Frobenius manifolds and Frobenius algebra-valued integrable systems

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Outline:

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Part D. Frobenius manifolds and Frobenius algebra-valued Integrable systems

Part A. Motivations

KdV equation : $\mathbf{4u_t - 12uu_x - u_{xxx} = 0}$

Lax pair: $L_t = [L_+^{\frac{3}{2}}, L], \quad L = \partial^2 + 2\,u$

tau function: $u = (\log \tau)_{xx}$

Bihamiltonian structure(BH): ($\{\cdot,\cdot\}_2 \Longrightarrow$ Virasoro algebra)

$$\left\{\tilde f, \tilde g\right\}_1 = 2\int \frac{\delta f}{\delta u}\frac{\partial}{\partial \mathbf{x}}\frac{\delta g}{\delta u}dx, \quad \left\{\tilde f, \tilde g\right\}_2 = -\frac{1}{2}\int \frac{\delta f}{\delta u}\left(\frac{\partial^3}{\partial \mathbf{x}^3} + \mathbf{2u}\frac{\partial}{\partial \mathbf{x}} + \mathbf{2}\frac{\partial}{\partial \mathbf{x}}\mathbf{u}\right)\frac{\delta g}{\delta u}dx.$$

Dispersionless $\Longrightarrow$ Hydrodynamics-type BH $\Longrightarrow$ Frobenius manifold

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Natural generalization

Lax pair/tau function/BH: KdV $\Longrightarrow$ GD$_n$ / DS $\Longrightarrow$ KP (other types)

$\{\cdot,\cdot\}_2$: Virasoro $\Longrightarrow$ W_n algbra $\Longrightarrow$ $W_\infty^{(N)}$ algebra

Lax pair/tau function/BH: dKdV $\Longrightarrow$ dGD$_n$ $\Longrightarrow$ dKP

$\{\cdot,\cdot\}_2$: Witt algebra $\Longrightarrow$ w_n algbra $\Longrightarrow$ $w_\infty^{(N)}$ algebra

FM: $A_1 \Longrightarrow A_{n-1} \Longrightarrow$ **Infinite-dimensional analogue (?)**

Other generalizations: **the coupled KdV**

$$\mathbf{4v_t - 12vv_x - v_{xxx} = 0, \quad 4w_t - 12(vw)_x - w_{xxx} = 0.}$$

tau function: $v = (\log \tau_0)_{xx}$, $w = (\frac{\tau_1}{\tau_0})_{xx}$.

R. Hirota, X.B.Hu and X.Y.Tang, J.Math.Anal.Appl.288(2003)326

BH: **A.P.Fordy, A.G.Reyman and M.A.Semenov-Tian-Shansky,** *Classical* r*-matrices and compatible Poisson brackets for coupled KdV systems*, Lett. Math. Phys. 17 (1989) 25–29.

W.X.Ma and B.Fuchssteiner, *The bihamiltonian structure of the perturbation equations of KdV Hierarchy.* Phys. Lett. A 213 (1996) 49–55.

P.Casati and G.Ortenzi, New integrable hierarchies from vertex operator representations of polynomial Lie algebras. J.Geom.Phys. 56(2006) 418–449.

Johan van de Leur,J.Geom.Phys., h57(2007)435–447.

$$\mathcal{F}_2^0 = \left\{ \begin{pmatrix} a_0 & 0 \\ a_1 & a_0 \end{pmatrix} \right\}, \quad \mathcal{F}_3^0 = \left\{ \begin{pmatrix} a_0 & 0 & 0 \\ a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \end{pmatrix} \right\}, \cdots, \mathcal{F}_m^0, \cdots$$

-valued **Lax pair**

When $m = 2$, it gives the Coupled KdV equation

$$\mathbf{4v_t - 12vv_x - v_{xxx} = 0}, \quad \mathbf{4w_t - 12(vw)_x - w_{xxx} = 0}.$$

Question: To construct BH for the coupled KP hierarchy and study the related W-type algebras and Frobenius manifolds.

Method: Adler-Gelfand-Dickey scheme

A conjectural construction :

$$\mathrm{tr}\,(A) = \text{the trace of } \begin{pmatrix} \frac{1}{m} & \frac{1}{m-1} & \cdots & 1 \\ 0 & \frac{1}{m} & \ddots & \frac{1}{2} \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{m} \end{pmatrix} A, \qquad A \in \mathcal{F}_m^0$$

Observation: $(\mathcal{F}_m^0, \mathrm{tr}\,, I_m)$ **is a Frobenius algebra.**

Part B. Some Known Facts about FM

Definition 1. *A* **Frobenius algebra** $\{\mathcal{A}, \circ, e, \mathrm{tr}\,\}$ *over* $\mathbb{K}$ *(=*$\mathbb{R}$ *or* $\mathbb{C}$*) satisfies the following conditions:*

(i). $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *is a commutative, associative algebra with unity* e*;*

(ii). $\mathrm{tr}\, \in \mathcal{A}^\star$ *defines a non-degenerate inner product* $\langle a, b\rangle = \mathrm{tr}\,(a \circ b)$ *.*

Since $\mathrm{tr}\,(a) = \langle e, a\rangle$ the inner product determines the form tr and visa-versa. This linear form tr is often called a **trace form (or Frobenius form)**.

Example 2. *Given* $\varepsilon \in \mathbb{R}$, *let* $\mathcal{F}_N^\varepsilon$ *be an* N*-dimensional commutative associative algebra over* $\mathbb{C}$ *with a basis* $e_1, e_2, \cdots, e_N$ *satisfying*

$$e_i \circ e_j = \begin{cases} e_{i+j-1}, & i+j \leq N+1; \\ \varepsilon e_{i+j-1-N}, & i+j > N+1. \end{cases}$$

We introduce N *"basic" trace forms* $\mathrm{tr} : \mathcal{F}_N^\varepsilon \to \mathbb{C}$ *defined by*

$$\mathrm{tr}\,(\sum_{j=1}^{N} a_j e_j) = a_k + a_N(1-\delta_k^N)\delta_0^\varepsilon, \quad k = 1, \cdots, N.$$

Obviously, all $\{\mathcal{F}_N^\varepsilon, \mathrm{tr}\,, e_1, \circ\}$ *are Frobenius algebras of the rank* N. *The algebra* $\mathcal{F}_N^\varepsilon$ *has a matrix representation as follows*

$$e_1 \mapsto \Theta^0 = \mathrm{Id_N}, \quad e_j \mapsto \Theta^{j-1}, \quad j = 2, \cdots, N,$$

where $\Theta = (\Theta_i^j) \in gl(N, \mathbb{C})$ *with the element* $\Theta_i^j = \delta_i^{j+1} + \varepsilon\delta_i^1\delta_N^j$ *and* $\Theta^0 = \mathrm{Id_N}$ *is the* N^{th}*-order identity matrix.*

Definition 3. *(B.Dubrovin) The set* $\{M, \circ, e, \langle\ ,\ \rangle, E\}$ *is a* **Frobenius manifold** *if each tangent space* T_tM *carries a smoothly varying Frobenius algebra with the properties:*

(i). $\langle\ ,\ \rangle$ *is a flat metric on* M*;*

(ii). $\nabla e = 0$*, where* ∇ *is the Levi-Civita connection of* $\langle\, ,\rangle$*;*

(iii). the tensors $c(u, v, w) := \langle u \circ v, w\rangle$ *and* $\nabla_z c(u, v, w)$ *are totally symmetric;*

(iv). A vector field E *exists, linear in the flat-variables, such that the corresponding group of diffeomorphisms acts by conformal transformation on the metric and by rescalings on the algebra on* T_tM *.*

These axioms imply the existence of **the prepotential** F which satisfies the WDVV-equations of associativity in the flat-coordinates of the metric (strictly speaking only a complex, non-degenerate bilinear form) on M. The multiplication is then defined by the third derivatives of the prepotential:

$$\frac{\partial}{\partial t^\alpha} \circ \frac{\partial}{\partial t^\beta} = c^\gamma_{\alpha\beta}(\mathbf{t}) \frac{\partial}{\partial t^\gamma}$$

where

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

and indices are raised and lowered using the metric $\eta_{\alpha\beta} = \langle \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta} \rangle$.

Witten-Dijgraff-Verlinde-Verlinde equations

2-D TFT: find free energy $F = F(t^1, \cdots, t^n)$ satisfying WDVV equations of associativity (1991):

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\delta \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\gamma},$$

with a quasihomogeneity condition

$$\mathcal{L}_E F = (3 - d)F + \text{quadratic polynomial in t},$$

and

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^1} = \eta^{\alpha\beta}$$

$(\eta^{\alpha\beta})$: constant nondegenerate.

Example 4. *Suppose* c_{ij}^k *are the structure constants for the Frobenius algebra* $\mathcal{A}$*, so* $e_i \circ e_j = c_{ij}^k e_k$ *and* $\eta_{ij} = \langle e_i, e_j \rangle$ *. For such an algebra one obtains a cubic prepotential*

$$
\begin{aligned}
F &= \frac{1}{6} c_{ijk} t^i t^j t^k \,, \\
&= \frac{1}{6} \mathrm{tr}\,(\mathbf{t} \circ \mathbf{t} \circ \mathbf{t})\,, \qquad \mathbf{t} = t^i e_i \,.
\end{aligned}
$$

The Euler vector field takes the form $E = \sum_i t^i \frac{\partial}{\partial t^i}$ *and* $E(F) = 3F$ *. The notation* $\mathcal{A}$ *will be used for both the algebra and the corresponding Frobenius manifold.*

On the Frobenius manifold M, its cotangent space T_t^*M also carries a Frobenius algebra structure, with an invariant bilinear form $\langle dt^\alpha, dt^\beta \rangle^* = \eta^{\alpha\beta}$ and a product given by

$$dt^\alpha \cdot dt^\beta = c_\gamma^{\alpha\beta} dt^\gamma, \quad c_\gamma^{\alpha\beta} = \eta^{\alpha\epsilon} c_{\epsilon\gamma}^\beta.$$

Let

$$g^{\alpha\beta} = i_E(dt^\alpha \cdot dt^\beta),$$

then $(dt^\alpha, dt^\beta)^* = g^{\alpha\beta}$ defines a symmetric bilinear form, called the intersection form, on T_t^*M.

The above two bilinear forms on T^*M compose a pencil $g^{\alpha\beta} + \mu\, \eta^{\alpha\beta}$ of flat metrics with parameter μ, hence they induces a bi-hamiltonian structure $\{\ ,\ \}_2 + \mu\{\ ,\ \}_1$ of hydrodynamic type on the loop space $\left\{S^1 \to M\right\}$.

Furthermore, on the loop space one can choose a family of functions $\theta_{\alpha,p}(t)$ with $\alpha = 1, 2, \ldots, n$ and $p \geq 0$ such that

$$\theta_{\alpha,0} = \eta_{\alpha\beta} t^{\beta}, \quad \theta_{\alpha,1} = \frac{\partial F}{\partial t^{\alpha}}, \quad \frac{\partial^2 \theta_{\alpha,p}}{\partial t^{\lambda} \partial t^{\mu}} = c^{\epsilon}_{\lambda\mu} \frac{\partial \theta_{\alpha,p-1}}{\partial t^{\epsilon}} \text{ for } p > 1.$$

The **principal hierarchy** associated to M is the following system of Hamiltonian equations

$$\frac{\partial t^{\gamma}}{\partial T^{\alpha,p}} = \left\{ t^{\gamma}(x), \int \theta_{\alpha,p}\, dx \right\}_1, \quad \alpha, \gamma = 1, 2, \ldots, n; \ p \geq 0,$$

in which x is the coordinate of S^1. This hierarchy can be written in a bi-hamiltonian recursion form if certain nonresonant condition is fulfilled. **(See ref.Dubrovin-Zhang 2001)**

Example 5. $F(u)=\frac{1}{6}u^3$, $E=u\frac{\partial}{\partial u}$, $e=\frac{\partial}{\partial u}$, $\eta^{11}=1$. *Denote* $T_n := n!T^{\alpha,n}$, $\theta_n := \theta_{\alpha,n}$, *then* $\theta_n = \frac{1}{(n+1)!}u^{n+1}$. *So the principal hierarchy is given by*

$$\frac{\partial u}{\partial T_n} = u_x u^n, \quad n=1,2,\cdots \quad \rightsquigarrow \text{ dispersionless KdV}$$

with two compatible Poisson structure

$$\mathbf{P_1} = \frac{\partial}{\partial \mathbf{x}}, \quad \mathbf{P_2} = \frac{\partial}{\partial \mathbf{x}}\mathbf{u} + \mathbf{u}\frac{\partial}{\partial \mathbf{x}}$$

which gives a flat pencil $\eta^{11} + \mu g^{11}$ *on the Frobenius manifold* $\mathbb{C}$.

Conversely, under certain conditions one can obtain the prepotential F *from the two compatible hydrodynamical Poisson structure.*

Part C. The Frobenius algebra-valued KP hierarchy

Problem: To study the Frobenius algebra-valued KP hierarchy.

For an $\mathcal{A}$-valued operator $P = \sum_i P_i \partial^i$, P_+ is the pure differential of the operator P and

$$P_- = P - P_+, \quad \text{res}(P) = P_{-1}, \quad P^* = \sum_i (-1)^i \partial^i P_i, \quad \partial = \frac{\partial}{\partial x}.$$

Let

$$L = \mathrm{I}_m \partial + U_1 \partial^{-1} + U_2 \partial^{-2} + \cdots \quad (1)$$

be an $\mathcal{A}$-valued ΨDO with coefficients U_1, U_2, $\cdots$ being smooth $\mathcal{A}$-valued functions of an infinitely many variables $t = (t_1, t_2, \cdots)$ and $t_1 = x$.

Definition. The $\mathcal{A}$-**KP hierarchy** is the set of equations

$$\frac{\partial L}{\partial t_r} = [B_r, L], \quad B_r = L^r_+, \quad r = 1, 2, \cdots \quad (2)$$

or equivalently,

$$\frac{\partial B_l}{\partial t_r} - \frac{\partial B_r}{\partial t_l} + [B_l, B_r] = 0. \quad (3)$$

Example 3.1.$[r = 2, l = 3]$ Using $B_2 = \mathrm{I}_m\partial^2 + 2U_1$ and $B_3 = \mathrm{I}_m\partial^3 + 3U_1\partial + 3U_2 + 3U_{1,x}$, the system (3) becomes

$$U_{1,t_2} = U_{1,xx} + 2U_{2,x}, \quad 2U_{1,t_3} = 2U_{1,xxx} + 3\,U_{2,xx} + 3\,U_{2,t_2} + 6\,U_1U_{1,x}.$$

If we eliminate U_2 and rename $t_2 = y$, $t_3 = t$ and $U = U_1$, we obtain

$$(4U_t - 12UU_x - U_{xxx})_x - 3U_{yy} = 0. \tag{4}$$

When we choose a $\mathcal{F}_2^0$-valued smooth function $U = \begin{pmatrix} u_0 & 0 \\ u_1 & u_0 \end{pmatrix}$, the system (4) reads the coupled KP equation (e.g., P.Casati and G.Ortenzi 2006)

$$\begin{cases} (4u_{0t} - 12u_0u_{0x} - u_{0xxx})_x - 3u_{0yy} = 0, \\ (4u_{1t} - 12u_0u_{1x} - 12u_{0x}u_1 - u_{1xxx})_x - 3u_{1yy} = 0. \end{cases}$$

Especially if $u_{0y} = u_{1y} = 0$, the coupled KP equation reduces to the coupled KdV equation (e.g.,A.P.Fordy et al 1989; Ma W.X et al,1996; Hirota-Hu-Tang 2003)

$$4u_{0t} - 12u_0u_{0x} - u_{0xxx} = 0, \quad 4u_{1t} - 12u_0u_{1x} - 12u_{0x}u_1 - u_{1xxx} = 0.$$

We have shown that (**Ian and Zuo 2015, JMP**)

(1). The $\mathcal{A}$-KP hierarchy has an $\mathcal{A}$-valued τ function

(2). The $\mathcal{A}$-KP hierarchy admits bi-Hamiltonian structures

(3). Local matrix generalizations of the classical W-algebras

Example 3.2. [The coupled KdV equation]. Assume that $\tau = \begin{pmatrix} \tau_0 & 0 \\ \tau_1 & \tau_0 \end{pmatrix}$ is its $\mathcal{F}_2^0$-valued τ-function, then we have

$$\begin{pmatrix} u_0 & 0 \\ u_1 & u_0 \end{pmatrix} = U = \frac{\partial}{\partial x}(\tau_x \tau^{-1}) = \begin{pmatrix} (\log \tau_0)_{xx} & 0 \\ (\frac{\tau_1}{\tau_0})_{xx} & (\log \tau_0)_{xx} \end{pmatrix}.$$

So $u_0 = (\log \tau_0)_{xx}$, $u_1 = (\frac{\tau_1}{\tau_0})_{xx}$. Taking a τ-function $\tau_0 = 1 + \exp(2ax + 2a^3 t)$ of the KdV equation, then for $A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$, the function $\tau = \mathrm{I}_2 + \exp(2Ax + 2A^3 t)$, i.e.,

$$\tau = \begin{pmatrix} 1 + \exp(2ax + 2a^3 t) & 0 \\ (2bx + 2b^3 t)\exp(2ax + 2a^3 t) & 1 + \exp(2ax + 2a^3 t) \end{pmatrix}$$

is a τ-function of the coupled KdV equation. Consequently, we obtain a solution given by

$$u_0 = \frac{a^2}{\cosh^2(ax + a^3 t)}, \quad u_1 = \left(\frac{(2bx + 2b^3 t)\exp(2ax + 2a^3 t)}{1 + \exp(2ax + 2a^3 t} \right)_{xx}.$$

Hamiltonian structures of $\mathcal{A}$-KP : the canonical AGD method and the trace form tr : $\mathcal{A} \longrightarrow \mathbb{K}$.

The space of functionals: $\widetilde{\mathcal{D}} = \left\{ \tilde{f} = \int \mathrm{tr}\, F(V)dx \middle| \mathrm{tr}\, F(V) \in \mathcal{D} \right\}$.

For $V = \sum_{q=1}^{m} v_q e_q$, the variational derivative $\frac{\delta F}{\delta V}$ is defined by

$$\mathbf{\tilde{f}(v+\delta v) - \tilde{f}(v) = \int \mathrm{tr} \left(\frac{\delta F}{\delta V} \circ \delta V + o(\delta V) \right) dx = \int \sum_{q=1}^{m} \left(\frac{\delta f}{\delta v_q} \delta v_q + o(\delta v) \right) dx,}$$

where $f(v) = \mathrm{tr}\, F(V)$, $\delta V = \sum_{q=1}^{m} \delta v_q e_q \in \mathcal{A}$ and $\frac{\delta f}{\delta v_q} = \sum_{j=0}^{\infty} (-\partial)^j \frac{\partial f}{\partial v_q^{(j)}}$.

The first and the second Poisson brackets of the $\mathcal{A}$-KP hierarchy associated with the $\mathcal{F}_m^0$-valued ΨDO L^n are given by

$$\begin{aligned}\left\{\tilde{f},\tilde{g}\right\}^{n(\infty)} &= \text{tr}\int \text{res}\, H^{n(\infty)}(\frac{\delta f}{\delta\mathcal{L}})\,\frac{\delta g}{\delta\mathcal{L}}\,dx \\ &= \text{tr}\int \text{res}\,\left([\mathcal{L}_-,(\frac{\delta f}{\delta\mathcal{L}})_+]_- - [\mathcal{L}_+,(\frac{\delta f}{\delta\mathcal{L}})_-]_+\right)\frac{\delta g}{\delta\mathcal{L}}\,dx\end{aligned}$$

and

$$\begin{aligned}\left\{\tilde{f},\tilde{g}\right\}^{n(0)} &= \text{tr}\int \text{res}\, H^{n(0)}(\frac{\delta f}{\delta\mathcal{L}})\,\frac{\delta g}{\delta\mathcal{L}}\,dx \\ &= \text{tr}\int \text{res}\,\left((\mathcal{L}\,\frac{\delta f}{\delta\mathcal{L}})_+\mathcal{L} - \mathcal{L}\,(\frac{\delta f}{\delta\mathcal{L}}\,\mathcal{L})_+\right)\frac{\delta g}{\delta\mathcal{L}}\,dx,\end{aligned}$$

where $\mathbf{\frac{\delta f}{\delta\mathcal{L}} = \sum_{i=-\infty}^{n-1}\partial^{-i-1}\frac{\delta F}{\delta V_i}}$. and $\tilde{f} = \text{tr}\int f(V)dx$, $\tilde{g} = \text{tr}\int g(V)dx$.

Theorem. The $\mathcal{A}$-KP hierarchy $\dfrac{\partial L}{\partial t_r} = [B_r, L]$ admits a bi-Hamiltonian representation given by

$$\frac{\partial \mathcal{L}}{\partial t_r} = H^{n(0)}\left(\frac{\delta h_r}{\delta \mathcal{L}}\right) = H^{n(\infty)}\left(\frac{\delta g_r}{\delta \mathcal{L}}\right) \tag{5}$$

with the Hamiltonians

$$\tilde{h}_r = \frac{n}{r}\mathrm{tr}\int \mathrm{res}\, L^r\, dx \quad \text{and} \quad \tilde{g}_r = -\frac{m}{r+n}\mathrm{tr}\int \mathrm{res}\, L^{n+r}\, dx,$$

where $H^{n(0)}(X) = (\mathcal{L}X)_+\mathcal{L} - \mathcal{L}(X\mathcal{L})_+$ and $H^{n(\infty)}(X) = [\mathcal{L}_-, X_+]_- - [\mathcal{L}_+, X_-]_+$.

Cor. The coupled KP ($\mathcal{F}^0_m$-KP) hierarchy defined in [CO2006] has at least m "basic" different local bi-Hamiltonian structures.

If we restrict to $V_{n-1}=0$, the first Hamiltonian structure automatically reduces to this submanifold, but the second one is reducible if and only if

$$\operatorname{res}\,[\mathcal{L},\frac{\delta f}{\delta\mathcal{L}}]=0. \tag{6}$$

We denote the corresponding reduced brackets by $\{\;,\;\}^{n(\infty)}$ and $\{\;,\;\}_D^{n(0)}$. Assume that $V_n=e$ and $X_i=\dfrac{\delta f}{\delta V_i}\in\mathcal{A}$, then the condition (6) is equivalent to

$$X_{n-1}=\frac{1}{n}\sum_{i=-\infty}^{n-2}\left(\binom{-i-1}{n-i}X_i^{(n-i-1)}+\sum_{j=i+1}^{n-1}\binom{-i-1}{j-i}(X_iV_j)^{(j-i-1)}\right).$$

Definition. In terms of the basis $\{v_{[i]q}\}$, the second Poisson bracket $\{\ ,\ \}^{n(0)}$ for L^n and the reduced bracket $\{\ ,\ \}_D^{n(0)}$ for L^n with the constraint $V_{n-1} = 0$ will provide two kinds of W-type algebras, we call them the $W_{\mathbb{KP}}^{(m,n)}$-algebra and the $W_{\infty}^{(m,n)}$-algebra respectively. Under the reduction $L_-^n = 0$, the corresponding algebras are called the $W_{\mathbb{GD}}^{(m,n)}$-algebra and the $W_{(m,n)}$-algebra respectively. All of them are local matrix generalizations.

Example 3.3. Taking $\varphi(x) = \operatorname{tr} V_{n-2}(x)$, the reduced Poisson bracket is given by

$$\{\varphi(x), \varphi(y)\}_D^{n(0)} = -\left(\frac{n^3 - n}{12}\partial^3 + \varphi\,\partial + \partial\,\varphi\right)\delta(x-y).$$

This means that both the $\mathrm{W}_\infty^{(m,n)}$-algebra and the $\mathrm{W}_{(m,n)}$-algebra contain the Virasoro algebra as its subalgebra.

Example 3.4. Consider the $\mathcal{A}$-KdV hierarchy with the Lax operator $L^2 = e\partial^2 + V$, i.e., $L^2_- = 0$. We denote $X = \partial^{-2}X_1 + \partial^{-1}X_0$ and $Y = \partial^{-2}Y_1 + \partial^{-1}Y_0$, and have

$$H^{2(\infty)} = [X, L^2]_+ = -2X_0'$$

and

$$H^{2(0)}(X) = (L^2 \circ X)_+ \circ L^2 - L^2 \circ (X \circ L^2)_+ = 2V \circ X_0' + X_0 \circ V' + \frac{1}{2}X_0'''.$$

Thus two compatible Poisson brackets of the $\mathcal{A}$-KdV hierarchy are given by

$$\left\{\tilde{f}, \tilde{g}\right\}^{2(\infty)} = 2\,\mathrm{tr} \int \frac{\delta f}{\delta V} \circ \frac{\partial}{\partial x}\frac{\delta g}{\delta V} dx$$

and

$$\left\{\tilde{f}, \tilde{g}\right\}^{2(0)}_D = -\frac{1}{2}\mathrm{tr} \int \frac{\delta f}{\delta V} \circ \left(e\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + 2\frac{\partial}{\partial x}V\right) \circ \frac{\delta g}{\delta V} dx.$$

In particular, if one chooses the algebra $\mathcal{A}$ to be the algebra $\mathcal{F}_2^0$, one obtains the $\mathcal{F}_2^0$-KdV equation for $V = ve_1 + we_2$ given by

$$\mathbf{4v_t - 12vv_x - v_{xxx} = 0, \quad 4w_t - 12(vw)_x - w_{xxx} = 0.} \tag{7}$$

Moreover, the system (7) can be written as

$$\begin{pmatrix} v \\ w \end{pmatrix}_{\mathbf{t}} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_2}{\delta v} \\ \frac{\delta H_2}{\delta w} \end{pmatrix} = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix} \begin{pmatrix} \frac{\delta H_1}{\delta v} \\ \frac{\delta H_1}{\delta w} \end{pmatrix}$$

with Hamiltonians

$$H_1 = \int_{\mathbb{S}^1} vw dx, \quad H_2 = \int_{\mathbb{S}^1} (\frac{3}{2}v^2 w + \frac{1}{4}vw_{xx})dx;$$

and

$$\begin{pmatrix} v \\ w \end{pmatrix}_{\mathbf{t}} = \begin{pmatrix} 0 & \partial \\ \partial & -\partial \end{pmatrix} \begin{pmatrix} \frac{\delta \widetilde{H}_2}{\delta v} \\ \frac{\delta \widetilde{H}_2}{\delta w} \end{pmatrix} = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 - J_0 \end{pmatrix} \begin{pmatrix} \frac{\delta \widetilde{H}_1}{\delta v} \\ \frac{\delta \widetilde{H}_1}{\delta w} \end{pmatrix}$$

with Hamiltonians

$$\widetilde{H}_1 = \int_{\mathbb{S}^1} (\frac{1}{2}v^2 + vw)dx, \quad \widetilde{H}_2 = \int_{\mathbb{S}^1} (\frac{3}{2}v^2 w + \frac{1}{4}vw_{xx} + \frac{1}{2}v^3 + \frac{1}{8}vv_{xx})dx,$$

where $J_0 = \frac{1}{4}\partial^3 + v\partial + \partial v$ and $J_1 = w\partial + \partial w$.

Example 3.5. [The $\mathcal{F}_m^0$-dBoussinesq hierarchy]. In this case, the Lax operator is given by

$$\mathcal{L} = \mathrm{I}_m p^3 + V_1 p + V_0, \quad V_0, V_1 \in \mathcal{F}_m^0.$$

The BH structure of the $\mathcal{F}_m^0$-dBoussinesq hierarchy is

$$\left\{\tilde{f}, \tilde{g}\right\}_1 = 3\,\mathrm{tr} \int (X_1 Y_0' + X_0 Y_1')dx, \quad \text{here} \quad X_k = \frac{\delta f}{\delta V_k}, Y_k = \frac{\delta f}{\delta V_k} \tag{8}$$

and

$$\begin{aligned} \left\{\tilde{f}, \tilde{g}\right\}_2 &= \frac{1}{3}\mathrm{tr} \int \left(X_0 Y_0' - X_0' Y_0\right) V_1^2\, dx + \mathrm{tr} \int \left(X_1' Y_1 - X_1 Y_1'\right) V_1\, dx \\ &+ \mathrm{tr} \int \left(2X_1' Y_0 - X_1 Y_0' + X_0' Y_1 - 2X_0 Y_1'\right) V_0\, dx. \end{aligned} \tag{9}$$

When $m = 1$, it is easy to get a Frobenius manifold as follows

$$\mathbf{F(V) = \frac{1}{2}V_0^2V_1 - \frac{1}{72}V_1^4}, \quad \mathbf{e = \frac{\partial}{\partial V_0}}, \quad \mathbf{E = V_0\frac{\partial}{\partial V_0} + \frac{2}{3}V_1\frac{\partial}{\partial V_1}}.$$

When $m = 2$, we write

$$V_0 = \begin{pmatrix} v_1 & 0 \\ v_2 & v_1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} v_3 & 0 \\ v_4 & v_3 \end{pmatrix} \in \mathcal{F}_2^0.$$

By using (8) and (9), we could get explicit formulas of $\{v_i(x), v_j(y)\}_k$, $k = 1, 2$. We can write

$$\{v_i(x), v_j(y)\}_1 = 3\eta^{ij}(v)\delta'(x-y) + h_1(v; v_x)\delta(x-y)$$

and

$$\{v_i(x), v_j(y)\}_2 = 3g^{ij}(v)\delta'(x-y) + h_2(v; v_x)\delta(x-y)$$

for known functions $h_k(v; v_x)$, where $\eta^{ij}(v)$ and $g^{ij}(v)$ form a flat pencil of metrics given by

$$(\eta^{ij}(v)) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

and

$$(g^{ij}(v)) = \begin{pmatrix} 0 & -\frac{2}{9}v_3^2 & 0 & v_1 \\ -\frac{2}{9}v_3^2 & \frac{2}{9}v_3^2 - \frac{4}{9}v_3v_4 & v_1 & v_2 - v_1 \\ 0 & v_1 & 0 & \frac{2}{3}v_3 \\ v_1 & v_2 - v_1 & \frac{2}{3}v_3 & \frac{2}{3}(v_4 - v_3) \end{pmatrix}.$$

Observation: A direct calculation gives $(F(v), e, E)$ as follows

$$e = \frac{\partial}{\partial v_1}$$

and

$$E = v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2} + \frac{2}{3}v_3\frac{\partial}{\partial v_3} + \frac{2}{3}v_4\frac{\partial}{\partial v_4}$$

and

$$\begin{aligned} F(v) &= \frac{1}{2}v_1^2 v_4 + \frac{1}{2}v_1^2 v_3 + v_1 v_2 v_3 - \frac{1}{18}v_3^3 v_4 - \frac{1}{72}v_3^4 \\ &= \mathrm{tr}\,(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{V}_{\mathbf{0}}^{\mathbf{2}}\mathbf{V}_{\mathbf{1}} - \frac{\mathbf{1}}{\mathbf{72}}\mathbf{V}_{\mathbf{1}}^{\mathbf{4}}). \end{aligned}$$

Here $\mathrm{tr}\,(\mathbf{a_1 e_1} + \mathbf{a_2 e_2}) = \mathbf{a_1} + \mathbf{a_2}$.

Part D. Frobenius manifolds and Frobenius algebra-valued Integrable systems

Question:

$$\left\{\ \text{Frobenius manifold}\ \right\} \quad \longleftarrow\!\!-\,-\,-\!\!\longrightarrow \quad \left\{\ \text{Principal hierarchy}\ \right\}$$

$$\Big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow$$

$$\left\{\ ????\ \right\} \quad \longleftarrow\!\!-\,-\,-\!\!\longrightarrow \quad \left\{\ \mathcal{A} - \text{valued principal hierarchy}\ \right\}$$

Definition.[Lifting map] Let $\{\mathcal{A}, \circ, e, \text{tr}\}$ be a Frobenius algebra over $\mathbb{K}$ with the basis $e = e_1, e_2, \cdots, e_m$ and f an analytic function on $\mathcal{M}$ (that is, analytic in the flat coordinates for $\mathcal{M}$). The $\mathcal{A}$-valued function $\widehat{f}$ is defined to be:

$$\widehat{f} = f|_{t^\alpha \mapsto t^{(\alpha i)} e_i}$$

with $\widehat{fg} = \widehat{f} \circ \widehat{g}$ and $\widehat{1} = e_1$. The evaluation $f^{\mathcal{A}}$ of $\widehat{f}$ is defined by

$$f^{\mathcal{A}} = \text{tr}\left(\widehat{f}\right),$$

where tr $\in \mathcal{A}^\star$ is the Frobenius form.

Theorem.(**[Ian-Zuo 2017]**) Let

$$u_t^\alpha = K^\alpha(u, u_x, \cdots), \qquad u = \{u^\alpha(x,t)\} \tag{10}$$

be a Hamiltonian system with the Hamiltonian $H[u]$, then the corresponding $\mathcal{A}$-valued system

$$\widehat{u_t^\alpha} = K^\alpha(\widehat{u, u_x,} \cdots) \tag{11}$$

is also Hamiltonian with the Hamiltonian $\mathcal{H}[\hat{u}] = \text{tr}\left(\widehat{H[u]}\right)$.

Cor. The $\mathcal{A}$-KP hierarchy admits local bi-Hamiltonian structures.

Theorem. (**[Ian-Zuo 2017]**) Let F be the prepotential of a Frobenius manifold $\mathcal{M}$ and let $\mathcal{A}$ be a Frobenius algebra with 1-form tr . Then the function

$$\mathbf{F}^{\mathcal{A}} = \text{tr}\left(\widehat{\mathbf{F}}\right)$$

defines a Frobenius manifold, namely the manifold $\mathcal{M} \otimes \mathcal{A}$.

(The tensor product was due to Kaufmann, Kontsevich and Manin.)

Remark. This construction could be generalized to TQFT and F-manifold.

References:

[1]. Ian Strachan and D.Zuo, Integrability of the Frobenius algebra-valued Kadomtsev-Petviashvili hierarchy. **J.Math.Phys.** 56 (2015), no. 11, 113509, 13 pp.

[2]. Ian Strachan and D.Zuo, Frobenius manifolds and Frobenius algebra-valued integrable systems, **Lett.Math.Phys.** 107(2017)996-1027.

THANKS