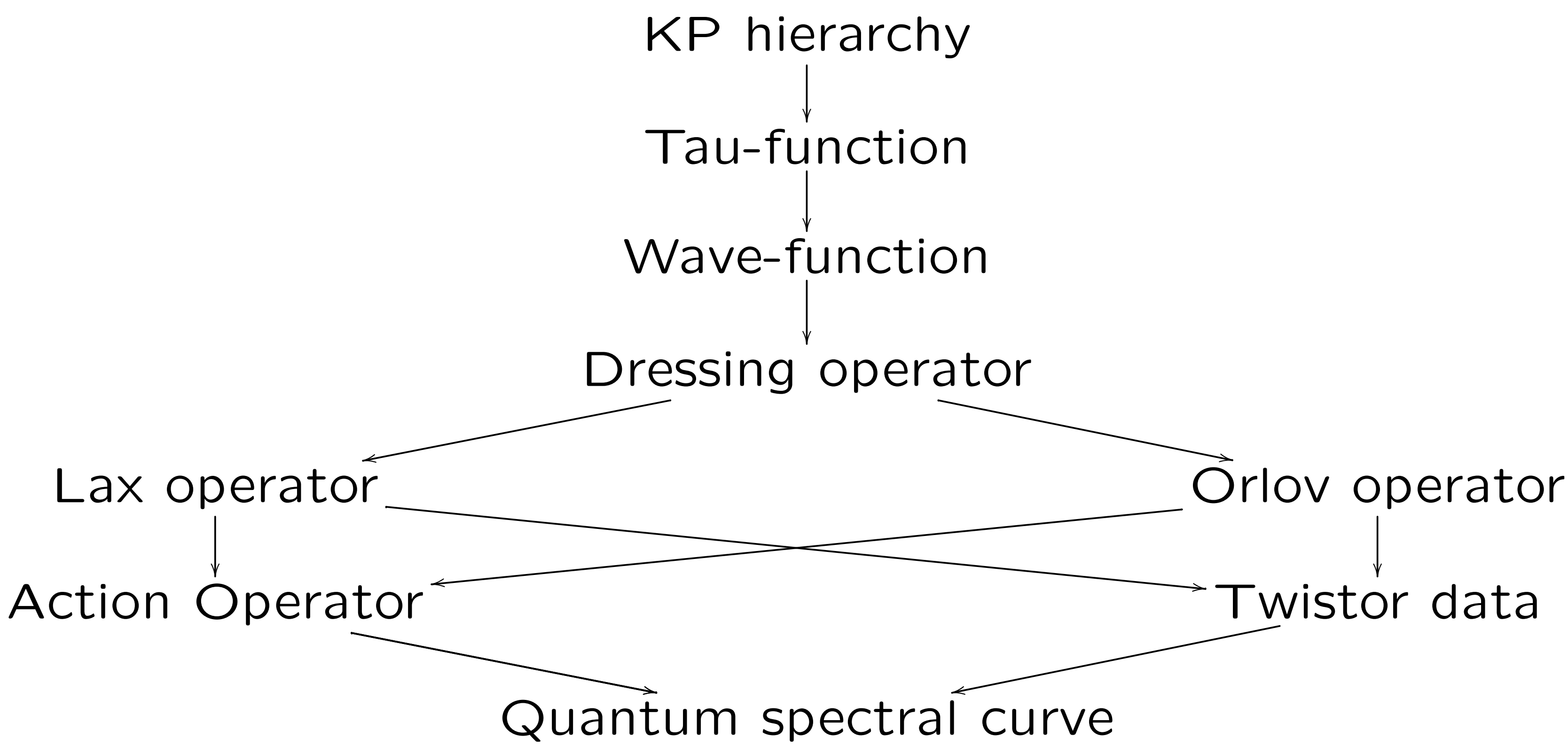
The latter is a system of partial differential equations:

$$\hbar\frac{\partial}{\partial t_k}L = [B_k, L], \quad k = 1, 2, \ldots,$$
$$B_k = (L^k)_+.$$

This system is equivalent to the Zakharov-Shabat zero-curvature equations:

$$\hbar\frac{\partial B_m}{\partial T_n} - \hbar\frac{\partial B_n}{\partial T_m} + [B_m, B_n] = 0.$$

Emergent geometry of KP hierarchy



Wave-function

By Sato's formula, one can define the wave function from the tau-function as follows:

$$w(\mathbf{T};\xi) = \exp\left(\hbar \sum_{n=1}^{\infty} T_n \xi^n\right) \cdot \frac{\tau(T_1 - \hbar\xi^{-1}, T_2 - \hbar\frac{1}{2}\xi^{-2}, \ldots; \hbar)}{\tau(T_1, T_2, \ldots; \hbar)}.$$

It is a formal solution of the form

$$w = \exp\left(\hbar^{-1} \sum_{n=1}^{\infty} T_n \xi^n\right) \cdot \left(1 + \frac{w_1}{\xi} + \frac{w_2}{\xi^2} + \cdots\right)$$

to the system

$$Lw = \xi \cdot w,$$
$$\hbar \frac{\partial w}{\partial T_n} = B_n w, \;\; n = 1, 2, \ldots.$$

The compatibility condition of this system is the KP system.

Dressing operator

The dressing operator W is defined by:

$$W := 1 + \sum_{n=1}^{\infty} w_j \partial_x^{-j}.$$

The dressing operator W and the wave-function w uniquely determine each other:

$$w = W \exp\left(\hbar^{-1} \sum_{n=1}^{\infty} T_n \xi^n\right).$$

The operator L and W are related by:

$$L := W \circ \hbar\partial_x \circ W^{-1}.$$

The evolution of the operator W is governed by the Sato equation:

$$\frac{\partial}{\partial T_k} W = -(L^k)_- W.$$

Orlov-Schulman operator

The Orlov-Schulman operator is defined by

$$M = W\left(\sum_{n=1}^{\infty} nT_n(\hbar\partial_x)^{n-1}\right)W^{-1}.$$

It can be written in the following form:

$$M = \sum_{n=1}^{\infty} nT_nL^{n-1} + \sum_{n=1}^{\infty} v_n(\mathbf{T};\hbar)L^{-n-1}.$$

This operator satisfies the following equations:

$$\hbar\frac{\partial M}{\partial T_n} = [B_n, M], \quad n = 1, 2, \ldots,$$
$$[L, M] = \hbar.$$

S-function

Write the logarithm of the wave-function as follows:

$$\log w(\mathbf{T};\xi) = \hbar^{-1}\sum_{n=1}^{\infty} T_n\xi^n + \hbar^{-1}\sum_{n=0}^{\infty} S_{n+1}(\mathbf{T};\hbar)\xi^{-n}.$$

By Sato's formula, we have

$$\hbar^{-1}\sum_{n=0}^{\infty} S_{n+1}(\mathbf{T};\hbar)\xi^{-n} = F(\mathbf{T}-\hbar[\xi^{-1}];\hbar) - F(\mathbf{T};\hbar).$$

Relations among operators by S-function

$$\hbar\partial = L + \sum_{n=1}^{\infty} \frac{\partial S_{n+1}}{\partial x} L^{-n},$$

$$M = \sum_{n=1}^{\infty} nT_n L^{n-1} - \sum_{n=1}^{\infty} nS_{n+1} L^{-n-1},$$

$$B_m = L^m + \sum_{n=1}^{\infty} \frac{\partial S_{n+1}}{\partial T_m} L^{-n}.$$

The $\widehat{S}$-operator and noncommutative Hamilton-Jacobi equation

Introduce an operator

$$\widehat{S} := \sum_{n=1}^{\infty} T_n L^n + \sum_{n=0}^{\infty} S_{n+1}(\mathbf{T};\hbar) L^{-n},$$

then one has

$$\hbar \widehat{S} w = (\sum_{n=1}^{\infty} T_n \xi^n + \sum_{n=0}^{\infty} S_{n+1}(\mathbf{T};\hbar)\xi^{-n})w.$$

One can regard $\widehat{S}$ as the *noncommutative action operator*. Formally one has

$$\partial_L \widehat{S} = M, \qquad \partial_{T_n} \widehat{S} = B_n.$$

Here L is the noncommutative generalized position, and M is the noncommutative conjugate momentum; T_n's are time variables, and B_n are the corresponding Hamiltonians.

Takasaki-Takebe's result on twistor data

Suppose that

$$f(\hbar, x, \hbar\partial_x) = \sum_{n\in\mathbb{Z}} f_n(\hbar, x)(\hbar\partial_x)^n,$$
$$g(\hbar, x, \hbar\partial_x) = \sum_{n\in\mathbb{Z}} g_n(\hbar, x)(\hbar\partial_x)^n$$

are two pseudo-differential operators with some homogeneity conditions, and that they satisfy the canonical commutation relation

$$[f, g] = \hbar.$$

Takasaki-Takebe's result on twistor data

Assume that there are two pseudo-differential operators L and M of the forms

$$L = \hbar\partial_x + \sum_{n=1}^{\infty} u_{n+1}(\hbar, t)(\hbar\partial_x)^{-n},$$

$$M = \sum_{n=1}^{\infty} n t_n L^{n-1} + \sum_{n=1}^{\infty} v_n(\hbar, t) L^{-n-1},$$

respectively, are given and that some homogeneity conditions are satisfied and

$$[L, M] = \hbar$$

Takasaki-Takebe's result on twistor data

If $f(\hbar, M, L)$ and $g(\hbar, M, L)$ are both differential operators, i.e.,

$$(f(\hbar, M, L))_- = (g(\hbar, M, L))_- = 0,$$

then L is a solution of the KP hierarchy, and M is the corresponding Orlov-Shulman operator.

The pair (f, g) above is called the twistor data of the tau-function of the KP hierarchy.

Conversely, any solution of the KP hierarchy possesses a twistor data.

Noncommutative special deformation

Write $P = f(M, L; \hbar)$ and $Q = g(M, L; \hbar)$. Then one has

$$[P, Q] = \hbar.$$

Hence one can also use the pair (P, Q) as noncommutative Darboux coordinates. In particular, one can express $\widehat{S}$ in terms of P and Q.

Let us rewrite $\widehat{S}$ as a noncommutative function on the noncommutative (P, Q)-plane. We will consider the following noncommutative curve:

$$Q = \partial_P \widehat{S}.$$

We will call this the *noncommutative special deformation*.

Special twistor data and string equation

Suppose that the operators P and Q satisfy:

$$
\begin{aligned}
P &= W(\hbar\partial_x)^n W^{-1}, \\
Q &= W\left(\frac{1}{n}x(\hbar\partial_x)^{1-n} + \sum_m c_m(\hbar)\cdot(\hbar\partial_x)^m\right)W^{-1}.
\end{aligned}
$$

In particular, $P = L^n$.

We call such a twistor data a special twistor data .

In this case the equation

$$[P, Q] = \hbar$$

is called the string equation .

Sato Grassmannian

In the above we are naturally led to the space $\mathbb{C}((\partial_x^{-1}))$ of pseudo-differential operators with constant coefficients. Let us recall Sato Grassmannian.

Denote by E the ring of pseudo-differential operators. Then $H = E/Ex$ is isomorphic as a vector space to the set $\mathbb{C}((\partial_x^{-1}))$. After a Fourier transform,

$$H \cong \mathbb{C}((z^{-1})),$$

with the action of $x^m \partial_x^n$ transformed to the action of $\partial_z^m z^n$.

Let $H_+ = \mathbb{C}[z]$ and $H_- = z^{-1}\mathbb{C}[[z^{-1}]]$. One has a decomposition $H = H_+ \oplus H_-$. Let $\pi : H \to H_+$ be the projection. The big cell $Gr^{(0)}$ of Sato grassmannian consists of linear subspaces V of H such that $\pi_+|_V : V \to H_+$ are isomorphisms.

Kac-Schwarz operator

One can use the dressing operator to associate an element V corresponding to the tau-function as follows:

$$V = W^{-1} H_+.$$

Since P and Q are differential operators, they satisfy:

$$PH_+ \subset H_+, \qquad QH_+ \subset H_+.$$

These conditions are equivalent to:

$$z^n V \subset V, \qquad AV \subset V,$$

where A is the Kac-Schwarz operator:

$$A = \hbar^{1-n} \frac{\partial}{\partial z^n} + \sum_m c_m(\hbar) \cdot (\hbar z)^m.$$

Kac-Schwarz operator characterization of Z_{WK}

Let $t_n = (2n+1)!!T_{2n+1}$, Z_{WK} becomes a series in $T_1, T_3. \dots,$, and a tau-function of the KP hierarchy.

According to Kac-Schwarz, the element $V \in Gr_{(0)}$ corresponding to Z_{WK} is given by a basis of the form $\{z^{2n}a(z), z^{2n+1}b(z)\}_{n\geq 0}$, where $a(z)$ and $b(z)$ are given by the following series respectively:

$$a(z) = \sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} z^{-3m},$$

$$b(z) = -\sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} \frac{6m+1}{6m-1} z^{-3m+1}.$$

This space is characterized as follows:

$$z^2 V \subset V, \qquad (\frac{1}{z}\partial_z - \frac{1}{2z^2} + z)V \subset V.$$

Twistor data and for Witten-Kontsevich tau-function

$$P = L^2, \qquad Q = L^{-1}M - \frac{1}{2}L^{-2} + L,$$

$$\widehat{S} = \sum_{n=1}^{\infty} T_n P^{n/2} + \sum_{n=0}^{\infty} S_{n+1}(\mathbf{T};\hbar) P^{-n/2},$$

Then the noncommutative special deformation is given by:

$$Q = \frac{1}{2}\sum_{n=1}^{\infty} nT_n P^{n/2-1} - \frac{1}{2}\sum_{n=0}^{\infty} S_{n+1}(\mathbf{T};\hbar) P^{-n/2-1}.$$

Set $T_{2n} = 0$ for $n \geq 1$ in the dispersionless version:

$$\mathcal{Q} = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)T_{2n+1}\mathcal{P}^{n-1/2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\partial F_0}{\partial T_{2n+1}}\mathcal{P}^{-n-3/2}.$$

This is the special deformation.

Fermionic picture from KP hierarchy point of view

Suppose that $V \in Gr^{(0)}$ is given by a normalized basis

$$\{f_n = z^n + \sum_{m \geq 0} a_{n,m} z^{-m-1}\}_{n \geq 0},$$

then after the boson-fermion correspondence the tau-function corresponds to:

$$|U\rangle = e^A |0\rangle,$$

where A is a linear operator on the fermionic Fock space:

$$A = \sum_{m,n \geq 0} a_{n,m} \psi_{-m-1/2} \psi^*_{-n-1/2}.$$

A formula for n-point function

For $n \geq 1$, the bosonic n-function associated to tau-function of the KP hierarchy is:

$$\sum_{j_1,\dots,j_n \geq 1} \frac{\partial^n F_U}{\partial T_{j_1} \cdots \partial T_{j_n}}\bigg|_{\mathbf{T}=0} \xi_1^{-j_1-1} \cdots \xi_n^{-j_n-1} + \frac{\delta_{n,2}}{(\xi_1 - \xi_2)^2}$$
$$= (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{i=1}^{n} \widehat{A}(\xi_{\sigma(i)}, \xi_{\sigma(i+1)}),$$

where $\widehat{A}(\xi_i, \xi_j)$ are the propagators:

$$\widehat{A}(\xi_i, \xi_j) = \begin{cases} i_{\xi_i,\xi_j} \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i < j, \\ A(\xi_i, \xi_i), & i = j, \\ i_{\xi_j,\xi_i} \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i > j. \end{cases}$$

Notations used in the formula

$$A(\xi,\eta)=\sum_{m,n\geq 0} a_{m,n}\xi^{-m-1}\eta^{-n-1},$$

$$i_{x,y}\frac{1}{(x-y)^n}=\sum_{k\geq 0}\binom{-n}{k}x^{-n-k}y^k.$$

The propagator for Witten-Kontsevich tau-function

Recall two solutions of Airy equation give rise to the Faber-Zagier series:

$$a(z) = \sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} z^{-3m},$$
$$b(z) = -\sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} \frac{6m+1}{6m-1} z^{-3m+1}.$$

I derive the following formula for the propagator:

$$\sum_{m,n\geq 0} a_{m,n} x^{-m-1} y^{-n-1} = -\frac{1}{x-y} + \frac{a(-x)\cdot b(y) - a(y)b(-x)}{x^2-y^2}.$$

First few terms of the propagator

The following are the first few terms of $A(x, y)$:

$$
\begin{aligned}
A(x,y) \;=\;& \frac{5}{24xy^3} - \frac{7}{24x^2y^2} + \frac{5}{24x^3y} \\
&+ \frac{385}{1152xy^6} - \frac{455}{1152x^2y^5} + \frac{385}{1152x^3y^4} \\
&- \frac{385}{1152x^4y^3} + \frac{455}{1152x^5y^2} - \frac{385}{1152x^6y} \\
&+ \frac{85085}{82944xy^9} - \frac{95095}{82944x^2y^8} + \frac{85085}{82944x^3y^7} \\
&- \frac{43505}{41472x^4y^6} + \frac{45955}{41472x^5y^5} - \frac{43505}{41472x^6y^4} \\
&+ \frac{85085}{82944x^7y^3} - \frac{95095}{82944x^8y^2} + \frac{85085}{82944x^9y} + \cdots .
\end{aligned}
$$

Formula for one-point function and two-point function

The bosonic one-point function for topological 2D gravity is:

$$\sum_j \frac{\partial F}{\partial T_j}\bigg|_{\mathbf{T}=0} \xi^{-j-1} = \sum_{g\geq 1} \frac{(6g-3)!!}{24^g g! \xi^{6g+1}}.$$

The bosonic two-point function for topological 2D gravity is:

$$\sum_{j,k} \frac{\partial^2 F}{\partial T_j \partial T_k}\bigg|_{\mathbf{T}=0} \xi_1^{-j-1} \xi_2^{-k-1} = -\widehat{A}(\xi_1,\xi_2)\widehat{A}(\xi_2,\xi_1) - \frac{1}{(\xi_1-\xi_2)^2}.$$

Example: Hurwitz numbers

Denote by $H_{g,\mu}$ the Hurwitz number of branched coverings of $\mathbb{P}^1$ of type μ by genus g Riemann surfaces.

These numbers can be computed by the Burnside formula:

$$\begin{aligned} Z_H : &= \exp \sum_{\mu \neq \emptyset} \sum_{g \geq 0} \frac{\lambda^{2g-2+l(\mu)+|\mu|}}{(2g-2+l(\mu)+|\mu|)!} H_{g,\mu} p_\mu \\ &= \sum_{\nu} \frac{\dim R_\nu}{|\nu|!} \cdot e^{\kappa_\nu \lambda/2} \cdot s_\nu. \end{aligned}$$

Here $p_\mu = \prod_{i=1}^{l(\mu)} p_{\mu_i}$ are the Newton functions, s_ν are the Schur functions.

Z_H is a tau-function of the KP hierarchy, with $T_n = \frac{p_n}{n}$. This is a special case of a result of Okounkov for two-Hurwitz numbers.

Spectral curve and quantum spectral curve for Hurwitz numbers

The spectral curve for Hurwitz numbers is the Lambert curve (Eynard et al):

$$x = ye^{-y}.$$

The quantum spectral curve for Hurwitz numbers is the quantum Lambert curve (Z.):

$$(\widehat{y} - \widehat{x}e^{\widehat{y}})\Phi = 0, \quad \widehat{x} = x\cdot, \quad \widehat{y} = \lambda x \frac{\partial}{\partial x},$$

where

$$\Phi = \sum_{n=0}^{\infty} e^{n(n-1)\lambda/2} \frac{x^n}{n!\lambda^n}.$$

Fermionic picture for Hurwitz partition function

$$U_H = \exp\Big(\sum_{m,n\geq 0} (-1)^n \frac{e^{(m(m+1)/2 - n(n+1)/2)\lambda}}{(m+n+1)\cdot m!\cdot n!} \cdot \psi_{-m-1/2}\psi^*_{-n-1/2}\Big)|0\rangle.$$

The one-point function:

$$G^{(1)}(\xi) = \sum_{n=1}^{\infty} \frac{1}{n!}(e^{n\lambda/2} - e^{-n\lambda/2})^{n-1}\xi^{-n-1}.$$

The two-point function:

$$
\begin{aligned}
& G^{(2)}(\xi_1,\xi_2) \\
= \; & \sum_{m>1} \frac{1}{(m+1)!}\left(q^{m/2}-q^{-m/2}\right)^{m+1} \xi_1^{-2}\xi_2^{-m-1} \\
+ \; & \sum_{m\geq 1} \frac{1}{(m+2)!}(q^{\frac{m}{2}}-q^{-\frac{m}{2}})^{m+1}\left(q^{m+1}-q^{-m}+mq-mq^{-1}\right)\xi_1^{-3}\xi_2^{-m-1} \\
+ \; & \sum_{m\geq 1} \frac{1}{(m+3)!}(q^{\frac{m}{2}}-q^{-\frac{m}{2}})^{m+1}(q^{3/2}-q^{-3/2})^2 \\
\cdot \; & \left[\frac{q^{2m+3}+q^{-2m-3}+(m+1)(q^{m+3}+q^{-m-3})-(m+2)(q^m+q^{-m})}{(q^{3/2}-q^{-3/2})^2}\right. \\
& \left. +\binom{m+2}{2}\right]\cdot \xi_1^{-4}\xi_2^{-m-1}+\cdots
\end{aligned}
$$

Example: Mariño-Vafa formula

For a partition $\mu = (\mu_1, \ldots, \mu_{l(\mu)})$, consider the Hodge integral:

$$\mathcal{C}_{g,\mu}(a) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\,\mathrm{Aut}(\mu)|}(a(a+1))^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i a + a)}{(\mu_i-1)!}$$
$$\cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-a-1)\Lambda_g^\vee(a)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}.$$

The Mariño-Vafa formula (Mariño-Vafa, Liu-Liu-Z., Okounkov-Pandharipande) states that ($q = e^{\sqrt{-1}\lambda}$):

$$
\begin{aligned}
Z_{MV} : &= \exp \sum_{|\mu|\geq 1} \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} \mathcal{C}_{g,\mu}(a) p_\mu \\
&= \sum_{|\nu|\geq 0} q^{a\kappa_\nu \tau/2} \sqrt{-1}^{|\nu|} s_\nu(q^\rho) s_\nu,
\end{aligned}
$$

$$
s_\nu(q^\rho) = s_\nu(q^{-1/2}, q^{-3/2}, \dots) = \frac{q^{\kappa_\nu/4}}{\prod_{x\in\nu}(q^{h(x)/2} - q^{-h(x)/2})}.
$$

This generalizes ELSV formula for Hurwitz numbers.

With $T_n = \frac{p_n}{n}$ Z_{MV} is a tau-function of the KP hierarchy (Z.).

Spectral curve and quantum spectral curve for Mariño-Vafa partition function

The spectral curve for MV partition function is the following curve (L, Chen, Z.):

$$x = y^a - y^{a+1}.$$

The quantum spectral curve for MV partition function is the quantum Lambert curve (Z.):

$$(1 - \widehat{y} - e^{\lambda/2}\widehat{x}\widehat{y}^{-a})\Phi = 0, \quad \widehat{x} = x\cdot, \quad \widehat{y} = e^{\lambda x \frac{\partial}{\partial x}},$$

where

$$\Phi = \sum_{n=0}^{\infty} \frac{e^{-an(n-1)\lambda/2 + n\lambda/2}}{\prod_{j=1}^{n}(1 - e^{j\lambda})} x^n.$$

Fermionic picture for MV partition function

$$
\begin{aligned}
U_{MV} &= \exp\Bigg(\sum_{m,n \geq 0} (-1)^n \sqrt{-1}^{m+n+1} \\
&\quad \cdot \frac{q^{(m(m+1)/2 - n(n+1)/2))(a+1/2)}}{[m+n+1]\cdot[m]!\cdot[n]!} \cdot \psi_{-m-1/2}\psi^*_{-n-1/2} \Bigg) |0\rangle, \\
q &= e^{\sqrt{-1}\lambda}, \\
[n] &= q^{n/2} - e^{-n/2}, \\
[n]! &= [n]\cdot[n-1]\cdots[1].
\end{aligned}
$$

Example: The conifold partition function

The full Mariño-Vafa conjecture states (Mariño-Vafa, Z.):

$$\exp \sum_{\mu \in \mathcal{P}^+} \sum_{g=0}^{\infty} \sum_{k=0}^{\infty} \sqrt{-1}^{l(\mu)} \tilde{F}_{g;k;\mu}^{(a)} \lambda^{2g-2+l(\mu)} e^{(|\mu|/2-k)t} p_\mu$$

$$= \sum_{\mu} s_\mu \cdot q^{a\kappa_\mu/2} \cdot \dim_q R_\mu.$$

The left-hand side is some open string partition function of the resolved conifold, the right-hand side is the quantum dimension :

$$\dim_q R_\mu = \prod_{1 \le i < j \le l(\mu)} \frac{[\mu_i - \mu_j + j - i]}{[j-i]} \cdot \prod_{i=1}^{l(\mu)} \frac{\prod_{j=1}^{\mu_i} [j-i]_{e^t}}{\prod_{j=1}^{\mu_i} [j - i + l(\mu)]}.$$

$$[n]_{e^t} = e^{t/2} q^{n/2} - e^{-t/2} q^{-n/2}.$$

Spectral curve and quantum spectral curve for conifold partition function

The spectral curve for conifold partition function is the following curve (Z.):

$$1 - y + xy^{a+1} - e^{-t}xy^a = 0.$$

The quantum spectral curve for conifold partition function is (Z.):

$$(1 - \widehat{y} + q^{1/2}\widehat{x}\widehat{y}^{a+1} - q^{1/2}e^{-t}\widehat{x}\widehat{y}^a)\Phi = 0, \quad \widehat{x} = x\cdot, \quad \widehat{y} = e^{-\sqrt{-1}\lambda x \frac{\partial}{\partial x}},$$

where

$$\Phi = \sum_{n=0}^{\infty} \prod_{j=1}^{n} \frac{1 - e^{-t}q^{-(j-1)}}{1 - q^{-j}} q^{an(n-1)/2 - n/2} x^n.$$

Fermionic picture for conifold partition function

$$U_{MV} \;=\; \exp\Bigg(\sum_{m,n\geq 0} (-1)^n \,\mathrm{dim}_q\, R_{(m|n)} \cdot \psi_{-m-1/2}\psi^*_{-n-1/2}\Bigg)|0\rangle, .$$

$$\begin{aligned}\mathrm{dim}_q\, R_{(m|n)} \;=\; & \frac{q^{(m(m+1)-n(n+1))/4}}{[m+n+1]\cdot[m]!\cdot[n]!} \\ & \cdot \prod_{j=0}^{m} (e^{t/2}q^{j/2} - e^{-t/2}q^{-j/2}) \\ & \cdot \prod_{k=1}^{n} (e^{t/2}q^{-k/2} - e^{-t/2}q^{k/2}).\end{aligned}$$

Example: Hermitian matrix model

For $N \geq 1$,

$$Z_N = \frac{\int_{\mathbb{H}_N} dM \exp\left(\mathrm{tr} \sum_{n=1}^{\infty} \frac{g_n - \delta_{n,2}}{n g_s} M^n\right)}{\int_{\mathbb{H}_N} dM \exp\left(-\frac{1}{2g_s} \mathrm{tr}(M^2)\right)}.$$

$\mathbb{H}_N$: space of Hermitian $N \times N$ matrices.

I will give a close formula for Z_N.

$$
\begin{aligned}
Z_N \;=\;& 1 + N^2 \cdot \frac{g_2}{2} + N g_s^{-1} \cdot \frac{g_1^2}{2} + (N + 2N^3) g_s \cdot \frac{g_4}{4} + 3N^2 \cdot \frac{g_3 g_1}{3} \\
+\;& (2N^2 + N^4) \cdot \frac{g_2^2}{8} + (2N + N^3) g_s^{-1} \cdot \frac{g_2 g_1^2}{4} + 3N^2 g_s^{-2} \cdot \frac{g_1^4}{4!} \\
+\;& (10N^2 + 5N^4) g_s^2 \frac{g_6}{6} + (5N + 10N^3) g_s \cdot \frac{g_5 g_1}{5} \\
+\;& (4N + 9N^3 + 2N^5) g_s \cdot \frac{g_4 g_2}{8} + (13N^2 + 2N^4) \cdot \frac{g_4 g_1^2}{8} \\
+\;& (3N + 12N^3) g_s \cdot \frac{g_3^2}{18} + (12N^2 + 3N^4) \cdot \frac{g_3 g_2 g_1}{6} \\
+\;& (6N + 9N^3) g_s^{-1} \cdot \frac{g_3 g_1^3}{18} + (8N^2 + 6N^4 + N^6) \cdot \frac{g_2^3}{48} \\
+\;& (8N + 6N^3 + N^5) g_s^{-1} \cdot \frac{g_2^2 g_1^2}{16} + (12N^2 + 3N^4) g_s^{-2} \cdot \frac{g_2 g_1^4}{48} \\
+\;& 15N^3 g_s^{-3} \cdot \frac{g_1^6}{720} + \cdots .
\end{aligned}
$$

Close formula for Hermitian matrix model partition function

As in other examples, we use the theory of symmetric functions
First change to Newton functions:

$$g_n = g_s^{1-n/2} p_n$$

next change to Schur functions:

$$s_\mu = \sum_\nu \frac{\chi_\mu(\nu)}{z_\nu} p_\nu, \quad p_\nu = \sum_\mu \chi_\mu(\nu) s_\mu.$$

Then one has a much more tractable expression:

$$
\begin{aligned}
Z_N &= 1 + \frac{N(N+1)}{2} s_{(2)} - \frac{N(N-1)}{2} \cdot s_{(1,1)} \\
&+ \frac{1}{8} N(N+1)(N+2)(N+3) \cdot s_{(4)} \\
&- \frac{1}{8} N(N+1)(N+2)(N-1) \cdot s_{(3,1)} \\
&+ \frac{1}{4} N^2(N+1)(N-1) \cdot s_{(2,2)} \\
&- \frac{1}{8} N(N+1)(N-1)(N-2) \cdot s_{(2,1,1)} \\
&+ \frac{1}{8} N(N-1)(N-2)(N-3) \cdot s_{(1^4)} + \cdots
\end{aligned}
$$

Close formula for partition function of Hermitian matrix model

$$Z = \sum_{n\geq 0}\sum_{|\lambda|=2n}(2n-1)!!\frac{\chi^{\lambda}_{(2^n)}}{\chi^{\lambda}_{(1^{2n})}}\cdot \dim R_\lambda^{U(N)}\cdot s_\lambda$$

$$= \sum_{n\geq 0}\sum_{|\lambda|=2n}(2n-1)!!\frac{\chi^{\lambda}_{(2^n)}}{\chi^{\lambda}_{(1^{2n})}}\cdot \prod_{x\in\lambda}\frac{N+c(x)}{h(x)}\cdot s_\lambda$$

$$(2n-1)!!\frac{\chi^{\lambda}_{(2^n)}}{\chi^{\lambda}_{(1^{2n})}} = (-1)^{n(n-1)/2}\frac{\prod_{f\ \ odd} f!!\prod_{f'\ \ even} f'!!}{\prod_{f\ \ odd, f'\ \ even}(f-f')}$$

where

$$f_i = \lambda_i + 2n - i,\quad i = 1,\ldots,2n.$$

Fermionic picture for Hermitian matrix model partition function

$$
\begin{aligned}
U &= \exp\Bigg(\sum_{n\geq 1}\sum_{p+q=2n-1}(-1)^p\cdot(-1)^{[(p+1)/2]}\binom{n-1}{[p/2]}\frac{(2n-1)!!}{(2n)!} \\
&\quad \cdot N\prod_{j=1}^{q}(N+j)\cdot\prod_{k=1}^{p}(N-k)\cdot\psi_{-q-1/2}\psi^*_{-p-1/2}\Bigg)|0\rangle \\
&= \exp\Bigg(N(N+1)/2\cdot\psi_{-3/2}\psi^*_{-1/2}+N(N-1)/2\cdot\psi_{-1/2}\psi^*_{-3/2} \\
&+ N(N+1)(N+2)(N+3)/8\cdot\psi_{-7/2}\psi^*_{-1/2} \\
&+ N(N+1)(N+2)(N-1)/8\cdot\psi_{-5/2}\psi^*_{-3/2} \\
&- N(N+1)(N-1)(N-2)/8\cdot\psi_{-3/2}\psi^*_{-5/2} \\
&- N(N-1)(N-2)(N-3)/8\cdot\psi_{-1/2}\psi^*_{-7/2}+\cdots\Bigg)|0\rangle
\end{aligned}
$$

One-point function for Hermitian matrix model partition function and Harer-Zagier numbers

$$
\begin{aligned}
G^{(1)}(\xi) &= 1 \cdot N^2 \xi^{-3} + (2\,N^3 + N)\xi^{-5} + (5\,N^4 + 10N^2)\xi^{-7} \\
&+ (14\,N^5 + 70N^3 + 21N)\xi^{-9} \\
&+ (42\,N^6 + 420N^4 + 483N^2)\xi^{-11} \\
&+ (132\,N^7 + 2310N^5 + 6468N^3 + 1485N)\xi^{-13} \\
&+ (429\,N^8 + 12012N^6 + 66066N^4 + 56628N^2)\xi^{-15} + \cdots
\end{aligned}
$$

The coefficients are the Harer-Zagier numbers $\epsilon_g(n)$ =number of ways to glue a $2n$-gon to get a Riemann surface of genus g:

$$
\epsilon_g(n) = \frac{(2n)!}{(n+1)!(n-2g)!} \cdot \text{coefficient of } x^{2g} \text{ in } \left(\frac{x/2}{\tanh(x/2)}\right)^{n+1}.
$$

A formula for Harer-Zagier numbers using Stirling numbers

For $n \geq 1$, we then have:

$$\sum_{g=0}^{[n/2]} \epsilon_g(n) N^{n+1-2g}$$

$$= \sum_{p+q=2n-1} (-1)^p \cdot (-1)^{[(p+1)/2]} \binom{n-1}{[p/2]} \frac{(2n-1)!!}{(2n)!}$$

$$\cdot N \prod_{j=1}^{q} (N+j) \cdot \prod_{k=1}^{p} (N-k).$$

Spectral curve and quantum spectral curve for Hermitian matrix model

The red numbers are Catalan numbers .

The spectral curve for Hermitian matrix model is the following curve (Z.):

$$x = -\frac{1}{\sqrt{2}}y + \frac{\sqrt{2}N}{y}.$$

Thank you very much for your attentions!