

SERGEI ARTEMOV

On the provability of consistency

The City University of New York, Graduate Center, USA

E-mail: sartemov@gc.cuny.edu

We revisit the foundational question concerning Peano arithmetic PA:

(1) *can consistency of PA be established by means expressible in PA?*

The usual answer to (1) is “No, by Gödel’s Second Incompleteness Theorem.” In that theorem (G2), Gödel used an arithmetization of contentual mathematical reasoning and established that the arithmetical formula representing PA-consistency is not derivable in PA. Applying G2 to (1), one makes use of the formalization thesis (FT):

FT: *any proof by means expressible in PA admits Gödel’s arithmetization.*

Historically, there has been no consensus on FT; Gödel (1931) and Hilbert (1934) argued against an even weaker version of FT with respect to finitary proofs, whereas von Neumann accepted it.

Note that the aforementioned negative answer to (1) is unwarranted: here is a counter-example to FT. Let $Ind(F)$ denote the induction statement for an arithmetical formula F . The claim C , “for each formula F , $Ind(F)$,” is directly provable by means of PA: given any F , argue by induction to establish $Ind(F)$. However, C is not supported by any arithmetization as a single formula since PA is not finitely axiomatizable.

We provide a positive answer to (1). We offer a mathematical proof of PA-consistency,

no finite sequence of formulas is a PA-proof of $0 = 1$,

by means expressible in PA, namely, by partial truth definitions. Naturally, this proof does not admit Gödel’s arithmetization either.