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ABSTRACTS

Semen Abramyan. *Higher Whitehead products in moment-angle complexes and substitution of simplicial complexes*

(Joint work with Taras Panov [2])

Higher Whitehead products are important invariants of unstable homotopy type. They have been studied since the 1960s in the works of homotopy theorists such as Hardie [7], Porter [12] and Williams [13].

The appearance of moment-angle complexes and, more generally, polyhedral products in toric topology at the end of the 1990s brought a completely new perspective on higher homotopy invariants such as higher Whitehead products. The homotopy fibration of polyhedral products

$$(D^2, S^1)^{\mathscr{K}} \to (\mathbb{C}P^\infty)^{\mathscr{K}} \to (\mathbb{C}P^\infty)^m \quad (1)$$

was used as the universal model for studying iterated higher Whitehead products in [1]. Here $(D^2, S^1)^{\mathscr{K}} = \mathscr{Z}_{\mathscr{K}}$ is the moment-angle complex, and $(\mathbb{C}P^\infty)^{\mathscr{K}}$ is homotopy equivalent to the Davis–Januszkiewicz space [3,4]. The form of nested brackets in an iterated higher Whitehead product is reflected in the combinatorics of the simplicial complex $\mathscr{K}$.

There are two classes of simplicial complexes $\mathscr{K}$ for which the moment-angle complex is particularly nice. From the geometric point of view, it is interesting to consider complexes $\mathscr{K}$ for which $\mathscr{Z}_{\mathscr{K}}$ is a manifold. This happens, for example, when $\mathscr{K}$ is a simplicial subdivision of sphere or the boundary of a polytope. The resulting moment-angle manifolds $\mathscr{Z}_{\mathscr{K}}$ often have remarkable geometric properties [10]. On the other hand, from the homotopy-theoretic point of view, it is important to identify the class of simplicial complexes $\mathscr{K}$ for which the moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ is homotopy equivalent to a wedge of spheres. We denote this class by B_Δ. The spheres in the wedge are usually expressed in terms of iterated higher Whitehead products of the canonical 2-spheres in the polyhedral product $(\mathbb{C}P^\infty)^{\mathscr{K}}$. We denote by W_Δ the subclass in B_Δ consisting of those $\mathscr{K}$ for which $\mathscr{Z}_{\mathscr{K}}$ is a wedge of iterated higher Whitehead products. The question of describing the class W_Δ was studied in [11] and formulated explicitly in Problem 8.4.5 [4]. It follows from the results of [11] and [5] that $W_\Delta = B_\Delta$ if we restrict attention to *flag* simplicial complexes only, and a flag complex $\mathscr{K}$ belongs to W_Δ if and only if its one-skeleton is a chordal graph. Furthermore, it is known that W_Δ contains directed MF-complexes [6], shifted and totally fillable complexes [8,9]. On the other hand, it has been recently shown in [1] that the class W_Δ is *strictly* contained in B_Δ. There is also a related question of *realisability* of an iterated higher Whitehead product w with a given form of nested brackets: we say that a simplicial complex $\mathscr{K}$ *realises* an iterated higher Whitehead product w if w is a nontrivial element of $\pi_*(\mathscr{Z}_{\mathscr{K}})$. For example, the boundary of simplex $\mathscr{K} = \partial\Delta(1, \dots, m)$ realises a single (non-iterated) higher Whitehead product $[\mu_1, \dots, \mu_m]$, which maps $\mathscr{Z}_{\mathscr{K}} = S^{2m-1}$ into the fat wedge $(\mathbb{C}P^\infty)^{\mathscr{K}}$.

In this talk we will discuss combinatorial approach to the question of realsability of an iterated higher Whitehead product. Using the operation of substitution of simplicial complexes, for any iterated higher Whitehead product w we describe a simplicial complex $\partial\Delta_w$ that realises w. Furthermore, for a particular form of brackets inside w the complex $\partial\Delta_w$ is the smallest one realising w. We also give a combinatorial criterion for the nontriviality of the product w.

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Andrei Alpeev. *Decay of mutual information for unique Gibbs measures on trees*

In a recent paper by A. Backhausz, B. Gerencser and V. Harangi, it was shown that factors of independent identically distributed random processes on trees obey certain geometry-driven inequalities. In particular, the mutual information shared between two vertices decays exponentially, and there is an explicit bound for this decay. I will show that all of these inequalities could be verbatim translated to the setting of factors of processes driven by unique Gibbs measures. As a consequence, we show that correlations decay exponentially for unique Gibbs measures on trees.

Mikhail Anikushin. *Convergence and Strange Nonchaotic Attractors in Almost Periodic Systems*

Our attention will be focused on an extension of a result of V. V. Zhikov [4] concerning almost periodic ODEs on the plane. The latter result states that under positive Lyapunov stability of the system (=equicontinuity on compact sets of the solving operators) either all bounded solutions are almost periodic and their frequencies lie in some common enumerable set or there is only one almost periodic solution. We show how one can extend this result to higher dimensional systems via using some reduction principle originated by R. A. Smith [1, 3]. For the differential equations that can be written as a control system in Lure form the condition allowing to reduce dimensions can be effectively verified by using frequency-domain methods. In fact, within the conditions of Zhikovs theorem when the first case is realized it is possible to obtain the convergence of all bounded in the future solutions to some of almost periodic ones. Along with direct applications we hope that this kind of results will be useful for understanding of sensitive dependence on initial conditions in almost periodic systems with a strange nonchaotic attractor (SNA). It was proved in [2] that some classical SNAs have sensitive dependence on initial conditions, however the largest Lyapunov exponent is negative (that is often accepted as non-chaoticness). To show sensitive dependence the authors used some dichotomy for transitive maps.

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Anton Ayzenberg. *Orbit spaces of equivariantly formal torus actions*

(Based on joint works with M. Masuda and V. Cherepanov)

Assume that a compact torus T acts on a smooth closed manifold X, and the action has nonempty finite set of fixed points. We assume that the action is cohomologically equivariantly formal, which is equivalent, under the given assumption on fixed points, to the vanishing of odd degree cohomology of X. We say that the action is in j-general position, if, at each fixed point x, every j of the weights of the tangent representation T_xX are linearly independent. In my talk, I will discuss the following results.

(1) If an equivariantly formal action on X is in j-general position, then its orbit space is $(j+1)$-acyclic. This result generalizes several known results listed below. 1. The result of Masuda–Panov asserting that the orbit space of an equivariantly formal torus action of complexity zero is an acyclic space. 2. The result of Buchstaber–Terzic asserting that $G_{4,2}/T^3 \cong S^5$ and $F_3/T^2 \cong S^4$, where $G_{4,2}$ is the Grassmann manifold of complex 2-planes in $\mathbb{C}^4$ and F_3 is the manifold of complete complex flags in $\mathbb{C}^3$. 3. The general result of Karshon–Tolman asserting that the orbit space of a complexity one Hamiltonian action in general position is homeomorphic to a sphere. 4. The result of the author, asserting that $\mathbb{H}P^2/T^3 \cong S^5$ and $S^6/T^2 \cong S^4$, where $S^6 = G_2/\mathrm{SU}(3)$ is the almost complex sphere. Now we have a homological explanation, why the orbit spaces of many natural manifolds are homeomorphic to spheres.

(2) The $(j+1)$-acyclicity of the orbit space is the only homological constraint on the orbit space of equivariantly formal actions. For any finite polyhedron L there exists an equivariantly formal action in j-general position whose orbit space is homotopy equivalent to the $(j+2)$-fold suspension of L. Moreover, such example can be found in the class of Hamiltonian torus actions of complexity one.

(3) Finally, for torus actions of complexity one in general position, we present a criterion of equivariant formality in terms of the topology of the orbit space.

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Malkhaz Bakuradze. *All extensions of C_2 by $C_{2^n} \times C_{2^n}$ are good for the Morava K-theory*

This talk is concerned with analyzing the 2-primary Morava K-theory of the classifying spaces BG of the groups G in the title. In particular it answers affirmatively the question whether transfers of Euler classes of complex representations of subgroups of G suffice to generate $K(s)^*(BG)$. Here $K(s)$ denotes Morava K-theory at prime $p = 2$ and natural number $s > 1$. The coefficient ring $K(s)^*(pt)$ is the Laurent polynomial ring in one variable, $\mathbb{F}_2[v_s, v_s^{-1}]$, where $\mathbb{F}_2$ is the field of 2 elements and $deg(v_s) = -2(2^s - 1)$. Not all finite groups are good as it was originally conjectured in [2]. For an odd prime p a counterexample to the even degree was constructed in [3]. The problem to construct 2-primary counterexample remains open.

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Polina Baron. *The Measure of Maximal Entropy for Minimal Interval Exchange Transformations with Flips on 4 Intervals*

The main object of our research (see [3] for more details) is the set of all minimal *interval exchange transformations with flips* on 4 subintervals (4–fIETs). An n–fIET is a piecewise isometry of an interval to itself with a finite number of jump discontinuities such that this isometry reverses the orientation of at least one of the intervals of continuity (called *a flip*). The interval exchange transformations with flips generalise the notion of the *interval exchange transformations* (n–IETs). Moreover, in a way, the fIETs are in between the IETTs and the linear involutions, which can be viewed as a further generalization. Interval exchange transformations with flips are a great tool to study billiards with flips in polygons (see [10]), oriented measured foliations on nonorientable surfaces (see [5]), vector fields on nonorientable surfaces (see [6]), and triangle tiling billiards (see [7]).

Let us give a more thorough definition of minimal interval exchange transformations with flips.

Definition 1. Let $S = (0, 1)\backslash\{a_1, \ldots, a_{n-1}\} = \sqcup_{i=0}^{n-1} I_i$, where $I_i = (a_i, a_{i+1})$, $a_0 = 0$, $a_n = 1$, $a_i \leqslant a_{i+1}$ $\forall i$, be a disjoint union of $n \geqslant 2$ oriented intervals. A bijection $T : S \to S$ is called an interval exchange transformation on n subintervals, or an n–fIET for short, if it acts as a translation on every subinterval I_i. T is called an interval exchange transformation with flips on n subintervals, or an n–fIET for short, if it also changes the orientation of at least one I_i.

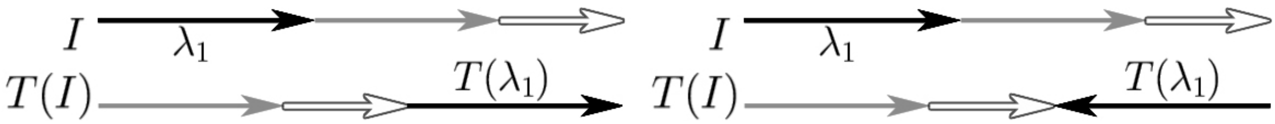

Figure 1. **Examples of IETs without (left) and with flips (right).**

Definition 2. A map $f : X \to X$ is called minimal if all its orbits are dense in X.

Several ergodic properties of linear involutions were uncovered by Boissy and Lanneau in [4]. The ergodic properties of IETs are well known and profoundly studied by many famous mathematicians, such as Yoccoz, Veech, Forni, Avila, and many more. It was proved that almost all irreducible IETs are minimal (see [8]) and, moreover, uniquely ergodic (see [9]). However, A. Nogueira proved that almost all fIETs are, in fact, not minimal (see [10]). In other words, the typical fIET has a periodic point. The ergodic properties of fIETs are in many ways unclear and yet to be studied.

Rauzy induction is a case of Euclidian type renormalization algorithm introduced for IETs in 1979 by Rauzy, and for fIETs in 1989 by Nogueira. Every n–fIET has a combinatorial type $\{\pi = (\pi_t, \pi_b), F = (F_t, F_b)\}$. Here π_t and π_b are permutations of intervals, and F_t, and F_b are sets of flipped intervals. Combinatorial types of irreducible fIETs are vertices of *Rauzy graph*. The oriented edges of this graph are given by the action of Rauzy induction. The connected components of the obtained graph are called *Rauzy diagrams*.

The parameter space σ for minimal 4–fIETs is a fractal (see [10] and [11]). We aim to study the invariant measures on σ. The measure of maximal entropy on σ is a projection via Abramov–Rokhlin formula (see [1]) of the measure of maximal entropy for the suspension flow associated with σ, and this suspension flow is a Teichmüller flow. We prove the following

Theorem 3. *In the case of interval exchange transformations with flips on 4 intervals, the Gibbs measure corresponding to the potential $\mu_0 = \varphi_{k_0} = -k_0 r$ is the unique measure of maximal entropy for the suspension flow.*

As a preparation to prove this theorem, we had to study the Rauzy graph for 4–fIETs and show that it contains only one, up to relabeling the intervals, minimal Rauzy diagram (that is, the component with minimal vertices). We proved that this Rauzy diagram is topologically transitive, and the topological Markov shift on this diagram is topologically mixing and satisfies big images and preimages property, which is an analogue to the recurrence of the Markov chain. The same result can be obtained experimentally with the aid of Paul Mercat's computer program.

Remark 4. It is notable that in the case of 6 intervals the computer program states that at least one of the Rauzy diagrams is not topologically transitive. Therefore, the topological Markov shift on this diagram is not topologically mixing and does not satisfy the big images and preimages property.

Our work is based upon the article [2] by Artur Avila, Pascal Hubert and Alexandra Skripchenko, who prove a very similar statement for a particular family of systems of isometries. To prove the main theorem, we exploit the results on fIETs obtained by Alexandra Skripchenko and Serge Troubetzkoy in [11], and the approach to thermodynamical formalism for countable Markov shifts developed by Omri Sarig.

This result is a first step towards exploring more advanced ergodic properties of minimal interval exchange transformations with flips. Currently, we are in the process of researching mixing for 4–fIETs. In collaboration with Alexandra Skripchenko, we aim to continue the research, obtain an analogue of the combinatorial criteria for minimal linear involutions (see [4]) in the case of the fIETs, and expand the results to higher numbers of intervals. We also hope to find geometrical interpretations of interval exchange transformations with flips.

Hypothesis 5. *For every minimal n–fIET there exists a translation surface corresponding to it.*

Acknowledgements. This research was done as a part of the author's undergraduate thesis. The author is deeply grateful to professor Alexandra Skripchenko for thoughtful guidance and supervision of the aforementioned thesis. The author also thanks Pascal Hubert, Paul Mercat and Olga Paris–Romaskevich for fruitful discussions and advice, and Paul Mercat especially for his wonderful computer program, mentioned earlier in the abstract.

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Yury Belousov. *A new algorithm of obtaining semimeander diagrams for knots*

A plane diagram of a knot is said to be semimeander if it is the union of two simple smooth arcs. Every knot has a semimeander diagram. We introduce a new algorithm of obtaining a semimeander diagram for a given knot. This algorithm is more efficient in comparison with known similar algorithms.

Introduction

Definition 1. A plane diagram of a knot is said to be *semimeander* if it is the union of two simple smooth arcs (an arc is said to be *simple* if it is non-self-intersecting).

Theorem 2. *Every knot has a semimeander diagram.*

This theorem has been independently discovered several times by different methods and in distinct terms (for detailed historical reference see [2]). Theorem 2 allows us to define a knot invariant: the semimeander crossing number.

Definition 3. Recall that the *crossing number* $\mathrm{cr}(K)$ of a knot K is the smallest number of crossings in any diagram of K. The *semimeander crossing number* $\mathrm{cr}_2(K)$ of K is the smallest number of crossings in any semimeander diagram of K.

The semimeander crossing number was introduced in [1], where the following estimate was proved:

Theorem 4. *For each knot K, the following inequality is fulfilled:*

$$\mathrm{cr}_2(K) \leqslant \sqrt[4]{6}^{\,\mathrm{cr}(K)}.$$

The key ingredient of the proof is an algorithm transforming a minimal diagram of a given knot K to a semimeander diagram with at most $\sqrt[4]{6}^{\,\mathrm{cr}(K)}$ crossings. This algorithm is based on two geometric transformations of diagrams, which could be seen in terms of the Gauss code. As it was recently understood, these transformations are special cases of a more general procedure. This procedure underlies the new algorithm, to the description of which we now pass.

The description of the algorithm

Suppose we are given a diagram D of a knot K and a simple arc J in D such that no endpoint of J is a crossing of D (the interior of J is allowed to contain crossings of D). A crossing x of D is said to be *reducible* (with respect to J) if there exists a simple curve γ with endpoints at a_i and b_i for some $i \in \{1, 2\}$ (see Fig. 1) such that the interior of γ intersects D only transversally and all intersections between D and γ lie on J. In this case, γ is called a *reduction curve* for x. The *reduction cost* for a reducible crossing x is the minimum number of intersection points between γ and J among all reduction curves γ for x.

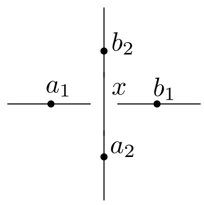

Figure 1. A neighborhood of a crossing

Now, for a given diagram D of a knot K we can obtain a semimeander diagram using the following steps.

(1) Choose a simple arc J in D such that no endpoint of J is a crossing of D.
(2) If J contains all crossings of D then D is semimeander. Otherwise, we find a reducible crossing x with the smallest reduction cost (say, p).
(3) Choose any reduction curve γ for x such that the number of intersections between γ and J is equal to p.
(4) Replace the arc of D connecting the endpoints of γ and containing x with γ (over/undercrossings in the new crossings should be set in the obvious way).
(5) Repeat steps 2–4 until our diagram becomes semimeander.

The proof that the described algorithm is correct is straightforward. Firstly, notice that step 4 reduces the number of crossings that do not lie on J. Secondly, if J does not contain all crossings of D, there exists at least one reducible crossing (this is so because two procedures introduced in [1] are special cases of this procedure and any of them can be applied to every non-semimeander diagram).

Remark 5. The algorithm can be rewritten in terms of the dual graph of a knot diagram. This allows us to easily find reduction costs for all crossings and also to find corresponding reduction curves (for example, using Dijkstra's algorithm).

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Mohan Bhupal. *Open book decompositions of links of minimally elliptic singularities*

I will discuss some results that allow one to obtain an explicit Milnor open book decomposition supporting the canonical contact structure on the link of any minimally elliptic singularity whose fundamental cycle Z satisfies $-3 \leqslant Z \cdot Z \leqslant -1$.

Pavel Bibikov. *Theory of differential invariants in algebraic classification problems*

(The results of this work were obtained in collaboration with V. V. Lychagin)

In the present work we suggest a new approach to study classification problems in classical invariant theory. We show the main ingredients of this approach on the following example. Consider the space of binary forms of degree n on the complex field $\mathbb{C}$. The group $\mathrm{GL}_2(\mathbb{C})$ acts on this space in the following way: the semi-simple part $\mathrm{SL}_2(\mathbb{C})$ acts by linear coordinate transformations, and center $\mathbb{C}^*$ acts by homotheties. The question is, when are two given binary forms equivalent with respect to this action?

This problem was set by Bool in 1841, and this was the debut of the classical invariant theory. A lot of great mathematicians such as Cayley, Hermite, Eisenstein, Gordan, Shioda, Hilbert, Noether, etc. took part in the solution of this problem for some small degrees n, and in 2010 this problem was solved for $n \leqslant 10$ and $n = 12$. Moreover, Popov proved that the number of polynomial invariants for this problem grows exponentially on n, so it is impossible to describe these invariants in the general situation.

Our approach uses not algebraic but differential-geometric methods. Namely, we consider binary forms as solutions of the Euler differential equation $xf_x + yf_y = nf$ and study the action of the group $\mathbf{GL}_2(\mathbb{C})$ on this differential equation. The invariants of such action are called *differential invariants*, and it is possible to describe the whole algebra of these invariants.

Theorem 1. *The algebra of differential invariants for the action of the group* $\mathrm{GL}_2(\mathbb{C})$ *on the Euler differential equation is freely generated by differential invariant* $H = \dfrac{u_{xx}u_{yy} - u_{xy}^2}{u^2}$ *and invariant derivation* $\nabla = \dfrac{u_y}{u}\dfrac{d}{dx} - \dfrac{u_x}{u}\dfrac{d}{dy}$.

Using this theorem it is possible to obtain an equivalence criterion for two binary forms. Namely, for a given non-zero binary forms f consider three differential invariants $I_1 := H$, $I_2 := \nabla H$ and $I_3 := \nabla^2 H$ and their restrictions $I_1(f)$, $I_2(f)$ and $I_3(f)$, which are just homogeneous fractions in two variables. Hence, there exists a unique (up to a multiplication on a constant) polynomial $\mathscr{F}_f$ of a lowest degree such that $\mathscr{F}_f(I_1(f), I_2(f), I_3(f)) \equiv 0$. This polynomial is called *the dependence polynomial of a binary form f.*

Theorem 2. *Two non-zero binary forms f_1 and f_2 of the same degree are* $\mathrm{GL}_2(\mathbb{C})$*-equivalent iff* $\mathscr{F}_{f_1} = \mathscr{F}_{f_2}$.

This result shows that it is appropriate to use the theory of differential invariants in different algebraic problems. We also show some generalizations of this idea. Namely, we consider the following problems.

1. What will happen, if we consider not complex but real binary forms?

2. What will happen, if we consider not binary but ternary forms (i.e. forms in three variables)? Or p-forms for arbitrary p?

3. What will happen, if we consider the linear action of another classical algebraic group (Sp, So, SL, etc.)?

4. What will happen, if we consider an arbitrary representation $\rho\colon G \to \mathrm{GL}(V)$ of a semi-simple group G in vector space G?

It appears that there exist exact answers to all these questions. We will briefly show the ideas for the last one.

Let G be a connected semi-simple complex Lie group, and let

$$\rho_\lambda\colon G \to \mathrm{GL}(V)$$

be its irreducible representation with highest weight λ.

First, let us fix a Borel subgroup B in group G and consider homogeneous complex flag manifold $M := G/B$.

Secondly, consider the action $B : G$ of Borel group B on G by the right shifts:

$$g \mapsto gb^{-1},$$

where $g \in G$ and $b \in B$.

Finally, let us define the bundle product $E := G \times_B \mathbb{C} = G \times \mathbb{C}/\sim$, where the equivalence relation $\sim$ is defined by the following:

$$(g, c) \sim (gb^{-1}, \chi_\lambda(b)c),$$

and where $\chi_\lambda \in \mathfrak{X}(T)$ is the character corresponding to the highest weight λ of the maximal torus $T \subset B$.

We introduce one–dimensional bundle

$$\pi^\lambda \colon E \to M, \qquad \pi^\lambda(g, c) = gB.$$

Holomorphic sections of this bundle are just holomorphic functions $f \colon G \to \mathbb{C}$, which satisfy the relation

$$f(gb) = \chi_\lambda(b) f(g),$$

for all $g \in G$ and $b \in B$.

Group G acts in bundle π^λ by left shifts. This action prolongs to the action on the space of holomorphic sections of bundle π^λ:

$$g(f)(g') = f(g^{-1}g').$$

According to Borel–Weil–Bott theorem, if λ is dominant weight of group G, then this action is isomorphic to representation ρ_λ.

Therefore, the study of orbits of irreducible representations of semisimple complex Lie groups with the highest weight λ is equivalent to the study of the orbits of these actions on the space of holomorphic sections of bundle π^λ.

Let us illustrate this idea in case $G = \mathrm{SL}_2(\mathbb{C})$. It is known, that dominant weights of group $\mathrm{SL}_2(\mathbb{C})$ equal $\lambda = \frac{n}{2}\alpha$, where α is the positive root of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and $n \geqslant 0$ is a non–negative integer. The Borel group $B = \mathrm{B}_2(\mathbb{C})$ consists of upper–triangular matrices, and character χ_λ acts on it in the following way:

$$\chi_\lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^n.$$

Then

$$M = \mathrm{SL}_2/\mathrm{B}_2 \simeq \mathbb{C}P^1.$$

If we denote the homogeneous coordinates on M by $(x : y)$, then holomorphic sections of bundle π^λ are just homogeneous polynomials of degree n in variables x and y.

Thus, the study of invariants of representations of group $\mathrm{SL}_2(\mathbb{C})$ is reduced to the classification of $\mathrm{SL}_2(\mathbb{C})$-orbits of binary forms.

Finally, we note that our results make it possible to separate the regular G-orbits for the algebraic action of the arbitrary reductive group G on algebraic manifold Ω, if there exists at least one regular G-orbit in Ω.

Indeed, according to the Sumihiro linearization theorem every affine algebraic manifold Ω with an algebraic action of algebraic group G allows embedding to a vector space V with a linear G-action as G-invariant submanifold. If this manifold contains regular G-orbit, then the restriction of G-invariants of V on Ω generate G-invariants of Ω.

Hence, our results can be applied to the calculation of invariants (differential and algebraic) and separation of the orbits for the action of connected semi-simple algebraic group G on an affine algebraic manifold Ω.

Michael Blank. *Topological properties of measurable semigroup actions*

We discuss two questions from this subject: recurrence property and shadowing of pseudo-trajectories. The first of them is based on the representation of trajectories of the semigroup as realizations of a certain Markov chain for which necessary and sufficient conditions for the recurrence were obtained recently in [1].

To this end it is worth to emphasize that a direct generalization of the recurrence notion does not work well in the case of the semigroup action and one needs to make proper corrections which will be discussed in detail. Examples of free semigroups such that each generator is recurrent a.e., but the corresponding semigroup has no strictly recurrent points; as well the opposite case when each generator has no recurrent points, but the semigroup is strictly recurrent everywhere, will be presented.

Remark that in the absence of continuity of the dynamical system under study the only classical result about the recurrence is the celebrated Poincare lemma, which claims that for any measurable set almost all points (with respect to a dynamically invariant measure) return to this set eventually. Even in the case of a single generator the support of the dynamically invariant measure may be very small or (in some exceptional cases) even in the compact phase space there are no invariant measures. Such exceptional situations are becoming typical for free semigroups having at least two measurable generators. Therefore technically the first step is to find a replacement for the Poincare lemma. We find necessary and sufficient conditions for this replacement.

To study the shadowing property one again needs to make some modifications to the standard definitions. We study both the cases of uniformly small perturbations and perturbations which are small only on average (like Gaussian noise). In both cases we find conditions under which free semigroups with Anosov type generators demonstrate both the (on average) shadowing and its absence. In a sense we have a kind of phase transition here because both types of behavior are observed in a one-parameter family of semigroups. Technically these results are based on a new way of proving the so called "direct product structure" property for Anosov type maps, especially in the non-local case. These constructions are new even in the case of a single map and allows to study non-invertible maps having local hyperbolic properties.

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DOI:10.17323/1609-4514-2019-19-1-37-50

Shimon Brooks. *Quantum chaos and rotations of the sphere*

(Based on joint work with E. Lindenstrauss and E. Le Masson)

We consider the round sphere $\mathbb{S}^2$, and study joint eigenfunctions of the Laplacian and an averaging operator over generators of a free subgroup satisfying mild non-degeneracy conditions. We show several ways in which these eigenfunctions are spread out, and do not concentrate too much in small sets, in contrast to eigenfunctions of the Laplacian alone, which can exhibit such localization.

The first result is a form of Quantum Ergodicity, which says that most joint eigenfunctions tend to equidistribute on the sphere. The second result gives non-trivial bounds on the L^p norms of an eigenfunction, analogous to the Hassell-Tacy bounds for Laplace eigenfunctions on a manifold of negative curvature. In both cases, the underlying technology is estimates for wave propagation on large graphs that do not have too many short cycles.

Victor Buchstaber. *Vladimir Abramovich Rokhlin and algebraic topology*

The book [1] incorporates 12 papers on algebraic topology written by Vladimir Abramovich Rokhlin. Add to them the survey [2], one of the first in the world surveys on bordism groups, and the paper [3], where the joint results of S. P. Novikov and V. A. Rokhlin are presented.

In connection with the title of the survey [2], we note that the well known terms "bordism" and "cobordism" appeared only in [4]. The "inner homology" groups in terminology of the survey [2] are actually the scalar groups of bordisms and cobordisms theories. In [4], references to the papers of V. A. Rokhlin are given, and the crucial notion of "exact Rokhlin sequence" is introduced in bordisms theory.

Topics of V. A. Rokhlin's papers include:

- The 2–torsions in bordism groups of oriented manifolds;
- The mappings of the $(n+3)$-dimensional sphere into the n-dimensional sphere;
- The embeddings of 3-dimensional manifolds in $\mathbb{R}^5$;
- The necessary and sufficient condition for an orientable closed manifold M^4 to be the boundary of the oriented manifold W^5;
- The formula $3\tau(M^4) = \langle P_1(M^4), [M^4]\rangle$, connecting the signature of a manifold M^4 with its first Pontryagin class;
- The divisibility by 16 of signature for any closed almost parallelizable manifold M^4;
- The external and the internal definitions of the characteristic Pontryagin cycles and the characteristic Pontryagin classes;
- The combinatorial invariance of the rational Pontryagin classes;
- Problems of homotopical and topological invariance of the rational Pontryagin classes;
- The integer Pontryagin classes and smoothing problems of combinatorial manifolds;
- Problems of realization of 2-dimensional cycles in 4-dimensional manifolds;

- The additivity of the signature with respect to the connected sum of manifolds M_1^{4k} and M_2^{4k} along the common connected component of their boundaries.

These papers were published in the period of the explosive development of mathematics (1951 – 1971). During these years, many fundamental problems of algebraic topology were solved. It became possible due to the revealment of the implicit connections of algebraic topology with real and complex algebraic geometries, functional analysis, the theory of differential equations, commutative and homological algebra. Moreover, those connections opened new areas of research in mathematics and physics. Rokhlin's papers significantly contributed to the success of algebraic topology (see [5] – [8]).

Many papers of V. A. Rokhlin were devoted to the theory of 4-dimension manifolds. Over the years, it became clear that this theory is fundamentally different from the theory of manifolds of other dimensions. The role of the results of V. A. Rokhlin in creating the rich and "wild world" of 4-dimension manifolds is amply presented in [9].

The talk will focus on the role of V. A. Rokhlin's results in the development of algebraic topology right up to the present moment. We will discuss the results of the papers [10] – [16], which consider the problems directly connected with results of V. A. Rokhlin, as well as new problems, the statements of which are closely related to the ideas of V. A. Rokhlin.

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Dmitri Burago. *Examples of exponentially many collisions in a hard ball system*

(Based on joint work with S. Ivanov)

20 years ago the topic of my talk at the ICM was a solution of a conjecture which goes back to Boltzmann and Ya. Sinai. It states that the number of collisions in a system of n identical balls colliding elastically in empty space is uniformly bounded for all initial positions and velocities of the balls. The answer is affirmative and the proven upper bound is (poly) exponential in n. Little was known about how many collisions can actually occur. In R^1, $n(n-1)/2$ is a realizable maximum. The only non-trivial (and counter-intuitive) example in higher dimensions I am aware of is an observation by Thurston and Sandri who gave an example of 4 collisions between 3 balls in R^2. Recently, Sergei Ivanov and me proved that there are examples with exponentially many collisions between n identical balls in R^3, even though the exponents in the lower and upper bounds do not

match. The example is not very explicit, we just prove its existence. A few related problems around entropy and other dynamical invariants will also be discussed.

Sergey Burian. *Dynamics of mechanisms near singular points*

Singular geometry theories could be applied to the studying of the motion of mechanisms with configuration space singularities. It is motivated by the following way: in the case when the configuration space X of mechanical system is a the smooth manifold, the motion equations in Lagrangian or Hamiltonian form could be interpretated as a vector fields on (co)tangent bundles of X. In general case, we could construct the "vector fields" on the "(co)tangent" bundle of the space X, which is not a smooth manifold.

The formulation of classical differential geometry constructions in the terms of observation functions algebra could be founded in the book [1]. Also there are point-based singular geometry methods which suppose that a singular space X has the topological structure: diffeology, differential spaces, Frölicher spaces. The overview and comparison of this theories is presented in the work [2].

In the previous works, the case of one-dimensional singularities was studied [3][4]. This means that the configuration space of the mechanism consists of some smooth curves with (locally) unique common point. Some problems of the using of the Frölicher space structure in the motion description are analyzed in [5]. In the current report we study the case of two-dimensional singularities.

The construction of one-dimensional singular pendulum is the following. We consider the planar mechanism (in the fixed plane Π) of double pendulum with rods AB with length l_1 and BC with length l_2. Suppose $l_1 > l_2$. The point A is fixed. Then we impose additional constraint to the displacement of the point C. The vertex C must lie on the fixed curve γ. In general points, configuration space of the planar singular pendulum in one-dimensional. The singular configurations of this mechanism could arise when two rods AB and BC become co-directed. If the distance function $d = |AC|$ has the local minimum or maximum at these configurations, then the mechanism could continue the motion in two different ways after coming across the singular configuration: by the curve c_1 or by the curve c_2. Curves c_1 and c_2 could have transversal intersection at common point s or they could have tangency singularity.

We could consider the tangent space at singular point s of the configuration space like as the classes of smooth curves in X. Then the space of directions is the union of two lines for the traversal intersection case and is one-dimensional linear space for the tangent case. There are difficulties for the construction of the choosing "smooth" coordinates, such as a map $h : U \to X$, where U is the subset of some Euclidean space.

Now we consider some modification of the planar singular pendulum. Assume that the plain Π contains vertical axes Oz in the (physical) space $\mathbb{R}^3$. The location of the plane Π in the space $\mathbb{R}^3$ is fixed. But the rods AB and BC move in the space. The motion of the vertex C is planar: C moves along the curve $\gamma \subset \Pi \subset \mathbb{R}^3$. The construction is presented in the Fig.1.

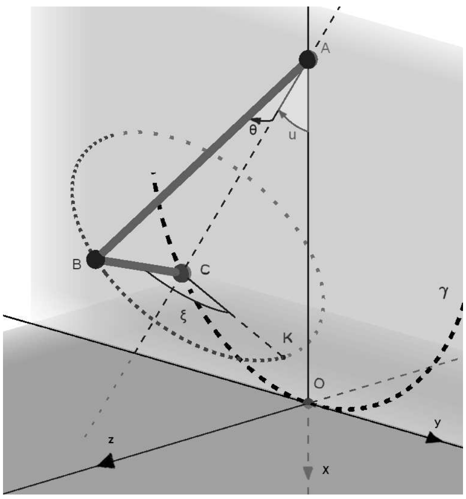

FIGURE 1. Space singular pendulum.

Consider the angles for of the space singular pendulum. Let u be the angle between the point C in the plane Π and axes Oz (or the line AO). Denote θ the angle between the lines AB and AC, and let ξ be the rotation angle of point B relative to the plane Π. We consider the case of oriented angles u, θ and ξ with the anti-clockwise orientation. The system of the space singular pendulum has two degrees of freedom in a general point.

Suppose that the curve γ could be parametrized by the angle u: then we have $\gamma = \gamma(u)$. For a given value of u, the location of the vertex C is unique defined. The coordinates of point B could be derived from the triangular ABC. They are given by the rotation angle ξ of triangular ABC. The couple (u, ξ) describes the geometric configuration of the space singular pendulum.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points B and C. We find the dependance of this coordinates on the variables u, θ and ξ. The coordinates of the point B could be founded in the following way. We start from the point $B' = (l_1 \cos\theta, 0, l+1 \sin\theta)$ on the plane Π. Then we rotate the point B' along the Oz-axis on the angle ξ and get B''. Finally, we rotate C'' for the angle u in the plane Π with the center A. To sum up, we get the following parametrization of the configuration space $X \subset \mathbb{R}^3$ of the space singular pendulum:

$$
\begin{aligned}
x_1^3 &= l_1 \cos\theta \cos u + \cos\xi (l_1 \sin u \sin\theta) - |AO|;\\
y_1^3 &= l_1 \cos\theta \sin u - \cos\xi (l_1 \cos u \sin\theta);\\
z_1^3 &= \sin\xi (l_1 \sin\theta);\\
x_2 &= d(u) \cos u;\\
y_2 &= d(u) \sin u;\\
z_2 &= 0.
\end{aligned}
\tag{1}
$$

For any ξ the coordinates $(0, 0, \xi)$ map to the one singular point s. This point is equal to the singular point of the planar singular pendulum. Consider the velocity vector of the system in the case $u \to 0$, $\theta \to 0$:

$$(\dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{x}_2, \dot{y}_2, \dot{z}_2)|_{(u=0,\theta=0)} = (0, l_1 - l_1\theta' \cos\xi, l_1\theta' \sin\xi, 0, d(0), 0)\dot{u}.$$

This means that the velocity vector of the point B is dependent on the limit value of ξ. In general case, the space X has *conical singularity* at singular point s. In the case of $\theta'(u) = 0$ we get that the velocity vectors of space singular pendulum are the linear shell of the vector

$$(0, l_1, 0, 0, d(0), 0).$$

In this case, the space X has *cuspidal singularity* at point s.

The angle θ could be founded from the triangular ABC:

$$\theta(u) = +\arccos\left(\frac{l_1^2 + d(u)^2 - l_2^2}{2l_1 d(u)}\right).$$

In order to derive the derivation of θ let us compute $\sin^2\theta$. From the geometric point of view, $\sin\theta = 0$ if the rods AB and BC are co-directed. In this case, two roots of the equation $\sin\theta = 0$ are $d_1 = l_1 + l_2$ and $d_2 = l_1 - l_2$. After some calculation we get:

$$\sin^2\theta = \frac{((l_1 + l_2) - d) \cdot (d - (l_1 - l_2)) \cdot (d - (-l_1 - l_2)) \cdot (d - (-l_1 - l_2))}{4l_1^2 d^2}.$$

Consider the curve γ with the properties: $d'(0) = d''(0) = d'''(0) = 0$, but $d^{(4)} \neq 0$. By the Hadamard lemma,

$$d(u) = (l_1 + l_2) - u^4 g(u),$$

so that $g(u)$ is a smooth function and the value of g at point $u = 0$ is positive. Then we can select a "singular" factor in $\theta'(u)$:

$$\theta'(u) = F(u) \cdot \frac{d'(u)}{\sqrt{(l_1 + l_2) - d(u)}} = F(u) \cdot \frac{u^3(4g(u) + ug'(u))}{u^2\sqrt{g(u)}}, \tag{2}$$

where $F(u)$ is smooth function, $F(0) \neq 0$. Therefore $\theta(u)$ is the smooth function too.

As a consequence, we could fix one smooth geometric branch θ (2) of the space singular pendulum. Then in the formulas (1) coordinates functions

$$\mathbf{r} = (x_1, y_1, z_1, x_2, y_2, z_2)$$

are smooth functions of the parameters (u, ξ).

PROPOSITION 1. *The coordinates (u, ξ) give us the smooth parametrization of the configuration space X of space singular pendulum.*

Suppose that points B and C are massive but rods AB and BC are mass-less. The dynamics of space singular pendulum could be derived by Lagrangian or Hamiltonian approach. The kinetic energy T is the smooth function as a square of $\dot{\mathbf{r}}$. Potential energy V is the smooth function of $\mathbf{r}$. has the smooth coefficients. This mean that the Lagrange equations for Lagrangian function $L = T + V$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

is the smooth vector field on the tangent bundle of the (u, ξ)-plane.

We could generalize this example of the space singular pendulum in the following way. Consider the space X in the arbitrary Euclidean space $\mathbb{R}^n$ with a unique conical (or cuspidal) singular point s. Suppose that we have the surjective map $h : U \to X$. Let $U = (x_1, \bar{x})$ and for all z the image $h(z, 0) = s$. The problem is to define vector field on the singular space X which is corresponded to the vector field on U. The motion in the U is smooth and thus the motion in $\mathbb{R}^6$ is smooth.

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Rezki Chemlal. *Measurable factors of one dimensional cellular automata*

We are interested in ergodic properties of one dimensional cellular automata. We show that an ergodic cellular automaton cannot have irrational eigenvalues. We show also that a cellular automaton with almost equicontinuous points according to Gilman's classification has an equicontinuous measurable factor.

Introduction

Let A be a finite set; a word is a sequence of elements of A. The length of a finite word $u = u_0 \ldots u_{n-1} \in A^n$ is $|u| = n$. We denote by $A^{\mathbb{Z}}$ the set of bi-infinite sequences over A. A point $x \in A^{\mathbb{Z}}$ is called a configuration. For two integers i, j with $i < j$ we denote by $x(i, j)$ the word $x_i \ldots x_j$.

For any word u we define the cylinder $[u]_l = \{x \in A^{\mathbb{Z}} : x(l, l + |u|) = u\}$ where the word u is at the position l. The cylinder $[u]_0$ is simply noted $[u]$. The cylinders are clopen (closed open) sets.

Endowed with the distance $d(x, y) = 2^{-n}$ with $n = \min\{i \geqslant 0 : x_i \neq y_i \,\text{or}\, x_{-i} \neq y_{-i}\}$, the set $A^{\mathbb{Z}}$ is a topological compact separated space.

The shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined as $\sigma(x)_i = x_{i+1}$, for any $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$. The shift map is a continuous and bijective function on $A^{\mathbb{Z}}$. The dynamical system $(A^{\mathbb{Z}}, \sigma)$ is commonly called *full shift.*

A cellular automaton is a continuous map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ commuting with the shift. By the Curtis-Hedlund-Lyndon, theorem [2] for every cellular automaton F there exist an integer r and a block map f from A^{2r+1} to A such that $F(x)_i = f(x_{i-r}, \ldots, x_i, \ldots x_{i+r})$. The integer r is called the radius of the cellular automaton.

Endowed with the sigma-algebra on $A^{\mathbb{Z}}$ generated by all cylinder sets and μ the uniform measure which gives the same probability to every letter of the alphabet, $(A^{\mathbb{Z}}, \mathbb{B}, F, \mu)$ is a measurable space. The uniform measure is invariant if and only if the cellular automaton is surjective [2].

A cellular automaton $(A^{\mathbb{Z}}, \mathbb{B}, F, \mu)$ is ergodic if there is no $F-$invariant subset of positive measure. It is said weakly mixing if $F \times F$ is ergodic. A cellular automaton $(A^{\mathbb{Z}}, \mathbb{B}, F, \mu)$ is mixing if , for any measurable $U, V \subset A^{\mathbb{Z}}$ we have $\lim\limits_{n \to \infty} \mu(U \cap F^{-n}(V)) = \mu(U)\mu(V)$.

A cellular automaton $(B^{\mathbb{Z}}, G)$ is a measurable factor of $(A^{\mathbb{Z}}, F)$, if there exists a surjective measurable map π from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ such that $\pi \circ f = g \circ \pi$.

We denote by L^2_μ the set of measurable functions $g : A^{\mathbb{Z}} \to \mathbb{C}$ for which $\|f\|_2 = \left(\int_{A^{\mathbb{Z}}} |g|^2 \, d\mu\right)^{\frac{1}{2}}$ is finite.

Let $(A^{\mathbb{Z}}, \mathbb{B}, F, \mu)$ be a cellular automaton where μ is an invariant measure. We say that the function $g \in L^2_\mu$ is a measurable eigenfunction associated to the measurable eigenvalue $\lambda \in \mathbb{C}$ if $g \circ F = \lambda F$ a.e.

By definition any eigenvalue must be an element of the unit circle. As any eigenvalue can be written in the form $\exp(2I\pi\alpha)$; we will say that an eigenvalue is rational if $\alpha \in \mathbb{Q}$ and irrational otherwise.

A cellular automaton is ergodic iff any eigenfunction is of constant module and weakly mixing iff it admits 1 as unique eigenvalue and that all eigenfunctions are constant.

Gilman's classification

Gilman [3,4] introduced a classification using Bernoulli measures which are not necessarily invariant.

Definition 1. Let F be a cellular automaton and $[i_1, i_2]$ a finite interval of $\mathbb{Z}$. For $x \in A^{\mathbb{Z}}$. We define $B_{[i_1,i_2]}(x)$ by:

$$B_{[i_1,i_2]}(x) = \{y \in A^{\mathbb{Z}}, \forall j : F^j(x)(i_1, i_2) = F^j(y)(i_1, i_2)\}.$$

For any interval $[i_1, i_2]$ the relation $\mathfrak{R}$ defined by $x\mathfrak{R}y$ if and only if $\forall j : F^j(x)(i_1, i_2) = F^j(y)(i_1, i_2)$ is an equivalence relation and the sets $B_{[i_1,i_2]}(x)$ are the equivalence classes.

Definition 2. Let (F, μ) a cellular automaton equipped with a shift ergodic measure μ, a point x is $\mu-$equicontinuous if for any $m > 0$ we have:

$$\lim_{n\to\infty} \frac{\mu\left([x(-n,n)] \cap B_{[-m,m]}(x)\right)}{\mu([x(-n,n)])} = 1.$$

We say that F is $\mu-$almost expansive if there exist $m > 0$ such that for all $x \in A^{\mathbb{Z}} : \mu\left(B_{[-m,m]}(x)\right) = 0$.

Definition 3. Let (F, μ) denote a cellular automaton equipped with a shift ergodic measure μ. Define classes of cellular automata as follows :
1- $(F, \mu) \in \mathscr{A}$ if F is equicontinuous at some $x \in A^{\mathbb{Z}}$.
2- $(F, \mu) \in \mathscr{B}$ if F is $\mu-$almost equicontinuous at some $x \in A^{\mathbb{Z}}$ but $F \notin \mathscr{A}$.
3- $(F, \mu) \in \mathscr{C}$ if F is $\mu-$almost expansive.

Statement of results

In the following (F, ν) will denote a surjective cellular automaton equipped with the uniform measure.

Lemma 4. *Let (F, ν) be a surjective cellular automaton of radius r with $\nu-$equicontinuous points. Then the set $\sigma^{-p}B_{[-r,r]}(x) \cap \left(B_{[-r,r]}(x)\right)$ is of positive measure for every integer p. Moreover for each point $y \in \sigma^{-p}B_{[-r,r]}(x) \cap \left(B_{[-r,r]}(x)\right)$ the sequence $F^k(y)(-r, r)$ is eventually periodic.*

Proposition 5. *Let (F, ν) be a surjective cellular automaton with $\nu-$ equicontinuous points then F have a measurable equicontinuous factor which is a cellular automaton.*

Proof. Let r be the radius of the cellular automaton and let x be a ν-equicontinuous point. Then we have $\nu\left(B_{[-r,r]}(x)\right) > 0$. Using Lemma 4 for every $y \in \sigma^{-p}B_{[-r,r]}(x) \cap \left(B_{[-r,r]}(x)\right)$ the sequence $F^k(y)(-r, r)$ is eventually periodic.

As the number of words of length $2r + 1$ is finite, so there exists a common period p and preperiod p_0 for all points of $\sigma^{-p}B_{[-r,r]}(x) \cap \left(B_{[-r,r]}(x)\right)$.

For some $y \in \sigma^{-p}B_{[-r,r]}(x) \cap \left(B_{[-r,r]}(x)\right)$ consider the finite set

$$\mathbb{P} = \left\{F^k(y)(-r, r) : p_0 \leqslant k \leqslant p_0 + p - 1\right\} = \{p_k : p_0 \leqslant k \leqslant p_0 + p - 1\}.$$

Let us define the measurable sets

$$\widetilde{W}_k = F^{-1}\{p_k\}, : p_0 \leqslant k \leqslant p_0 + p - 1.$$

Consider the alphabet $A = (\mathbb{Z}/(p + 1)\mathbb{Z})$ and let the function π be defined by:

$$\forall x \in A^{\mathbb{Z}} : \begin{cases} \pi(x)_i = (x_i + 1) \bmod p : x \in \widetilde{W}_k : k_0 \leqslant k \leqslant k_0 + p. \\ p : otherwise. \end{cases}$$

The function π is measurable and it is associated to the equicontinuous cellular automaton defined by:

$$\forall x \in (\mathbb{Z}/(p + 1)\mathbb{Z})^{\mathbb{Z}} : C(x)_i = \begin{cases} (x_i + 1) \bmod p \text{ if } x \neq p; \\ p \text{ if } x = p. \end{cases}$$ □

Proposition 6. *Let $\left(A^{\mathbb{Z}}, \mathbb{B}, F, \nu\right)$ an ergodic cellular automaton, then F cannot have any irrational eigenvalue.*

Proof. Suppose there is a subset G of $A^{\mathbb{Z}}$ such that $\nu(G) = 1$ and g is the eigenfunction associated to $e^{2i\pi\alpha}$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. As ν is ergodic the eigenfunction g is of constant module denoted by $r = |g(x)|$.

The function $h = \dfrac{g \circ \sigma^{-n}}{g}$ is an invariant function for F and by ergodicity it is constant.

Consider the family of sets $G_n(\varepsilon)$ defined by:

$$G_n(\varepsilon) = \{x \in G : \forall y \in G : x(-n, n) = y(-n, n) \Rightarrow |g(x) - g(y)| < \varepsilon\}.$$

As α is irrational for every δ small enough there exist an integer p such that: $B_\delta\left(e^{2i\pi p\alpha}\right) \cap B_\delta\left(e^{2i\pi(-p)\alpha}\right) = \varnothing$. Let us consider two words w_1, w_2 satisfying:

$$\begin{cases} \forall x \in [w_1] \cap G_n(\varepsilon) : g(x) \in B_\delta\left(re^{2i\pi(p+q)\alpha}\right). \\ \forall x \in [w_2] \cap G_n(\varepsilon) : g(x) \in B_\delta\left(re^{2i\pi(q)\alpha}\right). \end{cases}$$

Let be the words w_1uw_2 and w_1uw_1 with $|u| = n$; we have then:

$$\begin{cases} \forall x \in [w_1uw_2] \cap G_n(\varepsilon) : g(x) \in B_\delta\left(re^{2i\pi(p+q)\alpha}\right). \\ \forall x \in [w_2uw_1] \cap G_n(\varepsilon) : g(x) \in B_\delta\left(re^{2i\pi q\alpha}\right). \end{cases}$$

From an other side we have:

$$\begin{cases} \forall x \in [w_1 u w_2] \cap G_n(\varepsilon) : g\left(\sigma^{-n+|w_1|}(x)\right) \in B_\delta\left(re^{2i\pi q\alpha}\right). \\ \forall x \in [w_2 u w_1] \cap G_n(\varepsilon) : \ g\left(\sigma^{-n+|w_1|}(x)\right) \in B_\delta\left(re^{2i\pi(p+q)\alpha}\right). \end{cases}$$

Consequently:

$$\begin{cases} \forall x \in [w_1 u w_2] \cap E(\varepsilon) \Rightarrow h(x) \in B_\delta\left(e^{2i\pi p\alpha}\right). \\ \forall x \in [w_2 u w_1] \cap E(\varepsilon) \Rightarrow h(x) \in B_{s\delta}\left(e^{2i\pi(-p)\alpha}\right). \end{cases}$$

Thus h cannot be constant and F cannot be ergodic. □

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Vladimir Chernov. *Causality and Legendrian linking in higher dimensional and causally simple spacetimes*

Recently Nemirovski and myself showed that causal relation of two events in a globally hyperbolic spacetime is equivalent to the non triviality of the Legendrian link of the spheres in the space of all light rays formed by the lights rays through the two event points. The result was obtained under the condition that the universal cover of a Cauchy surface of the spacetime is not compact. In this talk we review these results and discuss the generalization of them to the case of Cauchy surfaces with compact universal coverings. The last results are based on the contact Bott–Samelson Theorem of Fraunfelder, Labrousse and Schlenk. If time permits we will discuss possible relations between causality and linking in causally simple spacetimes which are more general than globally hyperbolic ones.

Hichem Chtioui. *Prescribing scalar curvatures on n-dimensional manifolds, $4 \leqslant n \leqslant 6$*

(Based on joint work with A. Alghanemi, M. Soula)

We provide existence and multiplicity theorems for the scalar curvature problem on Riemannian manifolds of dimensions 4 and 5 and 6. Our approach is based on the critical points theory of Bahri and uses the positive mass theorem of Schoen-Yau.

Introduction

On a closed Riemannian manifold (M^n, g_0), $n \geqslant 3$, with a non negative scalar curvature R_{g_0}, let K be a given function. We address to the problem of finding suitable conditions on K to be realized as the scalar curvature of a conformal metric g on M^n. Setting $g = u^{\frac{4}{n-2}} g_0$, where u is a smooth positive function on M^n. The problem is then reduced to solve the nonlinear PDE

$$(1) \qquad \begin{cases} -L_{g_0} u &= K(x) u^{\frac{n+2}{n-2}}, \\ \quad u &> 0 \text{ on } M^n, \end{cases}$$

where $-L_{g_0} u = -\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u$ is the conformal Laplacien of (M^n, g_0).

In the current note, we deal with manifolds (M^n, g_0) of dimensions 4 and 5 which are not conformally diffeomorphic to S^n as well as manifolds (M^n, g_0) of dimensions 6 under some conditions on the Weyl tensor. Our aim is to provide existence and multiplicity results for the problem when the prescribed function K satisfies the so-called "β-flatness condition" near its critical points. The main novelty here is that the flatness order $\beta = \beta(y)$ allows in $(1, \infty)$ for any critical point y of K.

Let $G(\cdot, \cdot)$ be the Green function of $-L_{g_0}$ on M^n and let $H(\cdot, \cdot)$ be its regular part. We associate to any $a \in M^n$ a conformal metric $g_a = u_a^{\frac{4}{n-2}} g_0$ on M^n such that in the conformal normal coordinates $\{x_k\}_{1 \leqslant k \leqslant n}$ near a; we have

$$\det(g_a(x)) = 1 + O(|x-a|^N), \quad \forall x \in B(a, \rho_0).$$

Here $|x-a|$ denotes the geodesic distance between x and a with respect to the the metric g_a and N is a fixed positive integer larger enough.
It is known that in dimension $n = 3, 4, 5$ or if M^n is locally conformally flat the expression of the Green function $G(\cdot,\cdot)$ simplifies considerably in the conformal normal coordinates near a. Namely we have

$$G(a,x) = |x-a|^{2-n} + A_a + O(|x-a|), \text{ where } A_a = H(a,a).$$

$(\mathbf{f})_\beta$ Assume that K is a C^1-function satisfying at any of its critical point y the following: There exists a real number $\beta = \beta(y)$ such that in the conformal normal coordinates system near y, K is expressed as follows:

$$K(x) = K(y) + \sum_{k=1}^{n} b_k |x_k - y_k|^\beta + o(|x-y|^\beta),$$

with $b_k = b_k(y) \neq 0$, $\forall k = 1,\dots,n$, if $\beta(y) \leqslant n-2$. Moreover,

$$\begin{cases} \sum_{k=1}^{n} b_k(y) \neq 0, & \text{if } \beta(y) < n-2, \\ -n\dfrac{c_1}{w_{n-1}} \dfrac{\sum_{k=1}^n b_k}{K(y)^{\frac{n-2}{2}}} - A_y \neq 0, & \text{if } \beta(y) = n-2, \end{cases}$$

Here $c_1 = \displaystyle\int_{\mathbb{R}^n} |z_1|^\beta \frac{|z|^2-1}{(1+|z|^2)^{n+1}} dz$ and $w_{n-1} = |S^{n-1}|$.

Let $\mathscr{K}$ be the set of the critical points of K and define

$$\mathscr{K}_{<n-2} = \{y \in \mathscr{K},\ \beta(y) < n-2\},\ \mathscr{K}^+_{<n-2} = \{y \in \mathscr{K}_{<n-2},\ -\sum_{k=1}^{n} b_k(y) > 0\},$$

$$\mathscr{K}_{n-2} = \{y \in \mathscr{K},\ \beta(y) = n-2\},\ \mathscr{K}^+_{n-2} = \{y \in \mathscr{K}_{n-2},\ -\frac{nc_1}{w_{n-1}} \frac{\sum_{k=1}^n b_k}{K(y)^{\frac{n-2}{2}}} - A_y > 0\}$$

and

$$\mathscr{K}_{>n-2} = \{y \in K,\ \beta(y) > n-2\}.$$

For any p-tuple of distinct points $\tau_p := (y_{l_1},\dots,y_{l_p}) \in (\mathscr{K}^+_{n-2})^p$, $1 \leqslant p \leqslant \sharp\mathscr{K}^+_{n-2}$, we define the following symmetric matrix $M(\tau_p) = (m_{ij})_{1\leqslant i,j\leqslant p}$ such that

$$m_{ii} = m(y_{l_i}, y_{l_i}) = \frac{nc_1}{w_{n-1}} \frac{\sum_{k=1}^n b_k(y_{l_i})}{K(y_{l_i})^{\frac{n}{2}}} - \frac{A_{y_{l_i}}}{K(y_{l_i})},\ \forall i = 1,\dots,p.$$

$$m_{ij} = m(y_{l_i}, y_{l_j}) = -\frac{G(y_{l_i}, y_{l_j})}{[K(y_{l_i})K(y_{l_j})]^{\frac{n-2}{2}}},\ \forall i \neq j.$$

(A) Assume that for any p-tuple of distinct points $\tau_p \in (\mathscr{K}^+_{n-2})^p$, the least eigenvalue $\rho(\tau_p)$ of $M(\tau_p)$ is non zero.

Setting

$$C^\infty_{<n-2} := \{\tau_p = (y_{l_1},\dots,y_{l_p}) \in (\mathscr{K}^+_{<n-2})^p, p \geqslant 1 \textit{ and } y_{l_i} \neq y_{l_j}, \forall i \neq j\},$$
$$C^\infty_{n-2} := \{\tau_p = (y_{l_1},\dots,y_{l_p}) \in (\mathscr{K}^+_{n-2})^p, p \geqslant 1,\ y_{l_i} \neq y_{l_j}, \forall i \neq j \textit{ and } \rho(\tau_p) > 0\},$$

and define for any p-tuple of distinct points $\tau_p = (y_{l_1},\dots,y_{l_p})$

$$i(\tau_p) = p - 1 + \sum_{j=1}^{p} n - \tilde{i}(y_j),$$

where $\tilde{i}(y_j) = \sharp\{b_k(y_j), 1 \leqslant k \leqslant n, s.t., b_k(y_j) < 0\}$.
Here are our existence and multiplicity Theorems.

THEOREM 1. *Let $n = 4, 5$. Suppose (M^n, g_0) is not conformally diffeomorphic to S^n and K is a positive function on M^n for which (A) and $(f)_\beta$, $\beta \in (1,\infty)$ hold. If*

$$\sum_{\tau_p \in C^\infty_{<n-2} \bigcup C^\infty_{n-2} \bigcup\ (C^\infty_{<n-2} \times C^\infty_{n-2})} (-1)^{i(\tau_p)} - 1 \neq 0,$$

then (1.1) admits at least one solution.

Moreover for generic K and if $\beta \in (\frac{n-2}{2},\infty)$, the number of the solution of (1.1) is lager or equals to

$$\Big|\sum_{\tau_p \in C^\infty_{<n-2} \bigcup C^\infty_{n-2} \bigcup\ (C^\infty_{<n-2} \times C^\infty_{n-2})} (-1)^{i(\tau_p)} - 1\Big|.$$

Theorem 2. *Let $n = 6$. Suppose K is positive smooth function on M^n and denote y_0 a global maximum of K. If the Weyl tensor at y_0, $W_{g_0}(y_0)$ is non zero and all partial derivatives of K of order strictly less than $n-2$ vanish at y_0, then* (1.1) *has at least one solution.*

Moreover for generic K satisfying $(f)_\beta$-condition, with $\beta \in (\frac{n-2}{2}, \infty)$, if $W_{g_0}(y) \neq 0$ for any critical point y of K, then the number of the solutions of (1.1) *is larger or equals to*

$$\Big| \sum_{\tau_p \in C^\infty_{<n-2}} (-1)^{i(\tau_p)} - 1 \Big|.$$

Florin Damian. *On involutions without fixed points on the hyperbolic manifold*

We discuses methods of synthetic geometry that permit to construct easy new examples of hyperbolic manifolds and to describe their geometry. However in some cases we can find a manifolds with geodesic boundary. If the symmetry group of this boundary of co-dimension one, contains an involution without fixed point, then one can complete the construction.

In the communication we will give some example of hyperbolic n-manifold M^n ($n = 2, 3, 4, 5$) which possess such isometric involution. Some of them was obtained as metrical reconstruction of manifolds described in [1-4]. The factorization of these manifolds by the above involutions yield complete manifolds whose volume is two times less that the volume of the initial manifolds, for example for reconstructed Davis hyperbolic 4-manifold. Also we will give some examples of hyperbolic manifolds for which the constructed manifolds M^n are geodesical boundaries.

This investigation lead to an "intermediate" way of representing the hyperbolic manifold by an equidistant polyhedron [5] over compact basis as a submanifold of co-dimension one.

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Alexander Degtyarev. *Slopes of colored links*

(The subject is a joint work in progress with Vincent Florens and Ana G. Lecuona)

We introduce a new invariant, called *slope*, of a link in an integral homology sphere. More precisely, given a link $K \cup L \subset \mathbb{S}$ with a distinguished component K, the slope K/L is a $\mathbb{P}^1$-valued function defined on a dense Zariski open subset of the variety of *admissible characters*

$$\mathscr{A} := \{\omega\colon H_1(\mathbb{S} \smallsetminus L) \to \mathbb{C}^* \mid \omega[K] = 1\}.$$

The domain of definition contains all *unitary* admissible characters, and the value of K/L at each unitary character is real.

The slope function is rational (possibly identical ∞) on a Zariski open dense subset of the domain—at least away from the zeroes of the first nonvanishing order of L. Generically, the slope is the ratio of two Conway potentials:

$$(K/L)(\omega) = -\frac{\nabla'_{K\cup L}(1, \sqrt{\omega})}{2\nabla_L(\sqrt{\omega})} \in \mathbb{C} \cup \infty,$$

where $'$ stands for the derivative with respect to the first argument, *viz.* the one corresponding to K. Thus, K/L can be regarded as a multivariate generalization of the Kojima–Yamasaki η-function. However, the slope is still well defined and becomes really interesting when this ratio does not make sense (*i.e.*, a common root or, better yet, both potentials vanishing identically): l'Hôpital's rule does *not* apply, and our experiments with the link tables show that the slope can distinguish links with equal higher Alexander polynomials. Still, the slope is an invariant of the link group (together with the peripheral data). Among other approaches (*e.g.*, Seifert surfaces or, more generally, C-complexes), it can be computed by means of the Fox calculus. Combined with the Wirtinger presentation, this gives us a simple algorithm computing the slope in terms of the link diagram.

The original motivation for this work was our formula [1] for the multivariate signature (defined following the approach suggested by Rokhlin and Viro) of the splice $L' \cup L''$ of two colored links $K' \cup L' \subset \mathbb{S}'$ and $K'' \cup L'' \subset \mathbb{S}''$. The signature is almost additive:

$$\sigma_L(\omega', \omega'') = \sigma_{K' \cup L'}(\upsilon'', \omega') + \sigma_{K'' \cup L''}(\upsilon', \omega'') + \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega''),$$

where $\upsilon^* := \omega^*[K^*]$ and the correction term $\delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega'')$ depends only on the combinatorial characteristics of the links (their *linking vectors* λ', λ''). This formula holds unless $\upsilon' = \upsilon'' = 1$, *i.e.*, unless both characters ω', ω'' are admissible. In the exceptional case, which was left open in [1], the formula takes the form

$$\sigma_L(\omega', \omega'') = \sigma_{L'}(\omega') + \sigma_{L''}(\omega'') + \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega'') + \Delta\sigma(\kappa', \kappa''),$$

where the extra correction term

$$\Delta\sigma(\kappa', \kappa'') := \operatorname{sg}\kappa' - \operatorname{sg}\Bigl(\frac{1}{\kappa'} - \kappa''\Bigr)$$

depends on the slopes $\kappa^* := (K^*/L^*)(\omega^*)$. (For the purpose of this statement, we disambiguate $\infty - \infty$ to 0 and let $\operatorname{sg}\infty = 0$.) Note that this extra term is the only contribution of the knots K', K'' along which the links are spliced. Note also that both slopes are well defined and real as the characters involved are unitary.

Should time permit, I will also discuss further properties of the new invariant. For example, the slope is a concordance invariant away from the so-called *concordance roots*. The concept of slope extends to a special class of tangles; the corresponding signature formula generalizes and refines the skein relations for the signature.

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Ivan Dynnikov. *A method to distinguishing Legendrian and transverse knots*

(The talk is based on recent joint works with Maxim Prasolov and Vladimir Shastin)

A smooth knot (or link) K in the three-space $\mathbb{R}^3$ is called *Legendrian* if the restriction of the 1-form $\alpha = x\,dy + dz$ on K vanishes, where x, y, z are the standard coordinates in $\mathbb{R}^3$. If $\alpha|_K$ is everywhere non-vanishing on K, then K is called *transverse*.

Classification of Legendrian and transverse knots up to respectively Legendrian and transverse isotopy is an important unsolved problem of contact topology. A number of useful invariants have been constructed in the literature, but there are still small complexity examples in which the existing methods do not suffice to decide whether or not the given Legendrain (or transverse) knots are equivalent.

We propose a totally new approach to solving the equivalence problem for Legendrian and transverse knots, which allows to practically distinguish between non-equivalent knots in small complexity cases, and gives rise to a complete algorithm in the general case.

Nikolai Erokhovets. *Combinatorics and hyperbolic geometry of families of 3-dimensional polytopes: fullerenes and Pogorelov polytopes*

By a *polytope* we mean a class of combinatorial equivalence of 3-dimensional convex polytopes. A *k-belt* is a cyclic sequence of k faces such that faces are adjacent if and only if they follow each other, and no three faces have a common vertex. A simple polytope different from the *simplex* Δ^3 is *cyclic edge k-connected (ck-connected)*, if it has no l-belts for $l < k$, and *strongly ck-connected (c*k-connected)*, if in addition any its k-belt surrounds a face. By definition Δ^3 is c^*3-connected but not $c4$-connected. Any simple polytope (family $\mathscr{P}_s$) is $c3$-connected and at most c^*5-connected. We obtain a chain of nested families:

$$\mathscr{P}_s \supset \mathscr{P}_{aflag} \supset \mathscr{P}_{flag} \supset \mathscr{P}_{aPog} \supset \mathscr{P}_{Pog} \supset \mathscr{P}_{Pog^*}$$

The family of $c4$-connected polytopes coincides with the family $\mathscr{P}_{flag}$ of *flag* polytopes defined by the property that any set of pairwise adjacent faces has a non-empty intersection. The family of c^*3-connected polytopes we call *almost flag* polytopes and denote $\mathscr{P}_{aflag}$. Results by A.V. Pogorelov (1967) and E.M. Andreev (1970) imply that $c5$-connected polytopes (family $\mathscr{P}_{Pog}$ of Pogorelov polytopes) are exactly polytopes realizable in the Lobachevsky space $\mathbb{L}^3$ as bounded polytopes with right dihedral angles, and the realization is unique up to isometries. Andreev's result implies that flag polytopes are exactly polytopes realizable in $\mathbb{L}^3$ as polytopes with equal non-obtuse dihedral angles. An example of Pogorelov polytopes is given by *k-barrels* B_k, $k \geqslant 5$, see Fig. 1a). Results by T. Dŏslić (1998, 2003) imply that the family $\mathscr{P}_{Pog}$ contains *fullerenes*, that is simple polytopes with only pentagonal and hexagonal faces.

The family $\mathscr{P}_{aPog}$ of c^*4-connected polytopes we call *almost Pogorelov* polytopes, and the family $\mathscr{P}_{Pog^*}$ of c^*5-connected polytopes – *strongly Pogorelov*. G. D. Birkhoff (1913) reduced the 4-colour problem to the family $\mathscr{P}_{Pog^*}$.

A simple polytope with all faces except for the n-gon being pentagons and hexagons is called an *n-disk-fullerene*.

Proposition 1 ([1], [2]). *Any 3-disk-fullerene belongs to $\mathscr{P}_{aflag}$, any 4-disk-fullerene – to $\mathscr{P}_{aPog}$, and any 7-disk-fullerene – to $\mathscr{P}_{Pog}$. For each $n \geqslant 8$ there exist an n-disk-fullerene in $\mathscr{P}_{Pog^*}$ and an n-disk fullerene not in $\mathscr{P}_{aflag}$.*

T. E. Panov remarked that Andreev's results should imply that almost Pogorelov polytopes correspond to right-angled polytopes of finite volume in $\mathbb{L}^3$. Such polytopes may have 4-valent vertices on the absolute, while all proper vertices have valency 3.

Theorem 2 ([3]). *Cutting of 4-valent vertices defines a bijection between classes of combinatorial equivalence of right-angled polytopes of finite volume in $\mathbb{L}^3$ and almost Pogorelov polytopes different from the cube I^3 and the pentagonal prism $M_5 \times I$.*

We develop a theory of combinatorial construction of families of polytopes. The main idea is to build a family by a given set of operations from a small set of initial polytopes. A classical result by V. Eberhard (1891) states that any simple polytope can be obtained from the simplex Δ^3 by cuttings off vertices, edges and pairs of adjacent edges.

Proposition 3 ([3]). *A simple polytope belongs to $\mathscr{P}_{aflag}$ if and only if it can be obtained from the simplex with at most two vertices cut by cuttings off vertices, edges and pairs of adjacent edges not equivalent to cutting off a vertex of a triangle, and if and only if it is obtained by simultaneous cutting off a set of vertices of Δ^3 or a flag polytope.*

Results by A. Kotzig (1969) imply that a simple polytope is flag iff it can be obtained from I^3 by cuttings off edges and pairs of adjacent edges of at least hexagonal faces. The family $\mathscr{P}_{aPog}$ contains I^3, $M_5 \times I$, and the 3-dimensional *Stasheff polytope As^3*, which is the cube with three pairwise disjoint orthogonal edges cut. A result by D. Barnette (1974) implies that a simple polytope belongs to $\mathscr{P}_{aPog} \setminus \{I^3, M_5 \times I\}$ iff it can be obtained from As^3 by cuttings off edges not lying in quadrangles and pairs of adjacent edges of at least hexagonal faces. Unlike the case of flag polytopes, not any quadrangle of a polytope in $\mathscr{P}_{aPog}$ is obtained by cutting off an edge of a polytope of the same family. However, results by D. Barnette imply that if a polytope in $\mathscr{P}_{aPog}$ has quadrangles, then at least one quadrangle can be obtained in this way. A *matching* of a polytope is a set of its pairwise disjoint edges. A matching is *perfect*, if it covers all the vertices. Let P_8 be the cube with two disjoint orthogonal edges cut.

Theorem 4 ([3]). *Any almost Pogorelov polytope $P \neq I^3, M_5 \times I$ is obtained by cutting off a matching of a polytope in $\mathscr{P}_{aPog} \sqcup \{P_8\}$ producing all the quadrangles.*

A polytope in $\mathbb{L}^3$ is *ideal*, if all its vertices lie on the absolute. It has a finite volume.

Corollary 5 ([3]). *Any ideal right-angled polytope P is obtained from some polytope $Q \in \mathscr{P}_{aPog} \sqcup \{P_8\}$ by the contraction of edges of some perfect matching not containing opposite edges of any quadrangle.*

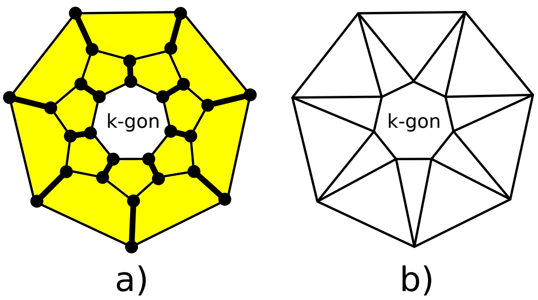

Figure 1. a) canonical perfect matching of the k-barrel; b) k-antiprism.

Example 6. The k-barrel has a canonical perfect matching drawn on Fig. 1a). The corresponding ideal polytope is called a *k-antiprism*, see Fig. 1b).

An operation of an *edge-twist* is drawn on Fig. 2. Two edges on the left lie in the same face and are disjoint. Let us call an edge-twist *restricted*, if both edges are adjacent to an edge of the same face. In the survey (2017) A. Yu. Vesnin combining results by I. Rivin (1996) on ideal polytopes and by G. Brinkmann, S. Greenberg, C. Greenhill, B.D. McKay, R. Thomas, P. Wollan (2005) on quadrangulations of a sphere stated that any ideal right-angled polytope can be obtained from a k-antiprism, $k \geqslant 3$, by edge-twists.

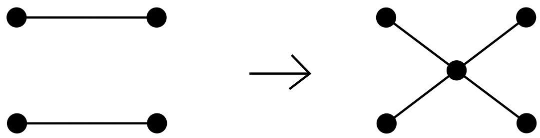

Figure 2. An edge-twist.

Theorem 7 ([3]). *A polytope is realizable as an ideal right-angled polytope iff it either is a k-antiprism, $k \geqslant 3$, or can be obtained from the 4-antiprism by restricted edge-twists.*

Results by I. Rivin (1994) imply that a realization of a polytope as an ideal polytope in $\mathbb{L}^3$ is unique up to isometrices.

Conjecture 8. *An edge-twist increases the volume of a right-angled polytope in $\mathbb{L}^3$.*

All k-barrels, $k \geqslant 5$, belong to $\mathscr{P}_{Pog*}$. Results by D. Barnette (1974,1977), J. W. Butler (1974) and results from [1] imply that a simple polytope different from these barrels belongs to $\mathscr{P}_{Pog}$ iff if it can be obtained from the 5- or the 6-barrel by cuttings off pairs of adjacent edges of at least hexagonal faces and connected sums with the 5-barrel (Fig. 3), and to the family $\mathscr{P}_{Pog*}$ iff it can be obtained from the 6-barrel by cuttings off pairs of adjacent edges of at least hexagonal faces. T. Inoue (2008) showed that both operations increase the hyperbolic volume and enumerated the first 825 bounded right-angled polytopes in the order of the increasing volume (2015).

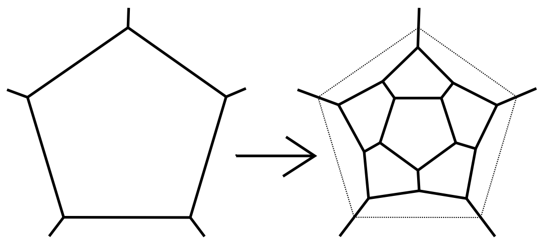

Figure 3. A connected sum with the 5-barrel.

For fullerenes there is a stronger result than for Pogorelov polytopes. There is a 1-parametric series of fullerenes obtained from the 5-barrel by connected sums with the 5-barrel along pentagons surrounded by pentagons. It consists of the 5-barrel and the so-called $(5,0)$-*nanotubes*. Results by F. Kardoš, R. Skrekovski (2008) and, independently, by K. Kutnar, D. Marušič (2008) imply that all the other fullerenes lie in $\mathscr{P}_{Pog*}$.

Theorem 9 ([1]). *Any fullerene different from the 5-barrel and the $(5,0)$-nanotubes can be obtained from the 6-barrel by a sequence of cuttings off pairs of adjacent edges of at least hexagonal faces in such a way that intermediate polytopes are either fullerenes or 7-disk-fullerenes with the heptagon adjacent to a pentagon.*

The difficulty is that the construction of the family $\mathscr{P}_{Pog*}$ does not guarantee that intermediate polytopes are close to fullerenes.

Theorem 10 ([2]). *A 7-disk-fullerene is not in $\mathscr{P}_{Pog*}$ iff it is obtained from a fullerene by a sequence of connected sums with the 5-barrel. Any 7-disk-fullerene from $\mathscr{P}_{Pog*}$ can be obtained from the 6-barrel by a sequence of cuttings off pairs of adjacent edges of at least hexagonal faces in such a way that intermediate polytopes have pentagonal, hexagonal and at most two heptagonal faces.*

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Sergey Finashin. *Chirality of real cubic fourfolds*

(Based on joint work with V. Kharlamov)

In our previous work [1] we have classified real non-singular cubic hypersurfaces in the 5-dimensional projective space up to equivalence that includes both real projective transformations and continuous variations of coefficients preserving the hypersurface non-singular. Here, we perform a finer classification giving a full answer to the chirality problem: which of real non-singular cubic hypersurfaces can not be continuously deformed to their mirror reflection.

Both deformation equivalence relations emerge naturally in the study of real non-singular projective hypersurfaces in the framework of 16th Hilbert's problem. More precisely, the *pure deformation equivalence* assigns hypersurfaces to the same equivalence class if they can be joined by a continuous path (called a *real deformation*) in the space of real non-singular projective hypersurfaces of some fixed degree. Another one is the *coarse deformation equivalence*, in which real deformations are combined with real projective transformations.

If the dimension of the ambient projective space is even, then the group of real projective transformations is connected, and the above equivalence relations coincide. By contrary, if the dimension of the ambient projective space is odd, this group has two connected components, and some of coarse deformation classes may split into two pure deformation classes. The hypersurfaces in such a class are not pure deformation equivalent to their mirror images and are called *chiral*. The hypersurfaces in the other classes are called *achiral*, since each of them is pure deformation equivalent to its mirror image.

The first case where a discrepancy between pure and coarse deformation equivalences shows up is that of real non-singular quartic surfaces in 3-space (achirality of all real non-singular cubic surfaces is due to F. Klein [6]). In this case it was studied in [4,5], where it was used to upgrade the coarse deformation classification of real non-singular quartic surfaces obtained by V. Nikulin [10] to a pure deformation classification.

Real non-singular cubic fourfolds is a next by complexity case. Their deformation study was launched in [1], where we classified them up to coarse deformation equivalence. Then in [2] we began studying of the chirality phenomenon and gave complete answers for cubic fourfolds of maximal, and almost maximal, topological complexity. The approach, which we elaborated and applied in [2] relies on the surjectivity of the period map for cubic fourfolds established by R. Laza [7] and E. Looijenga [9].

Recall that according to [1] there exist precisely 75 coarse deformation classes of real non-singular fourfold cubic hypersurfaces $X \subset P^5$ (throughout the paper X stands both for the variety itself and for its complex point set, while $X_{\mathbb{R}} = X \cap P^5_{\mathbb{R}}$ denotes the real locus). These classes are determined by the isomorphism type of the pairs $(\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X), h \in \mathbb{M}(X))$ where $\mathbb{M}(X) = H^4(X;\mathbb{Z})$ is considered as a lattice, $h \in \mathbb{M}(X)$ is the *polarization class* that is induced from the standard generator of $H^4(P^5;\mathbb{Z})$, and conj^* is induced by complex conjugation $\mathrm{conj} : X \to X$. This result can be simplified further and expressed in terms of a few simple numerical invariants. Namely, it is sufficient to consider the sublattice $\mathbb{M}^0_+(X) \subset \mathbb{M}(X)$, $\mathbb{M}^0_+(X) = \{x \in \mathbb{M}(X) : \mathrm{conj}^* x = x, xh = 0\}$, and to retain only the following three invariants: the rank ρ of $\mathbb{M}^0_+$, the rank d of the 2-primary part $\mathrm{discr}_2\mathbb{M}^0_+$ of the discriminant $\mathrm{discr}\mathbb{M}^0_+$, and the type, even or odd, of the discriminant form on $\mathrm{discr}_2\mathbb{M}^0_+$.

Thus, to formulate the pure deformation classification of real non-singular cubic fourfolds, it is sufficient to list the triples of invariants (ρ, d, parity) which specify the coarse deformation classes and to indicate which of the coarse classes are chiral, and which ones are achiral.

Theorem 1. *Among the 75 coarse deformation classes precisely 18 are chiral, and, thus, the number of pure deformation classes is 93. The chiral classes have pairs (ρ, d) satisfying $\rho + d \leqslant 12$. The only achiral classes with $\rho + d \leqslant 12$ are three classes with $4 \leqslant \rho = d \leqslant 6$ and one class with $(\rho, d) = (8, 4)$ and even pairity.*

A complete description of the pure deformation classes is presented in Table 1, where the coarse deformation classes are marked by letters c and a: by c, if the class is chiral, and by a, if it is achiral. We use ρ and d as Cartesian coordinates and employ bold letters to indicate even parity, while keeping normal letters for odd. For some pairs (ρ, d) there exist two coarse deformation classes, one with even discriminant form, and another with odd, and in this case, we put the even one in brackets.

In fact, the values of ρ and d determine the topology of the real locus of the cubic fourfold and are determined by it. Namely, for all pairs (ρ, d) except one the real locus of the fourfold is diffeomorphic to $\mathbb{RP}^4 \# a(S^2 \times S^2) \# b(S^1 \times S^3)$, where $a = \frac{1}{2}(\rho - d), b = \frac{1}{2}(22 - \rho - d)$. The exception is $(\rho, d, \text{parity}) = (12, 10, \text{even})$, in which case the real locus is diffeomorphic to $\mathbb{RP}^4 \sqcup S^4$ (see [3]). Comparing this with Table 1 we come to the following conclusion.

Corollary 2. *Chirality of a cubic $X \subset P^4$ is determined by the topological type of its real locus $X_{\mathbb{R}}$ unless $X_{\mathbb{R}} = \mathbb{RP}^4 \# 2(S^2 \times S^2) \# 5(S^1 \times S^3)$, or equivalently, $(\rho, d) = (8, 4)$. If $(\rho, d) = (8, 4)$, then X is achiral in the case of even parity, and chiral in the case of odd.* □

Table 1. Pure deformation classification via chirality

d \ ρ	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
11										a											
10									a		a(**a**)										
9								a		a		a									
8							a(**a**)		a		a(**a**)		a								
7						a		a		a		a		a							
6					a		a(**a**)		a		a(**a**)		a		a						
5				a		c		a		a		a		a		a					
4			a		c		c(**a**)		a		a(**a**)		a		a(**a**)		a				
3		c		c		c		c		a		a		a		a		a			
2	c		c(**c**)		c		**c**		c		a(**a**)		a		**a**		a		a(**a**)		
1		c		c						c		a						a		a	
0			**c**								**c**								**a**		

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David Gabai. *The 4-dimensional light bulb theorem; extensions and applications*

The 4-dimensional light bulb theorem asserts that a smooth 2-sphere in $S^2 \times S^2$ that is homotopic to $x \times S^2$ and intersects $S^2 \times y$ transversely once is smoothly isotopic to $x \times S^2$. We will discuss this theorem as well as extensions and applications.

Alexander Gaifullin. *Combinatorial computation of Pontryagin classes*

The definition of the Pontryagin classes of a manifold substantially uses the smooth structure on it. By a well-known example due to Milnor, integral Pontryagin classes are not invariant under PL homeomorphisms. However, in 1957 Rokhlin and Schwarz proved that rational Pontryagin classes are invariant under PL homeomorphisms. (A year later the same result was obtained independently by Thom.) Later this result was improved considerably by Novikov who showed that rational Pontryagin classes are topological invariants.

The Rokhlin-Schwarz-Thom result on combinatorial invariance of rational Pontryagin classes raised a problem on explicit computation of the rational Pontryagin classes of a manifold from a triangulation of it. In the context of a smooth manifold with smooth triangulation this problem was solved for the first Pontryagin class in a famous work of Gabrielov, Gelfand, and Losik (1975). However, their approach gave no answer in a purely combinatorial situation, i.e., for a triangulated manifold without given smoothing.

In 2004 the author suggested another approach based on the usage of bistellar moves, and constructed a purely combinatorial local formula for the first rational Pontryagin class of a triangulated manifold. More precisely, this result gave explicit description of all local combinatorial formulae for the first rational Pontryagin class, but no effective choice of a particular formula was made. Recently, Gorodkov and the author have constructed effectively a particular local combinatorial formula for the first rational Pontryagin class. The key ingredient is the study of the redistribution of the combinatorial Gaussian curvature of a triangulated 2-sphere under bistellar moves.

Mukta Garg. *Some stronger forms of transitivity in G-spaces*

The theory of dynamical systems attempts to describe the behaviour of all particles in the phase space. However, in a real physical system, sometimes it becomes almost impossible to determine the exact position of particles, so that, instead of points one tries to study the behaviour of open subsets of the space and accordingly estimate the dynamics of the underlying system. This leads to the notion of *topological transitivity.* The concept of topological transitivity was first used by G. D. Birkhoff in 1920. Topological transitivity is an important dynamical property because of its strong connection with chaos. It guarantees that for any pair of non-empty open sets, there is an iterate of the first that overlaps the second. Different forms of transitivity, like total transitivity, weakly mixing, strong mixing are defined in the literature. We have extended these notions to maps on topological transformation groups. Observing their interrelations, duly supported by examples/counterexamples, we have obtained several interesting results related to these notions including conditions for their equivalences.

Ilya Gekhtman. *Geometric and probabilistic boundaries of random walks, metrics on groups and measures on boundaries in negative curvature*

Consider a geometrically finite isometry group of a pinched negatively curved contractible manifold. There are two natural averaging procedures on this group: averaging with respect to balls in the manifold and taking a finitely supported random walk. These correspond to two natural measures on the boundary the Patterson-Sullivan measure (which for symmetric manifolds is in the Lebesgue measure class) and the harmonic measure which is the limit of convolution powers of the random walk. These two measures satisfy conformality properties with respect to two metrics on the lattice: the metric induced by the orbit map d and the so called Green metric d_G associated to the random walk, which is quasi-isometric to the word metric.

In turn, they correspond to two measures on the unit tangent bundle (the measure of maximal entropy and the harmonic invariant measure) and closed geodesics on the quotient manifold satisfy two different equidistribution properties with respect to the two measures.

We show that the harmonic and Patterson-Sullivan measures are singular unless the two metrics are roughly similar: $|d - c_1 d_G| < c_2$ for uniform constants c_1, c_2. Thus, they are always singular when the isometry group contains parabolics.

Everything can be generalized to geometrically finite actions on proper Gromov hyperbolic spaces, such as Hilbert geometries.

Furthermore, our techniques can be generalized to show that when the fundamental group contains parabolics, harmonic measures for finitely supported random walks are singular to any Gibbs measure associated to a Hoelder potential.

The latter involves proving a weighted analogue of Guivarch's fundamental inequality relating entropy, drift and volume growth which incorporates the potential function; such a formula is new even in the cocompact setting.

Most of this is based on two papers: one joint with Gerasimov-Potyagailo-Yang and another with Tiozzo.

Vladimir Golubyatnikov. *Non-uniqueness of periodic trajectories in some piece-wise linear dynamical systems*

Introduction

We study dynamical systems of a special type as models of gene networks functioning. One of the main aims of our considerations is description of conditions of existence, (non)-uniqueness and stability of cycles in the phase portraits of these dynamical systems. Biological interpretations of these models are exposed in [2,5]. The following piece-wise dynamical system is one of the typical models of this kind.

$$\begin{aligned} \frac{dX_1}{dt} &= L_1(X_4) - X_1; \quad \frac{dX_2}{dt} = \Gamma_2(X_1) - X_2; \\ \frac{dX_3}{dt} &= \Gamma_3(X_2) - X_3; \quad \frac{dX_4}{dt} = \Gamma_4(X_3) - X_4. \end{aligned} \tag{1}$$

Here, Γ_i are step-functions: $\Gamma_i(X_{i-1}) = \alpha_i > 0$ for $0 \leqslant X_{i-1}$; $\Gamma_i(X_{i-1}) = -1$ for $0 \geqslant X_{i-1} \geqslant -1$, $i = 2, 3, 4$, and the step-function L_1 is decreasing: $L_1(X_4) = \alpha_1 > 0$ for $-1 < X_4 < 0$, $L_1(X_4) = -1$ for $0 \leqslant X_4$. Non-negative quantities $X_i + 1$, denote concentrations of four components of the gene network, positive parameters α_i characterize velocities of synthesis of these components. Here and below $i = 1, 2, 3, 4$. Decreasing function L_1 describes negative feedback and increasing functions Γ_i correspond to positive feedbacks there.

The system (1) was constructed in [2], where the variables and equations were interpreted in details: the subtrahends in the equations correspond to degradations of the biological components, non-negative summand $L_1(X_4)$ in the first equation describes the velocity of synthesis of the "first" component with concentration $X_1 + 1$ as a function of concentration $X_4 + 1$ of the "previous" component in this circular gene network. Other equations of the system are interpreted similarly.

1. Discretization of the phase portrait of the system (1)

Let Q be the parallelepiped $[-1, \alpha_1] \times [-1, \alpha_2] \times [-1, \alpha_3] \times [-1, \alpha_4]$. The point $O = (0, 0, 0, 0)$ is located in its interior. Trajectories of the points of Q do not leave it as $t \to \infty$.

The proof of this fact consists in verifications of signs of the derivatives dX_i/dt on the faces $X_i = -1$ and $X_i = \alpha_i$ of Q, see for example [1,5] where this was proved for dynamical systems analogous to (1) in various dimensions.

The work [4] was devoted to so called symmetric dynamical systems: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Now, we study asymmetric case, whese all the parametres α_i are different in general.

The coordinate planes $X_i = 0$ subdivide the domain Q to 16 smaller parallelepipeds (or blocks) which we denote by binary multi-indices $\{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\}$, where $\varepsilon_i = 0$, if $X_i < 0$, and $\varepsilon_i = 1$, if $X_i > 0$.

It was shown in [1,4] that for arbitrary dimensions of the "billiards-like" systems of the type (1), their trajectories are piece-wise linear with angle points on the planes $X_i = 0$.

PROPOSITION 1. *For any two adjacent blocks B_1, B_2 of subdivision of Q, trajectories of points of their common 3-dimensional face $F = B_1 \cap B_2$ pass through this face in one direction only, either from B_1 to B_2 (denoted as $B_1 \to B_2$), or from B_2 to B_1 (denoted as $B_2 \to B_1$).*

As above, the proof consists in calculation of the signs of dX_i/dt at the points of the hyperplane $X_i = 0$ containing the face F.

Denote by G oriented graph with vertices $\{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\}$, i.e., four-dimensional cube with edges oriented according to the Proposition 1.

It was shown in [2] that the system (1) has a stable cycle $\mathscr{C}_1$ which passes through the blocks following the arrows of the State Transition Diagram:

$$
(2)\qquad
\begin{array}{ccccccc}
\{1111\} & \longrightarrow & \{0111\} & \longrightarrow & \{0011\} & \longrightarrow & \{0001\} \\
\uparrow & & & & & & \downarrow \\
\{1110\} & \longleftarrow & \{1100\} & \longleftarrow & \{1000\} & \longleftarrow & \{0000\}
\end{array}
$$

Actually, this is a subgraph of G. Let W_1 be the interior of the union of the blocks of (2). Trajectories of the points of interiors of these blocks can pass to one adjacent block only, according to the arrow. Hence, W_1 is a positively invariant domain of the system (1).

Denote by W_3 the interior of the union of remaining eight blocks, connected by the diagram

$$
(3)\qquad
\begin{array}{ccccccc}
\{1101\} & \longrightarrow & \{0101\} & \longrightarrow & \{0100\} & \longrightarrow & \{0110\} \\
\uparrow & & & & & & \downarrow \\
\{1001\} & \longleftarrow & \{1011\} & \longleftarrow & \{1010\} & \longleftarrow & \{0010\}
\end{array}
$$

As above, the arrows show possible transitions of trajectories from block to block.

PROPOSITION 2. *In contrast with the diagram (2), trajectories of the points of interiors of these blocks can pass to three adjacent blocks.*

This is why we write W_3 here. For example, trajectories of the points of the block $\{1001\}$ can pass to the block $\{1101\}$, as it is shown in the diagram (3), but some of these trajectories pass to the blocks $\{0001\}$ and $\{1000\}$ listed in the State Transition Diagram (2).

2. Description of trajectories in the domain W_3

Let $F_0 = \{1101\} \cap \{0101\}$, $X_1 = 0$; $\quad F_1 = \{0101\} \cap \{0100\}$, $X_4 = 0$;
$F_2 = \{0100\} \cap \{0110\}$, $X_3 = 0$; $\quad F_3 = \{0110\} \cap \{0010\}$, $X_2 = 0$;
$F_4 = \{0010\} \cap \{1010\}$, $X_1 = 0$; $\quad F_5 = \{1010\} \cap \{1011\}$, $X_4 = 0$;

$$F_6 = \{1011\} \cap \{1001\}, \quad X_3 = 0; \quad F_7 = \{1001\} \cap \{1101\}, \quad X_2 = 0;$$

be the common faces of adjacent bocks of the domain W_3. Denote by

$$\varphi_0 : F_0 \to F_1, \;\; \varphi_1 : F_1 \to F_2, \;\; \varphi_2 : F_2 \to F_3, \;\; \varphi_3 : F_3 \to F_4, \;\; \varphi_4 : F_4 \to F_5,$$
$$\varphi_5 : F_5 \to F_6, \quad \varphi_6 : F_6 \to F_7, \quad \varphi_7 : F_7 \to F_8, = F_0,$$

the shifts of the points of interiors of these faces along their trajectories of the system (1) inside each of the blocks of the diagram (3), and let $\Phi : F_0 \to F_0$ be the composition of these shifts. This is the Poincaré map of the cycle which we are going to detect.

REMARK 3. It follows from the Proposition 2 that trajectories of most of the points of the faces F_j hit the faces of the blocks listed in the diagram (2) and then remain in the interior of W_1, as it can be seen in the cases of analogous dynamical systems, see [1,4]. So, we assume that the mappings $\varphi_j : F_j \to F_{j+1}$ are defined on preimages of F_{j+1}. Here and below $j = 0, 1, \ldots, 7$.

In interior of each of the blocks $\{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\}$, the system (1) is solved explicitly, for example, in the block $\{1001\}$ of the diagram (3) we have

$$\frac{dX_1}{dt} = -1 - X_1, \;\; \frac{dX_2}{dt} = \alpha_2 - X_2, \;\; \frac{dX_3}{dt} = -1 - X_3, \;\; \frac{dX_4}{dt} = -1 - X_4; \tag{4}$$

$$X_1(t) = -1 + (X_1^0 + 1)e^{-t}; \; X_2(t) = \alpha_2(1 - e^{-t}); \; X_3(t) = -1 + (X_3^0 + 1)e^{-t}; \; X_1(t) = -1 + (X_4^0 + 1)e^{-t};$$

where $X_1^0 > 0$, $X_2^0 = 0$, $X_3^0 < 0$, $X_4^0 > 0$ are coordinates of a point of F_0.

Let $Oy_1^{(j)}y_2^{(j)}y_3^{(j)}$ be the coordinate system in the plane $X_i = 0$ containing the face F_j such that the positive octant of this coordinate system contains the face F_0, and the axes are enumerated as follows: $y_1^{(j)} = \pm X_{i+1} > 0$, $y_2^{(j)} = \pm X_{i+2} > 0$, $y_3^{(j)} = \pm X_{i+3} > 0$ in the interior of F_j.

Let $P^{(0)} \in \operatorname{int} F_0$ be a point with coordinates $X_1 = 0$, $y_1^{(0)} = X_2$, $y_2^{(0)} = -X_3$, $y_3^{(0)} = X_4$. Simple calculations with the system (4) and its analogues, see for example [1,3], show that the trajectory T of the point $P^{(0)}$ intersects the faces F_j of the diagram (3) at the points $P^{(j)} = T \cap F_j = (y_1^{(j)}, y_2^{(j)}, y_3^{(j)})$ defined by fractional linear functions:

$$\begin{gathered} P^{(1)} = \frac{M_0 P^{(0)}}{1 + y_3^0}; \;\; P^{(2)} = \frac{M_1 P^{(1)}}{\alpha_3 + y_3^{(1)}}; \;\; P^{(3)} = \frac{M_2 P^{(2)}}{1 + y_3^{(2)}}; \;\; P^{(4)} = \frac{M_3 P^{(3)}}{\alpha_1 + y_3^{(3)}}; \\ P^{(5)} = \frac{M_4 P^{(5)}}{1 + y_3^{(4)}}; \;\; P^{(6)} = \frac{M_5 P^{(5)}}{\alpha_3 + y_3^{(5)}}; \;\; P^{(7)} = \frac{M_6 P^{(6)}}{\alpha_2 + y_3^{(6)}}; \;\; P^{(8)} = \frac{M_7 P^{(7)}}{1 + y_3^{(7)}} \in F_0. \end{gathered} \tag{5}$$

The matrices M_j here are defined as follows:

$$M_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -\alpha_3 \end{pmatrix}; \; M_1 = \begin{pmatrix} 0 & 0 & 1 \\ \alpha_3 & 0 & -\alpha_1 \\ 0 & \alpha_3 & -1 \end{pmatrix}; \; M_2 = \begin{pmatrix} 0 & 0 & \alpha_3 \\ 1 & 0 & -\alpha_4 \\ 0 & 1 & -\alpha_1 \end{pmatrix} \ldots$$

remaining matrices have the form

$$M_j = \begin{pmatrix} 0 & 0 & D_j \\ C_j & 0 & -E_j \\ 0 & C_j & -G_j \end{pmatrix}$$

for some positive C_j, D_j, E_j, G_j, half of them is equal to 1.

The points $P^{(j)}$ are identified here with their position vectors $OP^{(j)}$. The formulae (5) describe projective transformations $F_j \to F_{j+1}$ defined on subsets of the faces F_j, as it was remarked above. In these faces, the rays containing the origin O are mapped to the next faces to the rays containing O.

3. Construction of the Poincaré map in the domain W_3

Let $M_* := M_7M_6M_5M_4M_3M_2M_1M_0$. Then the composition $\Phi : F_0 \to F_0$ is defined by

$$\Phi(P^{(0)}) = \frac{M_* P^{(0)}}{H(y_1^0, y_2^0, y_3^0)},$$

where H is a non-homogeneous linear function of coordinates of the point $P^{(0)}$ strictly positive on the face F_0. One can verify that for each matrix M_j its inverse matrix M_j^{-1} has non-negative elements, and that all elements of the matrix M_*^{-1} are positive. Hence, the matrix M_* has an eigenvector e_1 with positive coordinates, and its eigenvalue λ_1 is positive. Let ℓ be the ray codirectional with e_1. After one round along the diagram (3) trajectories of all points of ℓ return to ℓ and compose an invariant octahedral surface S with one vertex O.

Let z be a positive coordinate on the ray ℓ, then the transformation $\Phi : \ell \to \ell$ is described by

$$\Phi(z) = \frac{\lambda_1 z}{\beta + Kz}, \quad \text{where} \quad K > 0, \quad \text{and} \quad \beta := \alpha_1\alpha_2\alpha_3\alpha_4. \tag{6}$$

Characteristic polynomial of the matrix M_* has the form: $P(\lambda) = \lambda^3 - 3\beta\lambda^2 + I_2\lambda - \beta^3$. So,

$$P(\beta) = \beta(I_2 - 3\beta^2);\ \ P'(\beta) = I_2 - 3\beta^2;\ \ P'(\lambda) = 0 \text{ for } \lambda = \frac{3\beta \pm \sqrt{9\beta^2 - 3I_2}}{3}.$$

Proposition 4. $\det M_* = \beta^3$, $\operatorname{tr} M_* = 3\beta$. *If all eigenvalues of M_* are real then they coincide.*

Let $I_2 - 3\beta^2 > 0$. This inequality holds for example for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. In this case $P(\lambda)$ grows monotonically with λ, so the equation $P(\lambda) = 0$ has a unique real root λ_1 in the interval $(0, \beta)$, and the equation $z = \Phi(z)$ does not have positive solutions. Thus, trajectories of all points of the surface S are not periodic, and they tend to O when $t \to \infty$.

Let $I_2 - 3\beta^2 < 0$. This inequality holds for example for $\alpha_1 = 300$, $\alpha_2 = \alpha_3 = \alpha_4 = 1$. In this case $P(\beta) < 0$, $P'(\beta) < 0$, since $P(0) < 0$ and $P'(0) > 0$, the equation $P(\lambda) = 0$ has one root $\lambda_1 > \beta$. It follows from the previous proposition that the interval $(0, \beta)$ does not contain eigenvalues of the matrix M_*. This matrix does not have negative eigenvalues as well.

It follows from $\lambda_1 > \beta$ that for some positive z_0 we have $\Phi(z_0) = z_0$, thus, the point Z_0 of the ray ℓ with the coordinate z_0 returns to itself after one round along the arrows of the diagram (3). So, the equation (6) implies that such a point Z_0 is unique, and hence, the dynamical system (1) has one more cycle $\mathscr{C}_3 \subset S \subset W_3$.

If $I_2 = 3\beta^2$, then $P(\lambda) = (\lambda - \beta)^3$, and the matrix M_* has unique (up to proportionality) eigenvector e_1 corresponding to $\lambda_1 = \beta$. As above, the surface S is determined uniquely as well, and the equation $z = \beta z/(\beta + Kz)$ has only the zero solution. Thus, for $I_2 = 3\beta^2$ the system (1) does not have cycles in the domain W_3.

Theorem 5. *The surface $S \subset W_3$ contains a cycle $\mathscr{C}_3$ of the system* (1) *if and only if $I_2 - 3\beta^2 < 0$. In this case trajectories of points of S are attracted by $\mathscr{C}_3$.*

One can verify that the cycle $\mathscr{C}_1 \subset W_1$ has a nontrivial link in Q with the surface S, actually, this is the Hopf link. This follows from consideration of two cyclic subgraphs of G described by the diagrams (2) and (3) on the boundary of 4-dimensional cube. In the case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ this observation was done in [4].

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Evgeny Gordon. *On hyperfinite approximations of dynamical systems*

A hyperfinite set is a set whose cardinality is a natural number that is infinitely large in the sense of nonstandard analysis. In this talk we discuss some ergodic properties of permutations of hyperfinite sets and their applications to approximation of dynamical systems on Lebesgue spaces. Though the hyperfinite spaces inherit a lot of properties of finite spaces by the Transfer Principle of Nonstandard analysis and the proof of the Birkhoff Ergodic Theorem is trivial for finite spaces, the proof of BET for hyperfinite spaces is not easier than for the general case.

We show in this talk why it happens, and what actually does classical BET mean for hyperfinite spaces. We also demonstrate some new effects of behaviour of ergodic means on the (discrete) time intervals have the hyperfinite length comparable with the cardinality of the hyperfinite sets. We show how these effects can be used in investigation of finite approximation of standard dynamical systems. Some of these effects have natural interpretations for "very big" finite spaces. They can be even monitored in finite simulations of continuous dynamical systems. Their formulation in terms of nonstandard analysis are simple and intuitively clear, while their standard formulation are inappropriate and very complicated, if not to say unreadable.

Dmitry Gugnin. *Branched coverings of manifolds and n-valued Lie groups*

In the talk we present a new construction of branched coverings of a rather wide series of manifolds over the spheres. Even in the case of the covering space being the m-torus this construction is the first effective construction (before was known only Alexander's 1920 construction [1], which is ineffective and suites for an arbitrary orientable PL manifold).

Let us take the standard sphere (of nonzero dimension):

$$S^m = \{(x_1, \ldots, x_m, x_{m+1}) \in \mathbb{R}^{m+1} | x_1^2 + \ldots + x_m^2 + x_{m+1}^2 = 1\}.$$

Denote by $\tau \colon S^m \to S^m$ the standard involution of the sphere, which permutes its Northern and Southern poles: $\tau(x_1, \ldots, x_m, x_{m+1}) = (x_1, \ldots, x_m, -x_{m+1})$.

Consider the product of k spheres of arbitrary dimensions

$$S^{m_1} \times S^{m_2} \times \ldots \times S^{m_k}, \; k \geqslant 2.$$

On this manifold with the standard C^ω-structure we have the action of commuting involutions $\tau_1, \tau_2, \ldots, \tau_k$ (the direct product of involutions). Therefore, we get some concrete C^ω-action of the group $\mathbb{Z}_2^k$. Denote by G_k the subgroup of index 2 in $\mathbb{Z}_2^k$ consisting of even involutions.

Our manifold $S^{m_1} \times S^{m_2} \times \ldots \times S^{m_k}$ lies in

$$\mathbb{R}^{(m_1+1)+(m_2+1)+\ldots+(m_k+1)} =$$

$$\{(x_{1,1}, x_{1,2}, \ldots, x_{1,m_1}, x_{1,m_1+1}, x_{2,1}, \ldots, x_{2,m_2}, x_{2,m_2+1}, \ldots, x_{k,1}, \ldots, x_{k,m_k}, x_{k,m_k+1})\}$$

Denote by $K_{m_1,\ldots,m_k}$ the quotient space $S^{m_1} \times \ldots \times S^{m_k}/G_k$, and by π — the canonic projection $S^{m_1} \times \ldots \times S^{m_k} \to K_{m_1,\ldots,m_k}$.

Theorem 1. *(See [4]) The quotient space $K_{m_1,\ldots,m_k}$ is a topological sphere of dimension $m = m_1+m_2+\ldots+m_k$. Moreover, there exist a homeomorphism $\psi \colon K_{m_1,\ldots,m_k} \to S^m$, where S^m is the standard m-sphere such that the composition $\chi := \psi \circ \pi$ is obtained by the following explicit formula:*

$$\chi(x_{1,1}, \ldots, x_{1,m_1}, x_{1,m_1+1}, x_{2,1}, \ldots, x_{2,m_2}, x_{2,m_2+1}, \ldots, x_{k,1}, \ldots, x_{k,m_k}, x_{k,m_k+1}) :=$$

$$\frac{(x_{1,1}, \ldots, x_{1,m_1}, x_{2,1}, \ldots, x_{2,m_2}, \ldots, x_{k,1}, \ldots, x_{k,m_k}; x_{1,m_1+1} \cdot x_{2,m_2+1} \cdots x_{k,m_k+1})}{\sqrt{x_{1,1}^2 + \ldots + x_{k,m_k}^2 + x_{1,m_1+1}^2 x_{2,m_2+1}^2 \cdots x_{k,m_k+1}^2}}$$

Definition 2. (see [3]) Suppose X is a path-connected Hausdorff space. The n-th symmetric product $\mathrm{Sym}^n X$ is just the quotient X^n/S_n. An n-valued multiplication is just a continuous map $\mu \colon X \times X \to \mathrm{Sym}^n X$. We set $\mu(x, y) = x * y$ for all $x, y \in X$.

(*The unit axiom*) We say that an n-valued multiplication $\mu \colon X \times X \to \mathrm{Sym}^n X$ satisfies the unit axiom if there exists an element $e \in X$ such that $e * x = x * e = [x, x, \ldots, x]$ for all $x \in X$.

(*Associativity*) An n-valued multiplication $\mu \colon X \times X \to \mathrm{Sym}^n X$ is said to be associative if $(x * y) * z = x * (y * z) \in \mathrm{Sym}^{n^2} X$ for all $x, y, z \in X$.

(*Inverse map*) An n-valued multiplication $\mu \colon X \times X \to \mathrm{Sym}^n X$ is said to have an inverse map if there exists a continuous map $\mathrm{inv} \colon X \to X$ such that $\mathrm{inv}(x) * x \ni e$ and $x * \mathrm{inv}(x) \ni e$ for all $x \in X$.

If the three axioms above are satisfied, then the quadruple $(X, \mu, e, \mathrm{inv})$ is called an n-valued topological group.

Let us give a more strict

Definition 3. An n-valued topological group $(X, \mu, e, \mathrm{inv})$ is called an n-valued Lie group if X is a compact smooth m-manifold and the inverse map is an involution.

If one takes the case $m_1 = \ldots = m_k = 1$ in the above Theorem 1 he obtains the torus T^m and the group $G_k, k = m$, acting on this compact Lie group by automorphisms. Therefore, the quotient $T^m/G_m = S^m$ is a new example of coset n-valued topological group, $n = 2^{m-1}$ (see definitions in [3]).

Theorem 4. *(See [4]) For any integer $m \geqslant 2$ the sphere S^m admits the structure of a 2^{m-1}-valued abelian Lie group.*

Remark 5. The case $m = 2$ here is just the classical 2-valued group on $\mathbb{C}P^1$, discovered by Buchstaber in 1990 (see [2]).

Before Theorem 4 there was known a very little number of nontrivial examples of n-valued Lie groups, $n \geqslant 2$. Here is the list:

(1) $\mathbb{C}P^m, m \geqslant 1, n = (m + 1)!$ (abelian)
(2) $\mathrm{Sym}^m(T^2), m \geqslant 2, n = m!$ (abelian)
(3) $S^3, n = 2$ (nonabelian)

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Boris Gurevich. *On asymptotic behavior of equilibrium measures associated with finite sub-matrices of an infinite nonnegative matrix: new examples*

Every finite irreducible stochastic matrix determines a unique shift invariant probability Markov measure on a sequence space with finite alphabet. The same holds for some infinite stochastic matrices, which are said to be positive recurrent. Vere-Jones [1] generalized the notion of a positive recurrence to all nonnegative matrices and proved that every positive recurrent matrix A (he used a slightly different term) is similar to a stochastic matrix, also positive recurrent, which in its turn determines a shift invariant probability Markov measure μ^A on a sequence space with infinite alphabet.

In [2] (see also [3]) μ^A is proved to be an equilibrium measure (as well as a Gibbs measure) corresponding to a nearest neighbor interaction in a one-dimensional countable spin statistical physics system. This interaction is determined by the matrix A, and while the equilibrium measure is well-defined for every non-negative A, it exists if and only if A is positive recurrent. The set of positive recurrent matrices consists of two subsets, stable positive and unstable positive matrices, this separation is also meaningful from a point of view of statistical physics: if A is stable positive and A_n an arbitrary increasing sequence of finite sub-matrices of A that goes to A (we call such a sequence exhaustive), then μ^{A_n} tends to μ^A as $n \to \infty$; at the same time for every unstable positive A studied up to now, there are two exhaustive sequences A_n and A'_n of finite sub-matrices such that $\mu^{A_n} \to \mu^A$ and $\mu^{A'_n} \to 0$. All known and some new results in this direction will be presented in the talk.

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Tatsuya Horiguchi. *The topology of Hessenberg varieties*

Hessenberg varieties are subvarieties of the full flag variety introduced by De Mari-Procesi-Shayman around 1990. Particular examples are Springer fibres, the Peterson variety, and the permutohedral variety. The topology of Hessenberg varieties is related with other research areas, such as geometric representation, the quantum cohomology of flag varieties, hyperplane arrangements, and Stanley's chromatic symmetric function in graph theory. In this talk, I will give a survey of recent developments on Hessenberg varieties. The goal of this talk is to advertise that Hessenberg varieties can be studied from various viewpoints.

Ilia Itenberg. *Finite real algebraic curves*

(Based on joint work with E. Brugallé, A. Degtyarev and F. Mangolte)

The talk is devoted to *finite* real plane algebraic curves, that is, real plane algebraic curves with finitely many real points. We study the following question: *what is the maximal possible number $\delta(k)$ of real points of such a curve provided that it has given degree $2k$?* This question is related to the first part of Hilbert's 16-th problem (topology of real algebraic varieties) and to Hilbert's 17-th problem (more precisely, positivity of real polynomials vs. their representation as sums of squares).

The Petrovsky inequalities result in the upper bound

$$\delta(k) \leqslant \frac{3}{2}k(k-1) + 1.$$

Currently, this bound is the best known. Furthermore, being of topological nature, it is sharp in the realm of pseudo-holomorphic curves.

The exact value of $\delta(k)$ is known only for $k \leqslant 4$. The upper (Petrovsky inequality) and lower bounds for a few small values of k are as follows:

k	1	2	3	4	5	6	7	8	9	10
$\delta(k) \leqslant$	1	4	10	19	31	46	64	85	109	136
$\delta(k) \geqslant$	1	4	10	19	30	45	59	78	98	123

The cases $k = 1, 2$ are obvious (union of two complex conjugate lines or conics, respectively). A finite real sextic with 10 real points was constructed by D. Hilbert. In the talk, we prove the lower bounds for $k = 4$ and 5, as well as the asymptotic bound

$$\delta(k) \gtrsim \frac{4}{3}k^2.$$

(The best previously known asymptotic lower bound $\delta(k) \gtrsim \frac{10}{9}k^2$ was obtained by M. D. Choi, T. Y. Lam and B. Reznick.)

One can modify the above question asking for the maximal possible number $\delta_g(k)$ of real points of a finite real plane algebraic curve of given degree $2k$ and given geometric genus g. The upper bound

$$\delta_g(k) \leqslant k^2 + g + 1$$

is produced by a strengthening of the Petrovsky inequalities. We show that this bound is sharp for $g \leqslant k - 3$ (in particular, the bound is sharp for rational curves of degree $\geqslant 6$).

The constructions use the Viro patchworking technique and *dessins d'enfants*. Most results extend to curves in ruled surfaces.

Alexander Kachurovskii. *Fejer sums and the von Neumann ergodic theorem*

The Fejér sums of periodic measures and the norms of the deviations from the limit in the von Neumann ergodic theorem are calculated, in fact, using the same formulas (by integrating the Fejér kernels), so this ergodic theorem is, in fact, a statement about the asymptotics of the growth of the Fejér sums at zero for the spectral measure of the corresponding dynamical system. As a result, well-known estimates for the rates of convergence in the von Neumann ergodic theorem can be restated as estimates of the Fejér sums at the point for periodic measures. For example, natural criteria for the polynomial growth and polynomial decrease in these sums can be obtained. On the contrary, available in the literature, numerous estimates for the deviations of Fejér sums at a point can be used to obtain new estimates for the rate of convergence in this ergodic theorem. For example, for many dynamical systems popular in applications, the rates of convergence in the von Neumann ergodic theorem can be estimated with a sharp leading coefficient of the asymptotic by applying S. N. Bernstein's more than hundred-year old results in harmonic analysis.

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Vadim Kaimanovich. *Free and totally non-free boundary actions*

The fundamental Rokhlin Lemma states that all measure preserving transformations are "nearly periodic". In spite of this, any ergodic component is still either aperiodic or finite, so that the freeness assumption is virtually automatic in the classical ergodic theory of single transformations (i.e., of $\mathbb{Z}$-actions). This may be the reason why the presence of non-trivial point stabilizers for actions of general countable groups was mostly considered as an unpleasant nuisance until quite recently.

Below I present several results (joint with Anna Erschler, see [4], [5] for further background, references and details) on the properties of the stabilizers of the action of a countable group G on its Poisson boundary $\partial_\mu G$ determined by a step distribution μ. The main ones are:

- Theorem 3: *a full description of the kernels for effectively free boundary actions* (i.e., those that become free after quotienting out their kernel), in particular, of the class of groups admitting a free boundary action. This result encompasses both the characterization of amenability in terms of the triviality of the Poisson boundary obtained by Furstenberg, Kaimanovich – Vershik and Rosenblatt in the 70s and the recent identification of the class of groups admitting random walks with a non-trivial boundary by Frisch – Hartman – Tamuz – Vahidi Ferdowsi [6].
- Theorem 8: *the first example in which the boundary action is not effectively free, and, moreover, is totally non-free*, i.e., the stabilizers of almost all points of the Poisson boundary $\partial_\mu G$ are different, and therefore $\partial_\mu G$ can be identified with the space of subgroups $\mathrm{Sub}(G)$. In this example G is the infinite symmetric group (the group of finitely supported permutations of a countable set) endowed with an appropriate probability measure μ.

The Vershik transform

By $\mathrm{Sub}(G)$ we denote the space of all subgroups of a countable group G. It is endowed with the natural compact topology (as a subset of the power set 2^G) and with the left continuous action of G by conjugations.

Definition 1 ([9], Definition 4, also see the announcement [8]). The Vershik transform of a measure class preserving action $G \circlearrowright (X, m)$ is the map

$$\mathrm{V} : X \to \mathrm{Sub}(G) , \qquad x \mapsto \mathrm{Stab}\, x = \{g \in G : gx = x\} .$$

It is measurable and equivariant, so that the image $\mathrm{V}(m)$ is a quasi-invariant Borel measure on $\mathrm{Sub}(G)$. We refer to the measure $\mathrm{V}(m)$ and the action $G \circlearrowright (\mathrm{Sub}(G), \mathrm{V}(m))$ as the Vershik transforms of the measure m and of the action $G \circlearrowright (X, m)$, respectively (Vershik himself used the term characteristic measure).

Vershik formulated the problem of the description of all purely non-atomic invariant probability measures on $\mathrm{Sub}(G)$ ([9], Problem 1), and solved it for the infinite symmetric group ([9], Theorem 1). Invariant probability measures on $\mathrm{Sub}(G)$ were also virtually simultaneously introduced and used in [3] and [1] (the first arXiv versions of these papers were posted in 2010 and 2011, respectively). It is in the latter paper that the currently predominant probabilistically flavoured term "invariant random subgroup" (IRS) for the invariant measures of the action $G \circlearrowright \mathrm{Sub}(G)$ was coined (one does not talk about "invariant random points" instead of invariant measures when dealing with dynamical systems though). See the surveys in [1], Section 1, and [7], Section 10, for more history and the current state of the IRS theory.

In the language of Definition 1 an action $G \circlearrowright (X, m)$ is free (mod 0) if the Vershik transform $\mathrm{V}(m)$ is the δ-measure at the identity subgroup. More generally, we remind that an action $G \circlearrowright (X, m)$ is called effectively free with the kernel $H \in \mathrm{Sub}(G)$ if H is normal, its action on the space (X, m) is trivial, and the arising action of the quotient group G/H is free, i.e., equivalently, if $\mathrm{V}(m) = \delta_H$. In particular, a free action is effectively free with the trivial kernel $H = \{e\}$, whereas a trivial action is effectively free with the full kernel $H = G$. The opposite situation when the Vershik transform leaves the action space intact instead of collapsing it onto a single point is described in the following

Definition 2 ([9], Definition–Theorem 1). An action $G \circlearrowright (X, m)$ is totally non-free if its Vershik transform is an isomorphism of measure spaces, i.e., if the stabilizers of almost all points of the action space are different.

The Vershik transform of a totally non-free action is concentrated on the set $\mathrm{SN}(G)$ of self-normalizing subgroups. Conversely, the action on $\mathrm{Sub}(G)$ is totally non-free with respect to any quasi-invariant measure concentrated on $\mathrm{SN}(G)$, and in this case the Vershik transform is the identity map.

Notice that although Vershik considered the measure preserving actions only, his definitions are applicable to actions with a quasi-invariant measure as well (and this is how we have formulated them above).

The Poisson boundary

It is classically known since von Neumann and Bogolyubov that amenable groups are precisely the ones for which any continuous action on a compact space admits an invariant measure. Any non-amenable group has actions without invariant measures, and the simplest example of this kind is the action of a finitely generated free group on its boundary ($\equiv$ the space of infinite irreducible words).

It is always very convenient to have a natural "reference measure" to be able to work in the measure category, and in the absence of a conventional invariant measure one can consider instead "invariance in the mean", i.e., with respect to a weighted average of translations. The phenomenon of the absence of (holonomy) invariant measures is also very well known in the theory of foliations, and it was the main motivation for Garnett's notion of a harmonic measure of a Riemannian foliation (in which the holonomy invariance is replaced with the invariance with respect to the leafwise Brownian motion in precisely the same way).

A measure λ on an action space of a group G is called stationary with respect to a probability measure μ on G (or, μ-stationary, in short) if it is preserved by the convolution with μ, i.e., $\lambda = \sum_g \mu(g) g\lambda$. If the measure μ is non-degenerate, i.e., its support generates the whole group G as a semi-group, then any μ-stationary measure is necessarily quasi-invariant. The Poisson boundary $\partial_\mu G$ is in a sense universal among all spaces with a μ-stationary measure, in particular, if $\partial_\mu G$ is trivial (reduces to a single point), then any μ-stationary measure is necessarily invariant. It is defined as the space of ergodic components of the time shift on the path space of the associated random walk and is endowed with the μ-stationary harmonic measure ν issued from the group identity.

A countable group admits a random walk with a trivial Poisson boundary if and only if it is amenable. On the other hand, the Poisson boundary is trivial for any non-degenerate random walk on any hyper-FC-central (in particular, abelian or nilpotent) group. In somewhat different terms, for any group G and any non-degenerate measure μ the action of the hyper FC-centre $\mathrm{FC}_{\lim}(G)$ on the Poisson boundary $\partial_\mu G$ is trivial.

We remind that the union $\mathrm{FC}(G)$ of all finite conjugacy classes of a group G is its normal subgroup called the FC-centre. If it is trivial, then G is said to be a group with infinite conjugacy classes (ICC). The hyper-FC-centre $\mathrm{FC}_{\lim}(G)$ is the limit of the transfinite upper FC-series which consists of the kernels of the homomorphisms $G \to G_\alpha$ along the transfinite sequence $G_{\alpha+1} = G_\alpha / \mathrm{FC}(G_\alpha)$ starting from $G = G_0$. Equivalently, $\mathrm{FC}_{\lim}(G)$ is the minimal normal subgroup of G with the property that the associated quotient group is ICC. A group is called hyper-FC-central if it coincides with its hyper-FC-centre. This class contains all nilpotent groups, and a finitely generated group is hyper-FC-central if and only if it is virtually nilpotent. If $\mathrm{FC}(G)$ is finite, then the quotient group $G/\mathrm{FC}(G)$ is ICC, and the hyper-FC-centre $\mathrm{FC}_{\lim}(G)$ coincides with $\mathrm{FC}(G)$. In this case $\mathrm{FC}(G)$ is the maximal finite normal subgroup of G, i.e., coincides with the finite radical of G.

A striking recent result of Frisch – Hartman – Tamuz – Vahidi Ferdowsi [6] shows that, in fact, any countable group which is *not* hyper-FC-central does admit non-degenerate measures with a non-trivial Poisson boundary.

The results

We simultaneously generalize both the aforementioned characterization of amenability in terms of the triviality of the Poisson boundary and [6].

Theorem 3. *The following conditions on a subgroup $H \subset G$ are equivalent:*

(i) *there is a non-degenerate measure μ on G such that the action $G \circlearrowright \partial_\mu G$ is effectively free with the kernel H;*

(ii) *The subgroup H is amenable, normal, and the quotient G/H is either ICC or trivial.*

Moreover, the measure μ in condition (i) *can be chosen to be symmetric.*

Corollary 4. *Any countable group G admits a non-degenerate symmetric measure μ, for which the action $G \circlearrowright \partial_\mu G$ is effectively free with the kernel* $\mathrm{FC}_{\lim(G)}$.

Corollary 5. *A countable group G admits a non-degenerate symmetric measure μ such that the action $G \circlearrowright \partial_\mu G$ is free if and only if G is ICC.*

In view of Theorem 3 it is natural to ask whether there exist Poisson boundaries for which the boundary action would not be effectively free. It seems that in all previously known examples the boundary action is, indeed, effectively free, although the verification of this in concrete situations may require some effort. Here we state it for hyperbolic groups.

Proposition 6. *If μ is a non-degenerate probability measure on a non-elementary word hyperbolic group G, then the action $G \circlearrowright \partial_\mu G$ is effectively free with the finite kernel* $\mathrm{FC}(G)$.

In the torsion free case this was proved in [2], Proposition 1.3. In the general case one can use the observation (due to Olshanskii) that the kernel of the action of G on the hyperbolic boundary ∂G is its maximal finite normal subgroup. Therefore, it coincides with the FC-centre $\mathrm{FC}(G)$. Then the quotient group $G/\mathrm{FC}(G)$ is also hyperbolic and has the same Poisson boundary as G because of the finiteness of $\mathrm{FC}(G)$. Alternatively, Proposition 6 follows from the following general property:

Proposition 7. *Let μ be a non-degenerate probability measure on a countable group G, and let (B, λ) be an equivariant quotient of the Poisson boundary $(\partial_\mu G, \nu)$. If the point stabilizers of the action $G \circlearrowright (B, \lambda)$ are almost surely finite, then* $\mathrm{FC}(G)$ *is finite, and the action of G on both spaces $(B, \lambda), (\partial_\mu G, \nu)$ is effectively free with the kernel* $\mathrm{FC}(G)$.

Indeed, it is well-known that if μ is a non-degenerate measure on a word hyperbolic group G, then there exists an equivariant quotient map from $\partial_\mu G$ to the hyperbolic boundary ∂G, whereas the stabilizers of the action $G \circlearrowright \partial G$ are finite outside of a countable set of the fixed points of hyperbolic elements.

Finally, we shall give an example of a totally non-free Poisson boundary. Let $G = \mathfrak{S}(X)$ be the infinite symmetric group realized as the group of all finite permutations of an infinite countable set X. It is convenient to define the group operation on G in the *postfix notation* by putting $(g_1 g_2)(x) = g_2(g_1(x))$ and to let G act on X on the right as $x.g = g(x)$. Then, given a finite alphabet Θ, one has the left action $g\boldsymbol{\theta}(x) = \boldsymbol{\theta}(x.g)$ on the space Θ^X of all Θ-valued functions (configurations) on X.

Theorem 8. *Let $\boldsymbol{\chi} \in \Theta^X$ be a configuration on an infinite countable set X with infinite preimage sets. There exists a non-degenerate symmetric probability measure μ on the infinite symmetric group $G = \mathfrak{S}(X)$ such that*

(i) *for almost every sample path (g_n) of the associated random walk the translates $g_n \boldsymbol{\chi}$ pointwise converge to a random limit configuration $\boldsymbol{\chi}_\infty$;*

(ii) *the space of configurations Θ^X endowed with the resulting limit distribution is isomorphic to the Poisson boundary $\partial_\mu G$;*

(iii) *the action $G \circlearrowright \partial_\mu G$ is totally non-free.*

(i) This property is easy to arrange and just means that for any initial point $x \in X$ the sample path $(x.g_n)$ of the induced random walk on X is almost surely eventually confined to the same element of the preimage partition of $\boldsymbol{\chi}$ (which depends both on x and (g_n)).

(ii) This is the most difficult part of the argument. One has to make sure that the behaviour of the random walk at infinity is completely described just by the limit configurations $\boldsymbol{\chi}_\infty$, which is achieved by a judicious choice of the measure μ.

(iii) The stabilizer Stab $\boldsymbol{\theta}$ of a configuration $\boldsymbol{\theta} \in \Theta^X$ is uniquely determined by its preimage partition. Therefore, the Vershik transform V : $\boldsymbol{\theta} \mapsto$ Stab $\boldsymbol{\theta}$ has finite preimages, which (in the case of the Poisson boundary) is only possible if V is an isomorphism.

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Naohiko Kasuya. *Non-Kähler complex structures on* $\mathbb{R}^4$

Problem and Main Theorem

The author, Antonio J. Di Scala and Daniele Zuddas [2] constructed the first examples of non-Kähler complex surfaces which are diffeomorphic to $\mathbb{R}^4$. In this abstract, we give a brief explanation of the construction of the complex surfaces.

A complex manifold (M, J) is said to be Kähler if there exists a symplectic form ω compatible with the complex structure J, i.e., a symplectic form ω satisfying the following two conditions:

(1) $\omega(u, Ju) > 0$ for any nonzero vector $u \in TM$ (tamedness),
(2) $\omega(u, v) = \omega(Ju, Jv)$ for any vectors $u, v \in TM$ (J-invariance).

Since any complex manifold is locally Kähler, the problem is the global existence of a compatible symplectic form. In this sense, Kählerness or non-Kählerness is a global property of a complex manifold. Indeed, it is well-known that a compact complex surface is Kähler if and only if its first Betti number is even. In general, it follows from the Hodge theory that each odd degree Betti number of a compact Kähler manifold is even.

In the non-compact case, however, such statements are no longer true. It is easily checked that any open oriented connected 4-manifold admits a Kähler complex structure. Also in higher dimensions, there exist Stein manifolds whose first Betti number is odd by Eliashberg's theorem [4]. Thus, if one wants to construct non-compact non-Kähler complex manifolds, only topological informations are useless. On the other hand, we have the following lemma, which is one of important keys to our construction.

Lemma 1. *A complex manifold which contains a compact holomorphic curve representing trivial second homology is non-Kähler.*

The proof of this lemma is quite easy. If a Kähler manifold (M, J) contains a compact holomorphic curve C, then the integration $\mathrm{Int}_C\, \omega$ is positive for any compatible symplectic form ω. Hence, a compact holomorphic curve of a Kähler manifold represents a nontrivial second homology. Now, our problem is the following.

Problem 2. Is there any non-Kähler complex structure on $\mathbb{R}^{2n}$?

When $n = 1$, the answer is obviously negative, since any complex curve is Kähler. On the other hand, it has been showed by Calabi and Eckmann [1] that the answer is affirmative when $n \geqslant 3$. They constructed a uncountable family of complex structures on the product of two odd dimensional spheres in the following way.

Let $h_p\colon S^{2p+1} \to \mathbb{C}P^p$ and $h_q\colon S^{2q+1} \to \mathbb{C}P^q$ be the Hopf fibrations, and take the product map $h_{p,q}\colon S^{2p+1} \times S^{2q+1} \to \mathbb{C}P^p \times \mathbb{C}P^q$, which is a T^2 fiber bundle. Now take the standard atlas $\{U_{1i} \times U_{2j}\}$ $(0 \leqslant i \leqslant p, 0 \leqslant j \leqslant q)$ of $\mathbb{C}P^p \times \mathbb{C}P^q$, and the elliptic curve $S(\tau)$ of modulus τ. By an explicit holomorphic gluing of $U_{1i} \times U_{2j} \times S(\tau)$, they constructed a complex structure on $S^{2p+1} \times S^{2q+1}$ such that $h_{p,q}$ is a holomorphic elliptic bundle. This is the famous Calabi-Eckmann manifold $M_{p,q}(\tau)$. By removing a point on each sphere and taking the product, an open subset $E_{p,q}(\tau) \subset M_{p,q}(\tau)$ which is diffeomorphic to $\mathbb{R}^{2p+2q+2}$ is obtained. If p and q are both positive, then

$E_{p,q}(\tau)$ contains most elliptic fibers of $h_{p,q}$. Hence, $E_{p,q}(\tau)$ gives a non-Kähler complex structure on $\mathbb{R}^{2p+2q+2}$ by Lemma 1. Notice that their arguments are not valid when $n = 2$, since $M_{0,1}(\tau)$ is a Hopf surface and $E_{0,1}(\tau)$ is an open subset of $\mathbb{C}^2$. Therefore, another approach is needed for the case where $n = 2$.

In order to manage the case where $n = 2$, we focus on some example in the 4-dimensional topology found by Yukio Matsumoto [7] and Kenji Fukaya. It is a genus-one achiral Lefschetz fibration from S^4 to S^2 which has only two singularities of opposite signs. We call it the Matsumoto-Fukaya fibration. Since it contains one negative Lefschetz singularity, there seems no relation with complex geometry at first glance. However, if we remove a 4-ball containing the only one negative singularity from S^4, we can construct a complex structure on the complement such that the restriction of the fibration is holomorphic. Since the complement is diffeomorphic to $\mathbb{R}^4$ and contains elliptic curves as the regular fibers, it gives a non-Kähler complex structure on $\mathbb{R}^4$. This is the outline of our construction. Namely, our main theorem is the following.

THEOREM 3. *For any pair (ρ_1, ρ_2) of real numbers satisfying $1 < \rho_2 < \rho_1^{-1}$, there exist a complex manifold $E(\rho_1, \rho_2)$ and a surjective holomorphic map $f : E(\rho_1, \rho_2) \to \mathbb{C}P^1$ such that*

(1) *$E(\rho_1, \rho_2)$ is diffeomorphic to $\mathbb{R}^4$,*
(2) *the only singular fiber $f^{-1}(0)$ is an immersed holomorphic sphere with one node,*
(3) *a regular fiber is either an embedded holomorphic torus or an embedded holomorphic cylinder.*

Moreover, if $E(\rho_1, \rho_2)$ and $E(\rho_1', \rho_2')$ are biholomorphic, then $(\rho_1, \rho_2) = (\rho_1', \rho_2')$.

In the next section, we show the construction of $E(\rho_1, \rho_2)$. First, as an application of the Matsumoto-Fukaya fibration, we obtain a nontrivial topological decomposition of $\mathbb{R}^4$ into two pieces which gives the blueprint for the construction. According to the decomposition, we prepare some two pieces of complex surfaces and glue them analytically to obtain the complex surface.

The construction of $E(\rho_1, \rho_2)$

The Matsumoto-Fukaya fibration is a genus-one achiral Lefschetz fibration $f : S^4 \to S^2$, having two critical points of opposite signs. It can be seen as the composition of the Hopf fibration $h : S^3 \to S^2$ and its suspension $\Sigma h : \Sigma S^3 \to \Sigma S^2$.

Let $a_1 \in S^2$ be the positive critical value of f, and let a_2 be the negative one. Decompose the base space S^2 as the union of two disks D_1 and D_2 such that $a_j \in \operatorname{Int} D_j$, and put $N_j = f^{-1}(D_j)$. Then N_j is a tubular neighborhood of $F_j = f^{-1}(a_j) \subset S^4$ and the restriction $f_j = f_{|N_j} : N_j \to D_j \cong B^2$ is the (achiral) Lefschetz fibration having only one critical point. Now we remove a neighborhood $X \cong B^4$ of the negative singularity from N_2 so that the restriction of f_2 to $N_2 - \operatorname{Int} X$ is the total space of a trivial annulus bundle over D_2. Since $N_2 - \operatorname{Int} X$ is diffeomorphic to $B^2 \times A$, where $A = S^1 \times [0, 1]$, we obtain a decomposition of $S^4 - \operatorname{Int} X \cong B^4$ into the two pieces N_1 and $B^2 \times A$. Moreover, looking at the orientation reversing diffeomorphism between ∂N_1 and ∂N_2, we obtain the following (see [2] for details).

PROPOSITION 4. *If we glue $B^2 \times A$ to N_1 along $S^1 \times A$ so that for each $t \in \partial B^2 = -\partial D_1^2 \cong S^1$, the annulus $\{t\} \times A$ embeds in $f^{-1}(t) \cong T^2$ as a thickened meridian, and it rotates in the longitude direction once when $t \in S^1$ rotates once, then the resulting manifold is diffeomorphic to B^4, and so the interior is diffeomorphic to $\mathbb{R}^4$.*

We realize these pieces and the gluing by complex manifolds as follows.

We use the notations $\Delta(r_0, r_1) = \{z \in \mathbb{C} \mid r_0 < |z| < r_1\}$ and $\Delta(r) = \{z \in \mathbb{C} \mid |z| < r\}$. Fix positive numbers ρ_0, ρ_1 and ρ_2 such that $1 < \rho_2 < \rho_1^{-1} < \rho_0^{-1}$. We consider the quotient $(\mathbb{C}^* \times \Delta(0, \rho_1))/\mathbb{Z}$, where for any $n \in \mathbb{Z}$, the action is given by $n \cdot (z, w) = (zw^n, w)$. This elliptic fibration extends over $\Delta(\rho_1)$. Let us denote the completion by W_1. It is a fibered tubular neighborhood of a singular elliptic curve of type I_1 in the sense of Kodaira (see [6]). On the other hand, we put $W_2 = \Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$. W_1 and W_2 are the complex analytic models for $\operatorname{Int} N_1$ and $\operatorname{Int} N_2 - X$.

Next, we want to glue W_1 with W_2 analytically along $\Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \subset W_2$, so that the attaching map is isotopic to that of the Matsumoto-Fukaya fibration. We take an attaching region V in W_1 which is biholomorphic to the product $\Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1})$ as follows. Define a multi-valued holomorphic function $\varphi : \Delta(\rho_0, \rho_1) \to \mathbb{C}^*$ by

$$\varphi(w) = \exp\left(\frac{1}{4\pi i}(\log w)^2 - \frac{1}{2}\log w\right).$$

Then, it induces a holomorphic section of the elliptic fibration $f_1 : W_1 \to \Delta(\rho_1)$ restricted to $(\mathbb{C}^* \times \Delta(0, \rho_1))/\mathbb{Z}$. Hence we can define $V \subset W_1$ as the image of the holomorphic embedding $\Phi : \Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \to W_1$ given by $\Phi(z, w) = [(z\varphi(w^{-1}), w^{-1})]$. Using this embedding, we can define the complex manifold $E(\rho_1, \rho_2)$ by

$$E(\rho_1, \rho_2) = W_1 \cup_V W_2.$$

By construction, it is diffeomorphic to $\mathbb{R}^4$ and contains elliptic curves inside W_1. Hence it is non-Kähler by Lemma 1. Moreover, the elliptic fibration $f_1 : W_1 \to \Delta(\rho_1)$ and the second projection $f_2 : W_2 \to \Delta(\rho_0^{-1})$ are fit together to define the surjective holomorphic map $f\colon E(\rho_1,\rho_2) \to \mathbb{C}P^1$, where we see $\mathbb{C}P^1$ as the gluing of $\Delta(\rho_1)$ and $\Delta(\rho_0^{-1})$. Now, it is not so difficult to classify all the compact holomorphic curves in $E(\rho_1,\rho_2)$ and to prove that $E(\rho_1,\rho_2)$ and $E(\rho_1',\rho_2')$ are not biholomorphic if $(\rho_1,\rho_2) \neq (\rho_1',\rho_2')$.

There are many properties of $E(\rho_1,\rho_2)$ which have been already proved, but we omit them here for want of space (see [3,5] for details).

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Viatcheslav Kharlamov. *Real rational symplectic 4-manifolds*

The foundational results of Gromov–Taubes and Seiberg–Witten allowed to understand rather well the structure of rational and ruled symplectic 4-manifolds and, in particular, to prove that every such symplectic manifold is Kähler. The aim of our joint work with V. Shevchishin (work in progress) is to show at what extent the latter result can be extended to rational symplectic manifolds equipped with an anti-symplectic involution. Our approach is based on appropriate real versions of Lalonde–McDuff inflation and rational blow-ups. It shows, in particular, that the classification of anti-symplectic involutions on real rational symplectic 4-folds is very similar to that of the classification of real rational surfaces.

Cheikh Khoule. *Convergence of contact structures into integrable hyperplanes fields*

In this note we give a necessary and sufficient condition to the convergence of contact structures into codimension 1 foliation in $2n+1$-dimensional compact manifold. And I generalize of certain result of J. B. Etnyre.

Yuri Kifer. *Limit theorems for nonconventional polynomial arrays*

I'll discuss ergodic and limit theorems for sums of the form

$$\sum_{n=1}^{N}\prod_{j=1}^{\ell} T^{P_j(n,N)} f_j$$

(and more general ones) where $P_j(n,N)$, $j=1,...,\ell$, are polynomials in n and N taking on integer values, T is an invertible measure preserving transformation satisfying certain (depending on the problem) mixing assumptions and f_j, $j=1,...,\ell$, are bounded measurable (or more regular depending on the problem) functions.

Alexander Kolpakov. *A hyperbolic counterpart to Rokhlin's cobordism theorem*

(Joint work with Michelle Chu, University of California, Santa Barbara, USA)

A classical result by V. Rokhlin states that every compact orientable 3-manifold bounds a compact orientable 4-manifold, and thus the three-dimensional cobordism group is trivial. One can recast the question of bounding in the setting of hyperbolic geometry, which generated plenty of research directions over the past decades.

A hyperbolic manifold is a manifold endowed with a Riemannian metric of constant sectional curvature -1. Here and below all manifolds are assumed to be connected, orientable, complete, and of finite volume, unless otherwise stated. We refer to [14] for the definition of an arithmetic hyperbolic manifold.

A hyperbolic n-manifold $\mathscr{M}$ bounds geometrically if it is isometric to $\partial\mathscr{W}$, for a hyperbolic $(n+1)$-manifold $\mathscr{W}$ with totally geodesic boundary.

Indeed, some interest in hyperbolic manifolds that bound geometrically was kindled by the works of D. Long, A. Reid [10], [11], [12] and B. Niemershiem [15], motivated by a preceding work of M. Gromov [5], [6] and a question by F. Farrell and S. Zdravkovska [4]. This question is also related to hyperbolic instantons, as described by J. Ratcliffe and S. Tschantz [18], [19].

As [10] shows many closed hyperbolic 3-manifolds do not bound geometrically: a necessary condition is that the eta invariant of the 3-manifold must be an integer. The first known closed hyperbolic 3-manifold that bounds geometrically was constructed in [18] and has volume of order 200.

The first examples of knot and link complements that bound geometrically were produced by L. Slavich in [16], [17]. However, [8] implies that there are plenty of cusped hyperbolic 3-manifolds that cannot bound geometrically, with the obstruction being the geometry of their cusps.

In [1], M. Belolipetsky, T. Gelander, A. Lubotzky, and A. Shalev showed that the asymptotic growth rate of the number $\alpha_n(v)$ of all orientable arithmetic hyperbolic manifolds, up to isometry, with respect to volume v is super-exponential, in all dimensions $n \geqslant 3$. That is, there exist constants $A, B, C, D > 0$ such that $Av^{Bv} \leqslant \alpha_n(v) \leqslant Cv^{Dv}$. In our present work, we use the ideas of [1], [12], [13] together with the more combinatorial colouring techniques from [9] in order to prove the following facts:

Theorem 1. *Let $\beta_n(v)$ = the number of non-isometric compact arithmetic hyperbolic n-manifolds of volume $\leqslant v$ that bound geometrically. Then, if $2 \leqslant n \leqslant 8$, we have that $Av^{Bv} \leqslant \beta_n(v) \leqslant Cv^{Dv}$, for some constants $A, B, C, D > 0$.*

Theorem 2. *Let $\gamma_n(v)$ = the number of non-isometric cusped arithmetic hyperbolic n-manifolds of volume $\leqslant v$ that bound geometrically. Then, if $2 \leqslant n \leqslant 19$, or $n = 21$, we have that $Av^{Bv} \leqslant \gamma_n(v) \leqslant Cv^{Dv}$, for some constants $A, B, C, D > 0$.*

The proofs of both theorems above rely heavily on reflectivity of certain quadratic forms studied by È. Vinberg, I. Kaplinskaya [7], [20] and V. Bugaenko [2], [3].

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Sergey Komech. *Random averaging in ergodic theorem and boundary deformation rate in symbolic dynamics*

(Based on joint work with B. M. Gurevich and A. A. Tempelman)

A connection between the deformation rate of a small set boundary in the phase space of a dynamical system and the metric entropy of the system was claimed (not too rigorously) in physics literature [8], [9] and later studied for some discrete time systems in the mathematical papers [1], [4], [2].

First it was done for symbolic systems, specifically for topological Markov shifts and synchronized systems. We keep track of the boundary deformation of a small ball during a large time interval, our arguments resembled the proof of the Shannon–McMillan–Breiman theorem. Namely, let $X = (X, \rho)$ be a metric space and $B \subset X$. We denote the ε-neighborhood of B by $\mathscr{O}_\varepsilon(B)$. When $B = \{x\}$ is a point, $\mathscr{O}_\varepsilon(B)$ is a ball of radius ε centered at x, we denote it by $B(x, \varepsilon)$. Let $S : X \to X$ be a continuous map and $h_\mu(S)$ be the entropy of S with respect to an S-invariant probability measure μ. For the case of symbolic systems the following L_1 convergence holds (see [1], [4]):

$$\lim_{\varepsilon\to 0} \frac{1}{t(\varepsilon)} \ln \frac{\mu(\mathscr{O}_\varepsilon(S^{t(\varepsilon)}B(x,\varepsilon)))}{\mu(B(x,\varepsilon))} = h_\mu(S). \tag{1}$$

It is seen from (1) that t depends on ε. This is inevitable if we want to obtain valuable results: when first passing to the limit in t and then in ε, as is assumed in [8], we can get for large t the whole space X as $\mathscr{O}_\varepsilon(S^t B)$, and the ratio on the left-hand side of (1) ceases dependance on t. We use $t(\varepsilon)$ that satisfies the following conditions:

$$\lim_{\varepsilon\to 0} t(\varepsilon) = \infty, \quad \lim_{\varepsilon\to 0} t(\varepsilon)/\log\varepsilon = 0. \tag{2}$$

This dependence was introduced in [1] and appeared in all subsequent works on this subject.

For a smooth system on a Riemannian manifold it is reasonable to measure the boundary distortion in terms of the Lebesgue measure (even if it is not invariant) and to study the asymptotic behavior of the quantity

$$\frac{1}{t(\varepsilon)} \ln \frac{\mu(O_\varepsilon(f^{t(\varepsilon)}B(x,\varepsilon)))}{\mu(B(x,\varepsilon))}, \tag{3}$$

where f is a diffeomorphism of a compact Riemannian manifold M, μ is the Lebesgue measure (Riemannian volume), and t depends on ε as in (2).

For a torus automorphism with invariant Lebesgue measure, the convergence of (3) to the entropy at each point of the torus was proved in [5].

In [3] it was proved that (3) tends, at least for Anosov diffeomorphisms, to the sum of the positive Lyapunov exponents of an arbitrary f-invariant ergodic measure ν almost everywhere with respect to this measure. Therefore, if f-invariant measure ν is a Sinai-Ruelle-Bowen measure, then (3) tends to the entropy a.e.

Only quite recently a class of continuous time systems was considered (see [3]). These systems are suspension flows over discrete time dynamical systems (maps in the base). It turned out that in order to prove the abovementioned relationship between the deformation rate and the entropy for the flow, one has to prove this for the base map in a more general setting, namely, when the observation time depends on the center of the ball under consideration. This problem arises since the number of visits to the base along time depends on the center of the ball. So we revisited the results for Markov shifts and synchronized systems (although a Markov shift is a particular case of a synchronized system, the conditions we impose on invariant measures in these two cases are slightly different) and modified it in the following way, see [6]:

$$\lim_{\varepsilon\to 0} \frac{1}{t(x,\varepsilon)} \ln \frac{\mu(\mathscr{O}_\varepsilon(S^{t(x,\varepsilon)}B(x,\varepsilon)))}{\mu(B(x,\varepsilon))} = h_\mu(S), \tag{4}$$

where $t(x, \varepsilon)$ could "slightly" vary in x.

In order to establish this result it was necessary to prove a version of the statistical ergodic theorem with averaging over a random sequence of sets.

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Konstantinos Kourliouros. *Local classification of symplectic structures with respect to a Lagrangian variety*

We consider here a converse to a classical theorem of A. B. Givental concerning the local analytic classification (stability) of symplectic structures under diffeomorphisms preserving a Lagrangian variety. In particular we show that the class of (a primitive of) the symplectic form in the de Rham cohomology of the Lagrangian variety is a complete invariant of the classification problem (as conjectured by Y. C. de Verdière for plane curves), as long as the Lagrangian variety satisfies a formal-analytic comparison theorem in its 1st cohomology. We then present classes of Lagrangian varieties which satisfy this condition and others that don't. The problem is related to resolution of singularities and the vanishing of certain cohomology classes supported on the singular locus.

Tatyana Kozlovskaya. *Cyclically presented Sieradski groups and 3-manifolds*

We study the problem if a given presentation of group is geometric, i. e. it corresponds to a spine of a closed orientable 3-manifold. The Sieradski groups are defined in [1] by the following presentation: $S(m) = \langle x_1, x_2, \ldots, x_m \mid x_i x_{i+2} = x_{i+1}, i = 1, \ldots m\rangle$, where all subscripts are taken by $\bmod m$. The groups $S(m,p,q) = \langle x_1, \ldots, x_m \mid x_i\, x_{i+q} \cdots x_{i+(q-1)dq-q}\, x_{i+(q-1)dq} = x_{i+1}\, x_{i+q+1} \cdots x_{i+(q-1)dq-q+1}, i = 1, \ldots, m\rangle$ are called the generalised Sieradski groups. All subscripts are taken by $\bmod m$. Parameters p and q are co-prime integers such that $p = 1 + dq$, $d \in \mathbb{Z}$.

It was shown by Cavicchioli, Hegenbarth and Kim in [2] that Sieradski group presentation $S(m,p,q)$ correspond to spine of some 3 manifold which we denote by $M(m,p,q)$. Moreover, $M(m,p,q)$ are m-fold cyclic coverings of S^3 branched over the torus (p,q)-knot. If $m = 2n$ these manifolds have also symmetry of order n corresponding to another cyclic presentation of the same groups. Thus, $M(2n,3,2)$ are the n-fold cyclic coverings of the lens space $L(3,1)$ [3] and $M(2n,5,2)$ are the n-fold cyclic coverings of the lens space $L(5,1)$ [4]. The corresponding cyclic presentations are described in [3] and [4].

We consider the generalised Sieradski group S(2n, 7, 2) $n \geqslant 1$.

Theorem 1. *For $n \geqslant 1$ group S(2n,7,2) has a presentation with n generators $y_0, y_1, \ldots, y_{n-1}$ and defining relations*

$$\{y_i y_{i+1} y_{i+2} y_{i+3} y_{i+3} y_{i+4} y_{i+5} y_i y_{i+5}{}^{-1} y_{i+4}^{-1} y_{I+3}^{-1} y_{i+2} y_{i+3} y_{i+4}$$
$$y_{i+5} y_{i+4}^{-1} y_{i+3}^{-1} y_{i+2}^{-1} y_{i+1} y_{i+2} y_{i+3} y_{i+4} y_{i+3}^{-1} y_{i+2}^{-1} y_{y+1}^{-1}, \qquad i = 0, \ldots, n-1\}.$$

These presentation is geometric and corresponds to n-fold cyclic branched covering of the lens space $L(7,1)$.

For small n manifolds $S(2n,7,2)$ can be distinguished by using 3-Manifold Recognizer [5]. Namely, $S(2,7,2)$ is the Seifert manifold $(S^2, (2,1), (7,2), (7,2), (1,-1))$ and $S(6,7,2)$ is the Seifert manifold $(S^2, (6,1), (7,3), (7,3), (1,-1))$.

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Roman Krutowski. *Basic cohomology of moment-angle manifolds*

For any fan we may construct a corresponding moment-angle manifold which admits complex structures. These manifolds represent a wide family of non-Kähler manifolds with a holomorphic action of a compact torus. This action induces a canonical foliation on a moment-angle manifold which in case of a rational fan becomes a locally trivial bundle over a toric variety corresponding to the same fan. During the talk I am going to show how to compute basic de Rham and Dolbeault cohomology of any complex moment-angle manifold with respect to the canonical foliation. Furthermore, using this data I will describe how to approach the calculation of Dolbeault cohomology of complex moment-angle manifolds.

Victor Krym. *The Schouten curvature tensor for a nonholonomic distribution in sub-Riemannian geometry can be identical with the Riemannian curvature on a principal bundle*

1. The Schouten curvature

Distribution on a smooth manifold N is a family of subspaces $\mathscr{A}(x) \subset T_xN$, $x \in N$. General theory of the variational calculus with nonholonomic restrictions $\varphi(t, x, \dot{x}) = 0$ was published by G.A. Bliss [2]. If the restrictions are linear for velocities, $\omega_x(\dot{x}) = 0$, where ω is a 1-form, we get distributions [2,3].

The Romanian mathematician G. Vranceanu first introduced the term of the nonholonomic structure on a Riemannian manifold in 1928 [4]. The Dutch mathematician J.A. Schouten defined the connection and appropriate curvature tensor for horizontal vector fields on a distribution in 1930 [5,6,7].

Let us consider the distribution $\mathscr{A}$ of dimension m on a smooth manifold N of dimension n. Locally the distribution can be defined by its basis vector fields

$$e_k = \partial_k - \sum_{\alpha=m+1}^{n} A_k^\alpha \partial_\alpha, \quad k = 1, \ldots, m, \tag{1}$$

or by the family of differential 1-forms $\omega^\alpha = \sum_{s=1}^{m} A_s^\alpha dx^s + dx^\alpha$, $\alpha = m{+}1, \ldots, n$. The Lie brackets $[e_i, e_j] = \sum_{\alpha=m+1}^{n} F_{ij}^\alpha \partial_\alpha$, where F_{ij}^α is called the tensions tensor.

The metric tensor of a distribution is a square form $\langle \cdot, \cdot \rangle_x$ defined for horizontal vectors at any point $x \in N$. We study the *internal geometry* of a distribution therefore this metric should not be extended for all tangent bundle TN. Locally

$$g_{ij}(x) = \langle e_i, e_j \rangle_x, \quad i, j = 1, \ldots, m. \tag{2}$$

To define covariant differentiation ∇ on the distribution, we must introduce a symmetric Riemannian connection. The property of being Riemannian is defined in a standard way:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{3}$$

while the symmetry condition must be modified as [7]

$$\nabla_X Y - \nabla_Y X = \mathrm{pr}([X, Y]), \tag{4}$$

where $\mathrm{pr} = \sum_{k=1}^{m} e_k \otimes dx^k$ is the horizontal projection on the distribution [8,9,10]. To make this projection invariant with respect to transformations of coordinates, we must impose the following constraints: $\dfrac{\partial x^k}{\partial y^\alpha} = 0$,

$k = 1,\ldots,m$, $\alpha = m{+}1,\ldots,n$, and $\big(\frac{\partial x^\beta}{\partial y^\alpha}\big)_{\alpha,\beta=m+1,\ldots,n}$ is the identity matrix. The coordinates x^α are called *verticals*. Since these matrices form a group our definition is correct. Formally this is a restriction on a smooth structure of the manifold, but in fact a distribution with this projection is a principal bundle.

THEOREM 1. *Let a distribution be equipped with the horizontal projection. If this distribution and its metric tensor do not depend on the vertical coordinates then the Riemannian and pr-symmetric connection* ∇ *matches the Riemannian and symmetric connection of a manifold with metric* (2) *[11,12].*

For any point $x \in N$ and any three vectors $u, v, w \in \mathscr{A}(x)$ *the curvature map of a distribution* $\mathscr{A}$ at a point x is defined by the Schouten tensor [5,6,7]

$$R(u,v)w = \nabla_{\tilde u}\nabla_{\tilde v}\tilde w - \nabla_{\tilde v}\nabla_{\tilde u}\tilde w - \nabla_{\mathrm{pr}[\tilde u,\tilde v]}\tilde w - \mathrm{pr}\big[(1-\mathrm{pr})[\tilde u,\tilde v],\tilde w\big], \tag{5}$$

where $\tilde u$, $\tilde v$, $\tilde w$ are horizontal vector fields on a neighbourhood of x such that $\tilde u(x) = u$, $\tilde v(x) = v$, $\tilde w(x) = w$. The curvature does not depend on the way of expansion of u, v, w to vector fields.

THEOREM 2. *Let a distribution be equipped with the horizontal projection. If this distribution and its metric tensor do not depend on the vertical coordinates then its Schouten tensor matches the Riemannian curvature tensor of a manifold with metric* (2) *[11,12].*

If the distribution satisfies the conditions of Theorem 2 we say that the distribution satisfies the *cyclicity condition.*

2. The equations of geodesics

Let the distribution $\mathscr{A}$ on a manifold N^n be defined by the differential 1-forms $\omega^\alpha = \sum\limits_{s=1}^{m} A_s^\alpha dx^s + dx^\alpha$, $\alpha = m{+}1,\ldots,n$. The geodesics equations for a distribution with the cyclicity condition are [13]

$$a_0\frac{D\gamma'}{dt} + \sum_{\alpha=m+1}^{n} l_\alpha \hat F^\alpha\gamma' = 0, \tag{6}$$

where $\frac{D}{dt}$ – covariant derivative, (a_0, l) – the Lagrange multipliers, which cannot be altogether zero. Operator $\hat F^\alpha$ is the tensions tensor $F_{ij}^\alpha = \partial_j A_i^\alpha - \partial_i A_j^\alpha$ with the second index raised by the inverse metric tensor of the distribution. A geodesic γ is called *normal (or regular)*, iff there is the only one set of multipliers (a_0, l) with $a_0 = 1$ for γ. If $a_0 = 0$ the geodesic is *abnormal*. Here we consider regular geodesics only.

3. Variations and equations of variations

Let $\gamma : [t_0, T] \to N$ be a C^1-smooth horizontal path, $\omega(\gamma') = 0$, where ω is a 1-form. *Horizontal variation* of γ is a 1-parametric family of maps $\sigma(\cdot,\tau) : [t_1, t_2] \to N$, $|\tau| < \varepsilon$, if there are continuous vector fields $X = \frac{\partial\sigma}{\partial t}$, $Y = \frac{\partial\sigma}{\partial\tau}$, the central line is just $\sigma(t, 0) = \gamma(t)$ and the field X is horizontal, $X(t,\tau) \in \mathscr{A}(\sigma(t,\tau))$ and there are continuous second derivatives $\frac{\partial^2\sigma}{\partial t\partial\tau}$, $\frac{\partial^2\sigma}{\partial\tau\partial t}$ for all allowed t and τ.

Since the horizontality condition $\omega^\alpha(\frac{\partial\sigma}{\partial t}) = 0$ is fulfilled as an identity we can differentiate it for τ. We get $\sum\limits_{j,k=1}^{n} \frac{\partial\omega_k^\alpha}{\partial x^j}\frac{\partial\sigma^j}{\partial\tau}\frac{\partial\sigma^k}{\partial t} + \sum\limits_{k=1}^{n}\omega_k^\alpha\frac{\partial^2\sigma^k}{\partial t\partial\tau} = 0$. At $\tau = 0$ we obtain *the equations of variations along* γ*:*

$$\sum_{k=1}^{n}\omega_k^\alpha\frac{dY^k}{dt} + \sum_{j,k=1}^{n}\frac{\partial\omega_k^\alpha}{\partial x^j}\gamma'^k Y^j = 0, \quad \alpha = m{+}1,\ldots,n. \tag{7}$$

The vector field $Y(\cdot, 0)$ along γ is denoted as Y. These equations are of the form $\Phi^\alpha(Y', Y) = 0$, $\alpha = m{+}1,\ldots,n$. This is a system of $n - m$ differential equations. Since the rank of the matrix (ω_k^α) is $n - m$, the horizontal projection of the field Y is arbitrary and the vertical components Y^α are defined by the initial conditions.

4. The Jacobi equation

Let us assume that both the distribution and the metric tensor of the distribution are independent of vertical coordinates (the cyclicity condition). Hence the Lagrange multipliers are time-independent (constant). The Jacobi equation for this class of distributions can be written in geometric covariant form. The distribution is assumed to be totally nonholonomic.

Consider the minimization problem for the energy functional $J(\gamma) = \frac{1}{2}\int_{t_0}^{T}\langle\gamma',\gamma'\rangle\, dt$ for horizontal paths with fixed endpoints and time. This is the Lagrange problem. We shall omit the subscript α and the summation sign in sums involving Lagrange multipliers such as $\sum\limits_{\alpha=m+1}^{n} l_\alpha\omega^\alpha$ and $\sum\limits_{\alpha=m+1}^{n} l_\alpha F^\alpha$.

Definition 3. A pair $(\tilde{Y}, \lambda)$, where $\tilde{Y}$ is a vector field along a geodesic γ with Lagrange multipliers l, will be called a Jacobi field if $\tilde{Y}$ satisfies the variations equations (7) and its horizontal projection $Y = \mathrm{pr}\tilde{Y}$ satisfies the nonholonomic Jacobi equation [11,12]

$$\frac{D}{dt}\frac{DY}{dt} + R(Y, \gamma')\gamma' + l\hat{F}(\frac{DY}{dt}) + l(\nabla_Y \hat{F})(\gamma') + \lambda\hat{F}(\gamma') = 0. \tag{8}$$

The $\hat{F}$ operator is the tensions tensor F with the second index raised by the inverse metric tensor of the distribution. The equations (7), (8) together with $\lambda' = 0$ are a system of linear homogeneous differential equations with the variables $(\tilde{Y}, \lambda)$. The set of solutions of this system is a linear space. Therefore there are two types of Jacobi fields: for the first type $\lambda \equiv 0$ (zero vector) and for the second type of Jacobi fields $\lambda \neq 0$.

A horizontal vector field Y along a geodesic γ with Lagrange multipliers l will be called a *horizontal Jacobi field* iff it satisfies (8) with some multipliers λ.

Definition 4. Points $t_1, t_2 \in [t_0, T]$, $t_1 \neq t_2$, are said to be conjugated along a horizontal geodesic γ if there exists a nontrivial Jacobi field Y (with some λ) along γ which vanishes at these points: $Y(t_1) = 0$ and $Y(t_2) = 0$.

Theorem 5. *Let γ be a geodesic with the origin $x_0 = \gamma(t_0)$ and endpoint $x_1 = \gamma(T)$. The point $x_1 = \exp^l_{x_0}(u)$ is conjugated with the point x_0 along γ iff the rank of the differential $d_{(u,l)} \exp_{x_0}$ is not its maximum, i.e. iff (u, l) is a critical point of the mapping $(u, l) \mapsto \exp^l_{x_0}(u)$.*

By means of the curvature map described in this paper we proved that the energy functional $J(\gamma) = \frac{1}{2}\int_{t_0}^{T}\langle\gamma', \gamma'\rangle\, dt$ for horizontal paths attains its weak local minimum under the usual assumptions including the cyclicity condition [11, 12].

Theorem 6. *Let the cyclicity condition be satisfied for a distribution. Suppose that the metric tensor of a distribution is positive definite, a (regular) geodesic γ connects two given points x_0 and x_1, and there are no points on the semi-interval $(t_0, T]$ that are conjugated to t_0. Then, on the path γ, the energy functional attains its weak local minimum in the problem with fixed endpoints.*

Hence we propose the Jacobi equation for horizontal geodesics on a distribution in sub-Riemannian geometry which involves the curvature tensor of a distribution and its tensions tensor.

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Sergey Kryzhevich. *Invariant measures for interval translations and some other piecewise continuous maps*

1. Introduction

The famous Krylov–Bogolyubov Theorem claims that any continuous transformation of a compact metric space admits a Borel probability invariant measure. The similar statement for discontinuous maps is wrong.

Example 1. ([6], Exercise 4.1.1) Consider the map $T : [0,1] \to [0,1]$ given by the formula: $T(x) = x/2$ if $x > 0$; $T(0) = 1$. This map does not admit any Borel probability invariant measure.

Here we consider the problem of invariant measures for a special case of one-dimensional piecewise continuous maps.

Example 2. Consider the circle $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$. We represent it as a union of disjoint subsegments $M_j = [t_j, t_{j+1})$, $j = 0, \dots, n$, $t_0 = t_n$ and define the map S by the formula

$$S(t) = t + c_j \mod 1, \qquad t \in M_j.$$

Here c_j are real values. Such map is called *interval translation* (ITM) or, if it is one-to-one, it is called *interval exchange* (IEM).

Similarly one may consider interval translation maps on the segment $[0,1]$.

The Lebesgue measure is invariant for any interval exchange map. Moreover, the map S admits at most n Borel probability invariant non-atomic ergodic measures (see [6, §14.5,§ 14.6] for the basic theory of Interval Exchange maps and [9] for deeper results). The case of non-invertible ITMs, first considered by M. Boshernitzan and I. Kornfeld [2] is much more sophisticated.

Definition 3. We say that an Interval Translation Map S is *finite* if there exists a number $m \in \mathbb{N}$ such that

$$S^m(\mathbb{T}^1) = \bigcap_{k=1}^{\infty} S^k(\mathbb{T}^1).$$

Otherwise, the map S is *infinite*.

It was demonstrated in [2] that many ITMs are finite and thus may be restricted to interval exchange maps. However, there are examples with ergodic measures supported on Cantor sets. J. Schmeling and S. Troubetzkoy [8] provided some estimates on the number of minimal subsets for ITMs. H. Bruin and S. Troubetzkoy [5] studied ITMs of a segment of 3 intervals ($n = 3$). It was shown that in this case typical ITM is finite. In any case, results on Hausdorff dimension for attractors and unique ergodicity are given. These results are generalised in [3] for ITMs with arbitrary many pieces. There is an uncountable set of parameters leading to type ∞ interval translation maps but the Lebesgue measure of these parameters is zero. Furthermore conditions are given that imply that the ITMs have multiple ergodic invariant measures. H. Bruin and G. Clark [4] studied the so-called double rotations ($n = 2$ for maps of the circle). Almost all double rotations are of finite type. The parameters that correspond to infinite type maps, form a set of Hausdorff dimension strictly between 2 and 3. J. Buzzi and P. Hubert [1] studied piecewise monotonous maps of zero entropy and no periodic points. Particularly, they demonstrated that orientation-preserving ITMs without periodic points may have at most n ergodic probability invariant measures where n is the number of intervals. D. Volk [10] was studying ITMs of the segment. He demonstrated that almost all (w.r.t. Lebesgue measure on the parameters set) ITMs of 3 intervals is conjugated to a rotation or to a double rotation and, hence, are of finite type. B. Pires in his preprint [7] proved that almost all ITMs admit a non-atomic invariant measure (he assumed that the map does not have any connections or periodic points).

Represent the circle $\mathbb{T}^1$ as a finite union of closures of disjoint arcs M_j ($j = 1, \dots, n$) such that

$$\operatorname{Int} M_j \bigcap \operatorname{Int} M_k = \varnothing \qquad \text{if } j \neq k$$

and consider an interval translation map (see Example 2).

Observe that the measure Leb is not invariant for S unless this map is invertible almost everywhere.

Definition 4. We say that an arc $Q \subset \mathbb{T}^1$ is *periodic* if there is $k \in \mathbb{N}$ such that $S^k(Q) = Q$ and all iterations $S^l|_Q$ are continuous for each $1 \leqslant l \leqslant k$.

This implies that all points of Q are periodic.

Definition 5. We say that an arc $Q \subset \mathbb{T}^1$ is *eventually periodic* if there is $m \in \mathbb{N}$ such that $S^m(Q)$ is a periodic arc (we admit the case $Q = \mathbb{T}^1$).

2. Borel probability invariant measures for interval translation maps

Definition 6. A measure is called *non-atomic* if the measure of every singleton is zero.

Theorem 7. *Any interval translation map admits a Borel probability non-atomic invariant measure.*

Corollary 8. *For any map S the set $\Xi = \bigcap_{k=1}^{\infty}(S^k(\mathbb{T}^1))$ is uncountable.*

The next result demonstrates that any interval translation map endowed with a non-atomic invariant measure is metrically equivalent to an interval exchange map of the segment $[0, 1]$.

Theorem 9. *Let μ^* be the non-atomic invariant measure for an interval translation map S that exists by Theorem 7. Then the restriction $S|_{\mathrm{supp}\mu^*}$ is metrically equivalent to an interval exchange map $T : [0, 1] \to [0, 1]$ with the Lebesgue measure. The semi-conjugacy map is one-to-one everywhere, except a countable set.*

Definition 10. We call a point x *recurrent (Poisson stable)* with respect to the map S if there exists an increasing sequence $\{m_k \in \mathbb{N}\}$ such that $S^{m_k}(x) \to x$.

Theorem 11. *Let μ^* an the invariant measure for the mapping S. Then recurrent points are dense in* $\mathrm{supp}\,\mu^*$.

So, for any ITM, the number of recurrent points is infinite.

Detailed proofs of Theorems 7, 9, 11 may be found in arXiv:1812.04534. We just highlight the principal ideas of these proofs.

In Theorem 7, we approximate all parameters c_j and t_j (see Example 2) by rational values. For obtained maps (call them S_k), all points are eventually periodic with bounded periods, hence there exists a sequence of S_k invariant measures μ_k. Then we proceed to a weak-$*$ limit μ^* along a subsequence k_m. The main technical challenge is to prove that the measure μ^* is non-atomic and hence the measure of the discontinuity set is zero.

In Theorem 9, there is an evident candidate for the semi-conjugacy map: the distribution function for the measure μ^*. But the proof is still non-trivial cause it is not clear that what we obtained by the conjugacy is an interval **exchange** map, not just an ITM.

Theorem 11 is a trivial modification of its "continuous" analog (see [6]), readers can easily restore the proof.

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Shintarô Kuroki. *Flag Bott manifolds of general Lie type and their equivariant cohomology rings*

(This is a joint work with Shizuo Kaji, Eunjeong Lee and Dong Youp Suh, see the preprint [2])

Grossberg and Karshon introduce the notion of a *Bott tower* as the toric manifold which is obtained by the iterated $\mathbb{C}P^1$-bundles in [1]. More precisely, a Bott tower $\{B_j \mid j = 1, \ldots, m\}$ is a sequence of the fibre bundles $\mathbb{C}P^1 \longrightarrow B_i \xrightarrow{\pi_j} B_{j-1}$ such that B_j is the projectivization of the sum of two complex line bundles over B_{j-1}, where the initial manifold B_0 is a point. The top manifold B_m of the Bott tower $\{B_j \mid j = 1, \ldots, m\}$ is called a *Bott manifold*.

There are two ways of generalizations of Bott towers. One is introduced by Masuda-Suh in [4] called a *generalized Bott manifold*, which is a toric manifold diffeomorphic to an iterated complex projective bundles.

The other is introduced by Kuroki, Lee, Song and Suh in [3] called a *flag Bott manifold*, which is not a toric manifold but a manifold with a nice torus action (i.e., GKM manifold) diffeomorphic to an iterated bundle of flag manifolds.

In this talk, we introduce a class of iterated bundles with nice torus actions called a *flag Bott manifold of general Lie type* which contains both of the Bott manifolds and the flag Bott manifolds. We also give an explicit formula of the equivariant cohomology rings of flag Bott manifolds of general Lie type.

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Vladimir Lebedev. *Tame semicascades and cascades generated by affine self-mappings of the d-torus*

We give a complete characterization of the affine self-mappings φ of the torus $\mathbb{T}^d$ that generate tame, in the sense of Köhler, semicascades and cascades. Namely, we show that the semicascade generated by φ is tame if and only if the matrix A of φ satisfies $A^p = A^q$, where p and q are some nonnegative integers, $p \neq q$. For cascades the corresponding condition has the form $A^m = I$, where m is some positive integer and I is the identity matrix. The question of how to distinguish if the semicascade (cascade) generated by an affine self-mapping of the torus is tame was posed by A. V. Romanov.

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Keonhee Lee. *Spectral decomposition and Ω-stability of flows with expansive measures*

(This is joint work with N. Nguyen)

We discuss some recent and ongoing works on the dynamics of flows with various expansive measures. In particular, we present a measurable version of the Smale's spectral decomposition theorem for flows. More precisely, we prove that if a flow φ on a compact metric space X is invariantly measure expanding on its chain recurrent set $CR(\varphi)$ and has the invariantly measure shadowing property on $CR(\varphi)$ then φ has the spectral decomposition, i.e. the nonwandering set $\Omega(\varphi)$ is decomposed by a disjoint union of finitely many invariant and closed sets on which φ is topologically transitive. Moreover we show that a flow φ is invariantly measure expanding on $CR(\varphi)$ if and only if it is invariantly measure expanding on X. Using this, we characterize the measure expanding flows on a compact C^∞ manifold via the notion of Ω-stability.

Arkady Leiderman. *The separable quotient problem for topological groups*

The famous Banach-Mazur problem, which asks if every infinite-dimensional Banach space has an infinite-dimensional separable quotient Banach space, has remained unsolved for 85 years, though it has been answered in the affirmative for all Banach spaces $C(K)$, where K is a compact space; reflexive Banach spaces and even Banach spaces which are duals. The similar problem for general locally convex spaces has been answered recently in the negative, but has been shown to be true for large classes of locally convex spaces including all non-normable Frechet spaces. In our work we investigated the analogous problem of existing of separable quotients for topological groups. There are four natural questions: Does every non-totally disconnected topological group have a separable quotient group which is (i) non-trivial; (ii) infinite; (iii) metrizable; (iv) infinite metrizable.

We give positive answers for various important classes of topological groups.

THEOREM 1. *Let G be a topological group belonging to one of the following classes*:
(1) *all compact groups;* (2) *all locally compact abelian groups;* (3) *all σ-compact locally compact groups;* (4) *all abelian pro-Lie groups;* (5) *all σ-compact pro-Lie groups;* (6) *all pseudocompact groups.*
Then G admits an infinite separable metrizable quotient group.

However, for precompact groups we constructed a counter-example. We denote by $\mathbb{T}$ the circle group.

Theorem 2. *There exists an uncountable dense subgroup G of the compact abelian group $\mathbb{T}^{\mathfrak{c}}$ satisfying* $\dim G = 0$ *such that every quotient group of G is either the one-element group or non-separable.*

Free topological groups constitute a prominent class of topological groups — every topological group G is a quotient group of a free topological group, namely, $F(X)$, where X is a space homeomorphic to G. The group $F(X)$ contains X as a subspace which generates $F(X)$ algebraically and is characterized by the property that every continuous mapping of X to a topological group H extends to a continuous homomorphism of $F(X)$ to H. In the category of topological abelian groups, a similar object is known as the free abelian topological group which is denoted by $A(X)$. Our definition of free topological groups follows Markov's approach.

Theorem 3. *Let X be a Tychonoff space belonging to one of the following classes*:
(1) *all compact spaces;* (2) *all locally compact spaces;* (3) *all pseudocompact spaces;* (4) *all connected locally connected spaces.*
Then the groups $F(X)$ and $A(X)$ admit an open continuous homomorphism onto a nontrivial group (i.e. not finitely generated) with a countable network. In particular, $F(X)$ and $A(X)$ admit a nontrivial separable quotient group.

Recall that a topological space X is said to be *scattered* if every nonempty subset S of X has an isolated point relative to S.

Theorem 4. *Let $G(X)$ denotes $F(X)$ or $A(X)$. The following conditions are equivalent for a compact space X*:
(a) *The topological group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(b) *The topological group $G(X)$ admits a nontrivial metrizable quotient.*
(c) *The topological group $G(X)$ admits a nontrivial metrizable and separable quotient.*
(d) *X is not scattered.*

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Vladimir Leksin. *Serre duality of homotopy and homology properties of CW complexes*

In the talk the following assertion about some duality in the algebraic topology will be discus

Assertion. *Let X be line connected and simple connected a CW complex with integral homology groups and homotopy groups the finite type. If X has a finite number nonzero of homology groups, then X has the infinite number of nonzero homotopy groups. Conversely, if X has a finite number nonzero of homotopy groups, then X has the infinite number of nonzero of homology groups.*

The spheres $S^n, n \geqslant 2$, and Eilenberg–MacLane spaces $K(\mathbb{Z}, n)$, $n \geqslant 2$, give us many examples of CW complexes with described duality.

The proof of assertion in one side: {finite nonzero homology groups} $\Rightarrow$ {infinite nonzero homotopy grops} follows from the theorem Serre [1] and the theorem Umeda [3]. In other side:{finite nonzero homotomy groups} $\Rightarrow$ {infinite nonzero homology grops} for one floor Postnikov tower follows from calculations of J.-P. Serre [1] and H. Cartan [2]. For Postnikov tower with $n \geqslant 2$ floors we point the cases when the assertion holds [5].

For CW complexes when the assertion fulfilled then by Dold–Thom theorem [4], [6] the James operator of infinite symmetric product [6] transforms such CW complexes in CW complexes for which described duality also holds [5].

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Ivan Limonchenko. *On families of polytopes and Massey products in toric topology*

(Based on a joint work with Victor M. Buchstaber)

The study of Massey operation in cohomology of a differential graded algebra is well-known in algebraic topology and homological algebra. Nontrivial Massey products serve as an obstruction to Golodness of a local ring in homological algebra and to formality of a space in rational homotopy theory. According to Peter May, they also determine differentials in the Eilenberg–Moore spectral sequence and generate a kernel of the cohomology suspension homomorphism. Until now, few examples of manifolds M with nontrivial higher Massey products in $H^*(M)$ have been constructed.

In this talk, using the theory of direct families of polytopes, we introduce sequences of moment-angle manifolds over 2-truncated cubes $\{M_n\}_{n=1}^{\infty}$ such that for any $n \geqslant 2$: M_n is a submanifold and a retract of M_{n-1}, and there exists a nontrivial Massey product $\langle \alpha_1^n, \ldots, \alpha_k^n \rangle$ in $H^*(M_n)$ with $\dim \alpha_i^n = 3, 1 \leqslant i \leqslant k$ for each $2 \leqslant k \leqslant n$. As an application of our constructions, we examine nontriviality of differentials of the Eilenberg–Moore and Milnor spectral sequences for M_n.

Khudoyor Mamayusupov. *A parameter plane of cubic Newton maps with a parabolic fixed point at infinity*

Introduction, cubic parabolic Newton maps

We consider the Newton's method applied to the entire maps of the form $(az^2 + bz + c)\exp(dz + e)$. After appropriate conjugation, we see that it is enough to represent any such a cubic Newton map with a single complex number $\lambda \neq 0$, and denote these maps by $f_\lambda(z) = z^2 \dfrac{z + \lambda - 1}{(\lambda + 1)z - 1}$, and denote $\mathscr{F}$ the family of such a cubic Newton maps f_λ. We have $f_\lambda'(0) = f_\lambda'(\infty) = 0$ and $f_\lambda'(1) = 1$ at the three persistent fixed points. Thus the Julia sets are connected by Shishikura's theorem. The derivative $f_\lambda'(z) = \dfrac{z(2(\lambda+1)z^2 + (\lambda^2 - 4)z - 2\lambda + 2)}{((\lambda+1)z - 1)^2}$ shows that if $\lambda \neq \pm 1$ then there are two critical points that are not fixed by f_λ counted with multiplicities, and at least one of them always converges under the iterates of f_λ to the parabolic point at 1. We call the other critical point a "free" critical point.

In general, any cubic rational map with three fixed points two of which are superattracting and the third is multiple fixed point, then necessarily of multiplicity +1, can be conformally conjugated to the form f_λ.

The moduli space of $\mathscr{F}$ is double covered by the $\lambda-$parameter plane $\mathbb{C}\backslash\{0\}$, with identifications of λ and $-\lambda$, has an orbifold structure with a singular boundary locus $\{0, \infty\}$, both boundary points are of an elliptic type.

Since the parabolic fixed point 1 is persistent for the family $\mathscr{F}$, it makes sense to consider parameters $\lambda \in \mathbb{C}\backslash\{0\}$ such that the free critical point of f_λ belongs to basins of attracting cycles or the parabolic basin of $z = 1$ and call such an f_λ *stable*. More precisely, these stable maps are J-stable maps. Stable parameters form an open set in the $\lambda-$parameter plane and the connected components are called *stable components*. Our main result is that every stable component is a topological disk, with a unique center which is a cubic postcritically *minimal* Newton map, defined in [1,2,3]. For the main parabolic stable component, denoted H, charactered by maps f_λ for which the free critical point belongs to the immediate basin of 1, if there is no critical orbit relation then any two such maps are globally quasiconformal conjugate. If there is a critical orbit relation for f_λ then the quasiconformal conjugacy class of f_λ consists of the single f_λ.

For stable components parametrized by maps for which its free critical point belongs to the immediate basin of one of the two superattracting fixed points, it is well known that the value of the Bötcher map of the superattracting fixed point evaluated at the orbit point of the free critical point first visits the immediate basin gives a surjective conformal map from the stable component to the unit disk and parametrizes these types of stable components.

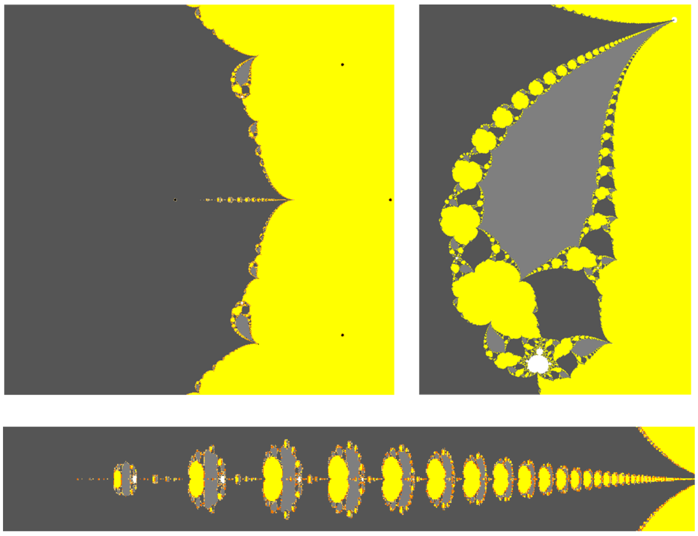

FIGURE 1. The parameter plane of cubic Newton maps. Left-top is the parameter plane with two zoom-ins on the right and the bottom: around a little island and a cascade of apple shaped regions. Yellow areas – the free critical point belongs to the parabolic basin, the main parabolic component H. Grey areas – the free critical point belongs to the basins of the two superattracting fixed points. White areas – the free critical point belongs to the basin of attracting periodic point of higher period. Black dots represent centers (pcm) and half-centers (pcnm) of the corresponding stable component.

Blaschke products and Model space

Denote $\beta_a(z) = \frac{1-\bar{a}}{1-a} \cdot \frac{z-a}{1-\bar{a}z}$ the unique automorphism of $\mathbb{D}$ sending $a \in \mathbb{D}$ to the origin and normalized to fix $z = 1$. For complex constants a_1, a_2, a_3 in $\mathbb{D}$, we define a cubic Blaschke product by $B(z) = \beta_{a_1}(z) \cdot \beta_{a_2}(z) \cdot \beta_{a_3}(z)$ normalized to fix 1. The result by Heins states that for a set of critical points in $\mathbb{D}$ we can find a Blaschke product of degree d that has critical points exactly at these points. The Blaschke product is unique up to post-composition by an automorphism of $\mathbb{D}$. Moreover, every Blaschke product can be normalized to fix 0 and 1 by a post-composition of an automorphism of $\mathbb{D}$. We want $z = 1$ to be a triple fixed point, the parabolic fixed point: around $z = 1$ the maps have the form $1 + (z-1) + A(z-1)^3 + o((z-1)^3)$, for a complex $A \neq 0$. It means that $B'(1) = 1$ and $B''(1) = 0$.

Denote $\mathscr{B}_3$ the model space consisting of cubic parabolic Blaschke products B such that $B(1) = 1$, $B'(1) = 1$, $B''(1) = 0$, and with critical points at 0 and w in $\mathbb{D}$. It is natural to parametrize $\mathscr{B}_3$ by the location of the critical point w of B.

PROPOSITION 1 (Normal forms for Blaschke products of $\mathscr{B}_3$). *Every cubic parabolic Blaschke product in the model space $\mathscr{B}_3$ has a normal form $B(z) = \beta_a(z^2 \cdot \beta_b(z))$ for a complex number $b \in \mathbb{D}$, where $a = \frac{(1-\bar{b})\left(3(1-b)+b(1-\bar{b})\right)}{(1-b)\left(-6+3b+8\bar{b}-3|b|^2-\bar{b}^2(3-b)\right)}$. The model space $\mathscr{B}_3$ is parametrized by a complex number $b \in \mathbb{D}$ (real 2 dimensional) and the dependence is real analytic. In particular, the other critical point w of a Blaschke product is given by $w = \frac{b}{4|b|^2}\left(3 + |b|^2 - \sqrt{(3+|b|^2)^2 - 16|b|^2}\right)$ depending real analytically on $b \in \mathbb{D}$. The location of the critical point w also parametrizes $\mathscr{B}_3$, which is complex analytic.*

Let $\mathscr{M} = \mathscr{M}(\mathscr{B}_3)$ denote the moduli space of $\mathscr{B}_3$ consisting of conformal conjugacy classes of maps in $\mathscr{B}_3$.

PROPOSITION 2. *The moduli space $\mathscr{M}$ of the model space $\mathscr{B}_3$ is an orbifold $\mathbb{D}/\Gamma$, a topological open disk, with an elliptic point of order 2 at the origin and a group action is by $\Gamma = \{\mathrm{id}, z \mapsto -z\frac{1-\bar{z}}{1-z}\}$ for $z \in \mathbb{D}$.*

For every cubic Newton map $f_a \in H$ corresponds a conformal class in the moduli space $\mathscr{M}$, and thus H can be identified with $\mathscr{M}$, which is a topological open disk.

THEOREM 3. *The main parabolic stable component H is simply connected with its unique center. The quasiconformal conjugacy classes of maps in H are of two types: type-I, a single class, which is topologically an infinitely punctured disk, and its punctures are of type-II. The set of type-II quasiconformal conjugacy classes is in one-to-one correspondence with the set of all cubic postcritically non-minimal Newton maps. Moreover, the boundary ∂H is a Jordan curve.*

The main result in summary is the following.

THEOREM 4. *Every stable component of the parameter plane of $\mathscr{F}$ is an open topological 2-cell with its unique center that is a cubic postcritically minimal Newton map. There exists a bijective map from the space of*

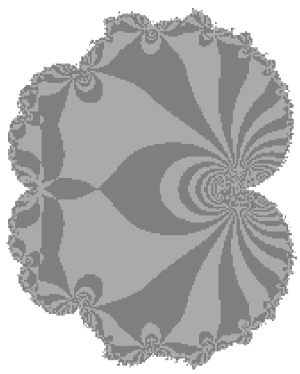

FIGURE 2. The main parabolic component H with a checkerboard structure in a suitable parametrization

Haïssinsky equivalent classes of centers (hyperbolic postcritically finite maps) of hyperbolic components of the space of standard cubic Newton maps of cubic polynomials to the centers of stable components of this space of parabolic cubic Newton maps. This bijection preserves the dynamics on the corresponding Julia sets and is obtained by parabolic surgery.

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Ciprian Manolescu. *Homology cobordism and triangulations*

The study of triangulations on manifolds in dimensions at least 5 is closely related to understanding the three-dimensional homology cobordism group $\Theta^3_{\mathbb{Z}}$. Indeed, in the 1970's, Galewski-Stern and Matumoto rephrased the problems of existence and classification of triangulations on high-dimensional manifolds in terms of questions about $\Theta^3_{\mathbb{Z}}$ and the Rokhlin homomorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$.

New information about $\Theta^3_{\mathbb{Z}}$ can be obtained from gauge theory. In the 1990's, using Yang-Mills theory, Fintushel-Stern and Furuta showed that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ subgroup. Later, Frøyshov showed it has a $\mathbb{Z}^\infty$ summand. More recent methods include Pin(2)-equivariant Seiberg-Witten Floer spectra and involutive Heegaard Floer homology. For example, one can show that there are no 2-torsion elements in $\Theta^3_{\mathbb{Z}}$ of Rokhlin invariant one. This implies the existence of non-triangulable manifolds in dimensions at least five.

Elena Martín-Peinador. *Locally quasi-convex groups and the Mackey–Arens Theorem*

(This is joint work with M. J. Chasco and V. Tarieladze)

The locally quasi-convex groups were defined by Vilenkin in the 50's of the last century. They are an important class of topological abelian groups which encloses as a subclass the locally convex topological vector spaces. Thus, central results of Functional Analysis may have extended versions for locally quasi-convex groups, which usually present some obstructions making the theory reacher. In order to deal with local quasi-convexity several authors have developed techniques with roots on numerical analysis, and there has been a great activity in this field in the last 25 years.

The Mackey–Arens Theorem is one of the most relevant results of linear Functional Analysis. It asserts that for a real topological vector space (X, τ), the set $\mathrm{LCT}(X, \tau)$ of all compatible locally convex topologies on X has a maximum, subsequently called the Mackey topology. In [1] the Mackey-Arens Theorem was studied within the category of abelian topological groups. The main problem left open in the mentioned paper was the existence of the analogue to the Mackey topology for abelian groups. Explicitly, if (G, τ) is an abelian topological group, is there a maximum in the family $\mathrm{LQC}(G, \tau)$ of all the locally quasi-convex topologies compatible with τ?. For a broad class of topological groups including the locally compact and the complete metrizable abelian groups, the existence was already established in [1].

Finally, in 2018 the question has been solved in the negative. Aussenhofer and Gabriyelyan (simultaneously) provided an example of a topological group which does not admit a Mackey topology: namely the free abelian topological group on a convergent sequence. Some other examples have just appeared.

In this lecture the Mackey theory for groups will be presented, stressing the similarities and differences with the classical Mackey theory for locally convex spaces.

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Mikiya Masuda. *Torus orbit closures in the flag varieties*

(Based on joint work with Eunjeong Lee (IBS) and Seonjeong Park (Ajou Univ.))

Let $\mathrm{Fl}(\mathbb{C}^n)$ be the flag variety consisting of complete flags in $\mathbb{C}^n$. It has the natural action of the torus $T = (\mathbb{C}^*)^n$ and the T-fixed point set $\mathrm{Fl}(\mathbb{C}^n)^T$ in $\mathrm{Fl}(\mathbb{C}^n)$ can be identified with the permutation group $\mathfrak{S}_n$ on n letters.

In this talk, we discuss the topology and combinatorics of T-orbit closures in $\mathrm{Fl}(\mathbb{C}^n)$. To an arbitrary T-orbit closure Y in $\mathrm{Fl}(\mathbb{C}^n)$, we associate a retraction

$$\mathscr{R}^g_Y\colon \mathfrak{S}_n \to Y^T \subset \mathfrak{S}_n$$

which maps an element in $\mathfrak{S}_n$ to its closest element in Y^T with respect to a metric on $\mathfrak{S}_n$. The retraction $\mathscr{R}^g_Y$ determines the fan of Y. On the other hand, for any subset $\mathscr{M}$ of $\mathfrak{S}_n$, we define a retraction

$$\mathscr{R}^a_{\mathscr{M}}\colon \mathfrak{S}_n \to \mathscr{M} \subset \mathfrak{S}_n$$

algebraically. It turns out that these two retractions agree when $Y^T = \mathscr{M}$ and they are related to Coxeter matroids in $\mathfrak{S}_n$.

Michael Megrelishvili. *Group actions on treelike compact spaces*

(This is a joint work with E. Glasner [2])

We study dynamical properties of group actions on treelike compact spaces and show that every continuous group action on a dendron is representable on a Rosenthal Banach space, hence also dynamically tame. Similar results are obtained for compact median pretrees using some results of A. V. Malyutin [5]. We show also that Helly's selection principle can be extended to bounded monotone sequences defined on median pretrees (e.g., dendrons or linearly ordered sets).

Dendron D is a connected compact space such that every pair of distinct points can be separated in D by a third point, [7]. *Dendrite* is a metrizable dendron.

Theorem 1. *Let D be a dendron. For every topological group G and a continuous action $G \curvearrowright D$, the dynamical G-system D admits a faithful representation on a Rosenthal Banach space. Hence, D is a tame G-space.*

As in [6, 3], a *representation* of a G-space X on a Banach space V is a pair

$$h : G \to \mathrm{Iso}(V),\ \ \alpha : X \to V^*,$$

where $h : G \to \mathrm{Iso}(V)$ is a continuous co-homomorphism into the linear isometry group $\mathrm{Iso}(V)$ (with its strong operator topology) and $\alpha : X \to V^*$ is a weak* continuous bounded G-mapping with respect to the dual action $G \times V^* \to V^*$. If α is an embedding then the representation is said to be *faithful*.

Representations on Banach spaces with "good" geometry lead to a natural hierarchy in the world of continuous actions $G \curvearrowright X$ of topological groups G on topological spaces X. In particular, representations on Banach spaces without a copy of l_1 (we call them *Rosenthal* Banach spaces) play a very important role in this hierarchy. According to the Rosenthal l_1-dichotomy [8], and the corresponding dynamical version of Bourgin-Fremlin-Talagrand dichotomy [3], there is a sharp dichotomy for metrizable dynamical systems; either their enveloping semigroup is of cardinality smaller or equal to that of the continuum, or it is very large and contains a copy of $\beta\mathbb{N}$.

When X is compact metrizable, in the first case, such a dynamical system (G, X) is called *tame*. By A. Köhler's [4] definition, tameness of a compact (not necessarily, metrizable) G-space X means that for every continuous real valued function $f : X \to \mathbb{R}$ the family of functions $fG := \{fg\}_{g\in G}$ is "combinatorially small"; namely, fG does not contain an *independent sequence*.

Recall that [8] a sequence $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ of functions on a set X is independent if $\exists\ a < b$ such that

$$\bigcap_{n\in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n\in M} f_n^{-1}(b, \infty) \neq \varnothing$$

for all finite disjoint subsets P, M of $\mathbb{N}$.

Definition 2. A *pretree structure* p on a set X is a *generalized interval system* $p : X \times X \to P(X),\ p(a, b) = [a, b] \subseteq X$ such that

(A0) $[a, b] \supseteq \{a, b\}$.
(A1) $[a, b] = [b, a]$.
(A2) If $c \in [a, b]$ and $b \in [a, c]$ then $b = c$.
(A3) $[a, b] \subseteq [a, c] \cup [c, b]$ for every $a, b, c \in X$.

This concept (see for example [10, 1, 5]) naturally unifies several important structures including betweenness relation on dendrons and semi-lattices (e.g., linear orders).

Following [5] we define the so-called *shadow topology* τ_s on (X, p). Given an ordered pair $(u, v) \in X^2, u \neq v$, let

$$S_u^v := \{x \in X : u \in [x, v]\}$$

be the *shadow* in X defined by the ordered pair (u, v). Pictorially, the shadow S_u^v is cast by a point u when the light source is located at the point v. The family $\mathscr{S} = \{S_u^v : u, v \in X, u \neq v\}$ is a subbase for the closed sets of the topology τ_s.

Example 3.

(1) Every dendron D is a compact (in its shadow topology) median pretree with respect to the standard betweenness relation (see [5, 7]).
(2) Every linearly ordered set is a median pretree. Its shadow topology is just the interval topology of the order.
(3) Let X be a $\mathbb{Z}$-*tree* (a median pretree with finite intervals $[u, v]$). Denote by $Ends(X)$ the set of all its ends. According to Malyutin [5, Section 12] the set $X \cup Ends(X)$ carries a natural τ_s-*compact* median pretree structure.

For every triple a, b, c in a pretree X the *median* $m(a, b, c)$ is the intersection

$$m(a, b, c) := [a, b] \cap [a, c] \cap [b, c].$$

When it is nonempty the median is a singleton. A pretree for which this intersection is always nonempty is called a *median pretree.*

Every median pretree is a *median algebra.* A map $f : X_1 \to X_2$ between two median algebras is monotone (i.e., interval preserving) if and only if f is median-preserving ([9, page 120]) if and only if f is convex ([9, page 123]). Convexity of f means that the preimage of a convex subset is convex.

Lemma 4. *Let (X, p) be a median pretree. Then the retraction map*

$$\varphi_{u,v} : X \to [u, v],\ x \mapsto m(u, x, v)$$

is monotone and continuous in the shadow topology for every $u, v \in X$.

For a τ_s-compact median pretree X we denote by $H_+(X)$ the topological group of monotone (equivalently, median-preserving) homeomorphisms. We treat $H_+(X)$ as a topological subgroup of the full homeomorphism group $\mathrm{Homeo}(X)$.

The following result generalizes Theorem 1. In the case of a dendron D we have $H_+(D) = \mathrm{Homeo}(D)$.

Theorem 5. *For every compact median pretree X and its automorphism group $G = H_+(X)$ the action of the topological group G on X is Rosenthal representable.*

By Example 3.3, Theorem 5 applies when X is a $\mathbb{Z}$-tree and we get

Corollary 6. *Let X be a $\mathbb{Z}$-tree. Denote by $Ends(X)$ the set of all its ends. Then for every monotone group action $G \curvearrowright X$ with continuous transformations the induced action of G on the compact space $\widehat{X} := X \cup Ends(X)$ is Rosenthal representable.*

Such compact spaces $\widehat{X}$ as in Corollary 6 are often zero-dimensional. So, at least, formally this case cannot be deduced from the dendron's case.

Theorem 7. *Let X be a median pretree. Then every pair of monotone (equivalently, convex) real valued functions $f_i : X \to \mathbb{R}$, $i \in \{0, 1\}$ is not independent.*

Using results of Rosenthal [8] and Theorem 7 we get

Theorem 8. *(Generalized Helly's selection principle) Let X be a median pretree (e.g., dendron or a linearly ordered set) and $\{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ be a bounded sequence of monotone real valued functions. Then there exists a pointwise converging subsequence.*

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Sergey Melikhov. *Brunnian link maps in the 4-sphere*

A *link map* is a map $X_1 \sqcup \dots \sqcup X_m \to Y$ such that the images of the X_i are pairwise disjoint, and a *link homotopy* is a homotopy whose every time instant is a link map. For example, link maps $S^p \sqcup S^q \to S^{p+q+1}$ are classified (up to link homotopy) by the linking number, and link maps $S^1 \sqcup S^1 \sqcup S^1 \to S^3$ are classified by Milnor's triple $\bar{\mu}$-invariant. A nontrivial link map $S^2 \sqcup S^2 \to S^4$ was constructed by R. Fenn and D. Rolfsen (1986) using that each component of the Whitehead link is null-homotopic in the complement ot the other one. P. Kirk (1988) introduced an invariant of link maps $S^2 \sqcup S^2 \to S^4$ with values in the infinitely generated free abelian group $\mathbb{Z}[x] \oplus \mathbb{Z}[y]$ and found its image. According to a 2017 preprint by R. Schneiderman and P. Teichner, the long-standing problem of injectivity of Kirk's invariant has an affirmative solution.

A natural extension of Kirk's invariant to link maps of m copies of S^2 in S^4 was described by U. Koschorke (1991) and takes values in $\big(\mathbb{Z}[\mathbb{Z}^{m-1}/t]\big)^m$, where $t(\vec{v}) = -\vec{v}$. We show that the Kirk–Koschorke invariant is not injective for $m > 2$. To prove this, we introduce a new "non-abelian" invariant of m-component link maps in S^4 with values in $\big(\mathbb{Z}[RF_{m-1}/T, c]\big)^m$, where RF_k is the Milnor free group (RF_2 is also known as the discrete Heisenberg group), $T(g) = g^{-1}$ and c stands for conjugation. Loosely speaking, the new invariant is related to the Kirk–Koschorke invariant in the same way as Milnor's $\bar{\mu}$-invariants of link homotopy are related to pairwise linking numbers.

For link maps $S^2 \sqcup S^2 \sqcup S^2 \to S^4$ we also find the image of their Kirk–Koschorke invariant. The main step is a new elementary construction of Brunnian link maps $S^2 \sqcup \dots \sqcup S^2 \to S^4$: they are described by an explicit link-homotopy-movie (just like the Fenn–Rolfsen link map), which is closely related to the minimal solution of the Chinese Rings puzzle. The existence of nontrivial Brunninan link maps in S^4 (of more than two components) was established previously by Gui-Song Li (1999), but his construction requires a lot more of 4-dimensional imagination (it is based on iterated Whitney towers and a process of their desingularization) and does not suffice to generate the image of the Kirk–Koschorke invariant.

Our interest in finding the images of invariants of link maps in S^4 actually comes from a study of classical links. In arXiv:1711.03514 = JKTR 27:13 (2018), 1842012, the computation of the image of Kirk's invariant of link maps $S^2 \sqcup S^2 \to S^4$ is applied to reprove the Nakanishi–Ohyama classification of links $S^1 \sqcup S^1 \to S^3$ up to self C_2-moves. Now, our computation of the image of the Kirk–Koschorke invariant of link maps $S^2 \sqcup S^2 \sqcup S^2 \to S^4$ has the following application: Two links $S^1 \sqcup S^1 \sqcup S^1 \to S^3$ that are link homotopic to the unlink are related by C_2^{xxx}-moves (=self C_2-moves) and $C_3^{xx,yz}$-moves (of Goussarov and Habiro) if and only if they have equal $\bar{\mu}$-invariants (with possibly repeating indices) of length at most 4.

Grigory Mikhalkin. *Real algebraic curves in the plane and in the 3-space: indices and their extremal properties*

(The talk is based on a series of joint works with Stepan Orevkov)

Several types of indices, i.e. integer numbers invariant under isotopies, are known for plane and spatial real algebraic curves. For plane curves an index is given by the area of the logarithmic image of the curve (this area turns out to be integer under certain assumptions). For spatial curves an index is given by Viro's encomplexed writhe. Some other indices are given by natural framings of real algebraic curves in the 3-space. It turns out that these indices are subject to extremal properties. In particular, topology of curves of a given degree with the maximal possible values of theses indices can be explicitly described.

Andrey Mikhovich. *On p–adic variation of Segal theorem*

By definition, a finite type presentation of a discrete group G is an exact sequence

$$1 \to R \to F \xrightarrow{\pi} G \to 1 \tag{1}$$

in which $F = F(X)$ is the free group with a finite set X of generators, and R is a normal subgroup in F generated by a finite number of defining relations $r \in R$. By a pro–p–group one calls a group isomorphic to inverse limit of finite p–groups. This is a topological group (with the topology of direct product) which is a compact totally disconnected group. For such groups one has a presentation theory which is in many aspects similar to the combinatorial theory of discrete groups. By analogy with finite type presentation of a discrete group, we shall say that a pro–p–group G is given by a finite type pro–p–presentation if G is included into an exact sequence (1) in which F is a free pro–p–group with finite number of generators, and R is a closed normal subgroup topologically generated by a finite number of elements in F, contained in the Frattini subgroup of the group F.

For discrete groups, p will run over all primes, while for pro–p–groups p is fixed. Let G be a (pro–p)group with a finite type (pro–p)presentation (1), and $\overline{R} = R/[R, R]$ be the corresponding G–**module of relations (relation module)**, where $[R, R]$ is the commutant, and the action of G is induced by conjugation by F on R.

Introduce the notation $\mathbb{Z}_{((p))}$ for $\mathbb{Z}$ in the case of discrete groups and $\mathbb{Z}_p$ in the case of pro–p–groups.

Definition 1. We shall call the (pro–p)presentation (1) quasirational (QR–(pro–p)presentation) if one of the following three equivalent conditions is satisfied:

(i) for each $n > 0$ and for each prime $p \geqslant 2$, the F/RU-module $R/[R, RU]$ has no p–torsion (p is fixed for pro–p–groups and runs over all primes $p \geqslant 2$ in the discrete case) for any normal subgroups $U \trianglelefteq F$ of finite index $\mid G/U \mid = p^n$, $n \in \mathbb{N}$;

(ii) the quotient module of coinvariants $\overline{R}_G = \overline{R}_F = R/[R, F]$ has no torsion;

(iii) $H_2(G, \mathbb{Z}_{((p))})$ has no torsion.

The proof of equivalence of conditions (i)–(iii) is contained in [1, Proposition 4] and [2, Proposition 1]. QR–presentations are curious in particular due to the fact that they contain aspherical presentations of discrete groups and their subpresentations (so they are important in the study of asphericity), and also pro–p–presentations of pro–p–groups with one defining relation [3] (the study of J.-P. Serre's question).

In [3] the importance of study of p–adic permutation representations of finite p–groups for asphericity questions is shown. In particular it is intersting to verify a difference between Burnside and Representations rings of a group. Let G be a finite group and $A(G)$ be its Burnside ring, that is a Grothendieck ring of isomorphism classes of finite G–sets with disjoint union taken as commutative addition and Cartesian product of G–sets as multiplication [5, 1.1]. We also need $R_k(G)$ - the representation ring (over a field k) of a finite group G, it is a Grothendieck ring of isomorphism classes of finite-dimensional k–representations of G. Consider a natural homomorphism of rings

$$h_k : A(G) \to R_k(G),$$

which assign for any isomorphism class of G–sets $[X]$ the corresponding class $h_k([X])$ in the representation ring $R_k(G)$ of a finite group G. According to Segal theorem [5, Theorem 4.4.1] for any finite p–group G the natural homomorphism

$$h_{\mathbb{Q}} : A(G) \to R_{\mathbb{Q}}(G)$$

is the epimorphism of rings. However this is no longer true when for the homomorphism of rings

$$h_{\mathbb{Q}_2} : A(G) \to R_{\mathbb{Q}_{\not\geq}}(G)$$

in the case of 2–groups [4]. We develop Burnside ring theory for pro–p–groups with connection to the structure of relation modules.

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Dmitry Millionshchikov. *Massey products and representation theory*

Massey products in cohomology have found a lot of interesting applications in topology and geometry. For instance, the existence of a non-trivial Massey product in $H^*(M, \mathbb{R})$ is an obstruction for a (symplectic) manifold M to be Kähler. An initial data of almost all known examples of non-Kähler symplectic manifolds are nilmanifolds that admit some non-trivial Massey products in their cohomology. We consider n-fold Massey products in the cohomology of finite dimensional positively graded graded Lie algebras (and of the corresponding nilmanifolds) with rational structure constants. Feigin, Fuchs and Retakh proposed to used $(n+1)$-dimensional graded modules V over positively graded Lie algebra $\mathfrak{g}$ for the construction of defining systems for n-fold Massey products $\langle a_1, \ldots, a_n \rangle$ in $H^2(\mathfrak{g})$, where $a_1, \ldots, a_n$ are all cohomology classes from $H^1(\mathfrak{g})$. We will discuss old and new problems and results related to this approach.

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Aleksandr Mishchenko. *Geometric description of the Hochschild cohomology of Group Algebras*

A description of the algebra of outer derivations of a group algebra of a finitely presented discrete group is given in terms of the Cayley complex of the groupoid of the adjoint action of the group. This task is a smooth version of Johnson's problem concerning the derivations of a group algebra. It is shown that the algebra of outer derivations is isomorphic to the group of the one-dimensional cohomology with compact supports of the Cayley complex over the field of complex numbers.

On the other hand the group of outer derivation is isomorphic to the one dimensional Hochschild cohomology of the group algebra. Thus the whole Hochschild cohomology group can be described in terms of the cohomology of the classifying space of the groupoid of the adjoint action of the group under the suitable assumption of the finiteness of the supports of cohomology groups.

The report presents the results partly obtained jointly with A. Arutyunov, and also with the help of A. I. Shtern.

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Nikolai Mnev. *On local combinatorial formulas for Euler class of triangulated spherical fiber bundle*

Suppose we have a PL spherical fiber bundle with a fiber S^n triangulated over the base simplicial complex. The bundle determines $n+1$ dimensional Euler characteristic class in the base. Local combinatorial formula for the Euler class is a universal combinatorial function of elementary triangualeted S^n-bundles over $n+1$ simplices universally representing Euler cocycle of the bundle in simplicial cohomology of the base. Such functions exist for rational coefficients in cohomology. They can be constructed as explicit local chain-level formulas for Gysin homomorphism in Gysin sequence of the bundle. To get an access to local chain combinatorics of spectral sequence of the bundle we may use Guy Hirsh homology model of the bundle as a local system and then applying homology perturbation theory obtain local formulas as certain measure of twisting in combinatorial Hodge structure of the elementary bundle. The answer can be interpreted and evaluated statistically as certain combinatorial counting using Catanzaro–Chernyak–Klein higher Kirchhoff theorems.

Egor Morozov. *Surfaces containing two parabolas through each point*

We prove that any surface in $\mathbb{R}^3$ containing two arcs of parabolas with axes parallel to Oz through each point has a parametrization $\left(\frac{P(u,v)}{R(u,v)}, \frac{Q(u,v)}{R(u,v)}, \frac{Z(u,v)}{R^2(u,v)}\right)$ for some $P, Q, R, Z \in \mathbb{R}[u,v]$ such that P, Q, R have degree at most 1 in u and v, and Z has degree at most 2 in u and v (under some additional technical assumptions). This result can be considered in the context of describing surfaces containing two isotropic circles through each point in isotropic geometry (in the Euclidean case all such surfaces are described in the recent paper of M. Skopenkov and R. Krasauskas). We also consider some other problems about surfaces containing isotropic circles and lines through each point.

Michele Mulazzani. *The complexity of orientable graph manifolds*

(Joint work with Alessia Cattabriga)

Graph manifolds, introduced and classified by Waldhausen in [10] and [11], are compact 3-manifolds obtained by gluing Seifert fibre spaces along toric boundary components.

S. Matveev in [7] (see also [8] and [9]) introduced the notion of complexity for compact 3-dimensional manifolds, as a way to measure how "complicated" a manifold is. Indeed, for closed irreducible and $\mathbb{P}^2$-irreducible manifolds the complexity coincides with the minimum number of tetrahedra needed to construct the manifold, with the only exceptions of S^3, $\mathbb{RP}^3$ and $L(3,1)$, all having complexity zero. Moreover, complexity is additive under connected sum and it is finite-to-one in the closed irreducible case. The last property has been used in order to construct a census of manifolds according to increasing complexity: for the orientable case, up to complexity 12, in the Recognizer catalogue (available at `http://matlas.math.csu.ru/?page=search`) and for the non-orientable case, up to complexity 11, in the Regina catalogue (available at `https://regina-normal.github.io`).

Upper bounds for the Matveev complexity of infinite families of 3-manifolds are given in [6] for lens spaces, in [5] for closed orientable Seifert fibre spaces and for orientable torus bundles over the circle, in [4] for orientable Sefert fibre space with boundary and in [1] for non-orientable compact Seifert fibre spaces. All the previous upper bounds are sharp for manifolds contained in the Recognize and Regina catalogues. Very little is known for the complexity of graph manifolds: in [2] and [3] upper bounds are given only for the case of graph manifolds obtained by gluing along the boundary two or three Seifert fibre spaces with disk base space and at most two exceptional fibres.

We present an upper bound for the Matveev complexity of the whole class of closed connected orientable prime graph manifolds that is sharp for all 14502 graph manifolds of the Recognizer catalogue.

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Oleg Musin. *Borsuk–Ulam type theorems for f–neighbors*

(The talk is based on joint work with Andrei Malyutin)

We introduce and study a new class of extensions for the Borsuk–Ulam theorem. Our approach is based on the theory of Voronoi diagrams and Delaunay triangulations. One of our main results is as follows.

THEOREM 1. *Let $\mathbb{S}^m$ be a unit sphere in $\mathbb{R}^{m+1}$ and let $f\colon \mathbb{S}^m \to \mathbb{R}^n$ be a continuous map. Then there are points p and q in $\mathbb{S}^m$ such that*

- $\|p-q\| \geqslant \sqrt{2 \cdot \frac{m+2}{m+1}}$;
- *$f(p)$ and $f(q)$ lie on the boundary ∂B of a closed metric ball $B \subset \mathbb{R}^n$ whose interior does not meet $f(\mathbb{S}^m)$.*

Note that $\sqrt{2 \cdot \frac{m+2}{m+1}}$ is the diameter of a regular simplex inscribed in $\mathbb{S}^m$.

The Borsuk–Ulam theorem states that every continuous map of a Euclidean n-sphere $\mathbb{S}^n$ to Euclidean n-space $\mathbb{R}^n$ of the same dimension sends a pair of antipodal points to the same point. Theorem 1 is an extension of the Borsuk–Ulam theorem in the sense that the latter says that the images of two extremely distant points are extremely close and the former says that the images of two quite distant points are close in a certain sense. A number of extensions and generalizations of the Borsuk–Ulam theorem exploit the symmetry argument and variate the condition on the initial points while preserving the condition of coinciding images. As a rule, these extensions and generalizations are not applicable to the case $\mathbb{S}^m \to \mathbb{R}^n$ with $m < n$. We study a new class of extensions covering the case with $m < n$. In these extensions, the condition of coinciding images is replaced with weaker conditions on the images "to be close in a certain sense". We consider several concepts of "closeness".

DEFINITION 2. Let $f\colon \mathbb{S}^m \to \mathbb{R}^n$ be a map.

- We say that two points a and b in $\mathbb{S}^m$ are *topological f–neighbors* if $f(a)$ and $f(b)$ are connected by a path in $\mathbb{R}^m$ whose interior does not meet $f(\mathbb{S}^m)$.
- We say that a and b are *visual f–neighbors* if the interior of the line segment with endpoints at $f(a)$ and $f(b)$ does not meet $f(\mathbb{S}^m)$.
- We say that a and b are *spherical f–neighbors* if $f(a)$ and $f(b)$ lie on the boundary of a metric ball whose interior does not meet $f(\mathbb{S}^m)$.

The concept of spherical f–neighbors can be generalised as follows.

DEFINITION 3. Let $f\colon X \to M$ be a map of metric spaces. We say that a subset Z of X is a *family of spherical f–neighbors* if $f(Z)$ lie on the boundary of a metric ball $B \subset M$ whose interior does not meet $f(X)$.

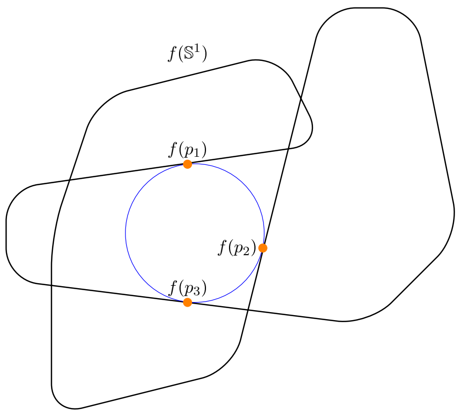

FIGURE 1. Images of spherical f–neighbors

Theorem 1 is a consequence of the following theorem.

THEOREM 4. *Let $\mathbb{S}^m$ be a unit sphere in $\mathbb{R}^{m+1}$ and let $f\colon \mathbb{S}^m \to \mathbb{R}^n$ be a continuous map. Then each point inside of $\mathbb{S}^m$ is contained in the convex hull of a family of spherical f–neighbors.*

Theorem 4 can be seen as an extension of the Borsuk–Ulam theorem if we reformulate the latter as follows: if $\mathbb{S}^m$ is a unit sphere in $\mathbb{R}^{m+1}$ and $f\colon \mathbb{S}^m \to \mathbb{R}^m$ is a continuous map, then for each point p inside of $\mathbb{S}^m$ there exist two points a and b in $\mathbb{S}^m$ such that $f(a) = f(b)$ and p is contained in the line segment with endpoints at a and b.

Theorem 4 has the following generalization.

Theorem 5. *Let Q be a compact subset in $\mathbb{R}^m$, let ∂Q be the boundary of Q, and let $f\colon \partial Q \to \mathbb{R}^n$ be a continuous map. Then every point of Q is contained in the convex hull of a family of spherical f–neighbors.*

We prove Theorems 4 and 5 in the framework of the theory of Voronoi diagrams and Delaunay triangulations. Another approach with non-null-homotopic coverings allows us to extend the theory to the case of maps to arbitrary contractible metric spaces. We give two examples.

Theorem 6 (cf. Theorem 1). *Let $\mathbb{S}^m$ be a unit sphere in $\mathbb{R}^{m+1}$ and let $f\colon \mathbb{S}^m \to M$ be a continuous map to a contractible metric space M. Then there are spherical f–neighbors p and q in $\mathbb{S}^m$ with*

$$\|p - q\| \geqslant \sqrt{\frac{m+2}{m}}.$$

Note that $\sqrt{\frac{m+2}{m}}$ is the distance between the centers of two $(m-1)$-dimensional faces of the same m-dimensional facet of a regular simplex inscribed in $\mathbb{S}^m$.

Theorem 7. *Let f be a continuous map from the boundary of an m-dimensional cube to a contractible metric space. Then there are spherical f–neighbors lying on disjoint faces of the cube.*

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Abdigappar Narmanov. *On the group of diffeomorphisms of foliated manifolds*

The diffeomorphism groups of smooth manifolds are of great importance in differential geometry and in analysis. The fundamental works in this area are the studies of V. I. Arnold, H. Omori, A. M. Lukatsky [1,2,4]. Intensive development of the theory of groups of diffeomorphisms began after the work of V. I. Arnold, in which it was shown that the motions of an ideal incompressible fluid are geodesic on a group of diffeomorphisms that preserve volume element.

It is known that the group $Diff(M)$ is topological group in compact open topology ([5], page 270), [3].

We will denote by (M, F) manifold M with $k-$dimensional foliation F on M [6].

In this talk we investigate some subgroups of the group $Diff_F(M)$ of diffeomorphisms of the foliated manifold (M, F).

Let $L(p)$ be a leaf of the foliation F passing through point the p, T_pF- the tangent space to the the leaf $L(p)$ at p. We get subbundle (smooth distribution) $TF = \{T_pF : p \in M\}$ of the tangent bundle TM of the manifold M.

Let us denote by $V(M), V(F)$ the set of smooth sections of bundles TM, TF respectively.

Definition 1. If for the some C^r- diffeomorphism $\varphi : M \to M$ the image $\varphi(L_\alpha)$ of any leaf L_α of foliation F is a leaf of foliation F, we say that the f is C^r- diffeomorphism of foliated manifold and write as $f : (M, F) \to (M, F)$.

Example 2. Let $M = R^2(x, y)-$ Euclidean plane with the Cartesian coordinates (x, y). Leaves L_α of foliation F are given by the equations $x^2 - y = \alpha = const$. Then the plan diffeomorphism $\varphi : R^2 \to R^2$ determined by the formula

$$\varphi(x, y) = (x, y + \lambda f(x, y))$$

is diffeomorphism of foliated plane (R^2, F), for every $\lambda \in R$, such that $\lambda \neq 1$. It sends a leaf L_α to $L_{(1-\lambda)\alpha}$. It is easy to check that in fact $\varphi : R^2 \to R^2$ is an isometry of foliated plane (R^2, F).

Example 3. Let $(M, F)-$ foliated manifold, where $F-$ is $k-$ dimensional smooth foliation where $0 < k < n$. Recall a vector field X is called a foliated field if for every vector field Y, tangent to F, Lie brocket $[X, Y]$ also is tangent to F. It is known that flow of every foliated field consists of diffeomorphisms of foliated manifold (M, F) [6].

For foliated plane from Example 2 vector field $X = (x^2 - y)\frac{\partial}{\partial y}$ is foliated field and its flow consists of diffeomorphisms $\varphi^t : (x, y) \in R^2 \to (x, x^2 - e^{-t}(x^2 - y)) \in R^2$ of foliated plane (R^2, F). Every diffeomorphism $\varphi^t : (x, y) \in R^2 \to (x, x^2 - e^{-t}(x^2 - y)) \in R^2$ sends a leaf L_α to $L_{e^{-t}\alpha}$.

Let's denote as $Diff_F(M)-$ the set of all C^r diffeomorphisms of foliated manifold (M, F), where $r \geqslant 0$. The group $Diff_F(M)$ is subgroup of $Diff(M)$ and therefore it is topological group in compact open topology.

Theorem 4. *Let (M, F) is foliated manifold where M is a smooth connected finite-dimensional manifold. Then the group $Diff_F(M)$ is a closed subgroup of $Diff(M)$ in compact open topology.*

The closedness of the set $Diff_F(M)$ allows us to state the following corollary.

Corollary 5. *Factor space $Diff(M)/Diff_F(M)$ is regular homogeneous topological space.*

We will introduce some topology on the group $Diff_F(M)$, which depends on foliation F and coincides with compact open topology when F is n-dimensional foliation.

Let $\{K_\lambda\}$ be a family of all compact sets where each K_λ is a subset of some leaf of foliation F and let $\{U_\beta\}-$ family of all open sets on M. We consider for each pair K_λ and U_β set of all mappings $f \in G^r_F(M)$, for which $f(K_\lambda) \subset U_\beta$. This set of mappings we denote through $[K_\lambda, U_\beta] = \{f : M \to M | f(K_\lambda) \subset U_\beta\}$.

It isn't difficult to show that every possible finite intersections of sets of the form $[K_\lambda, U_\beta]$ forms a base for some topology. This topology we call foliated compact open topology or in brief $F-$compact open topology. Let's denote as $Diff^0_F(M)$ set of all C^r diffeomorphisms $g \in Diff_F(M)$ of foliated manifold (M, F), such that $g(L_\alpha) = L_\alpha$ for every L_α leaf of foliation F. Flow of every tangent vector field consists of diffeomorphisms of foliated manifold (M, F), which belong to the group $Diff^0_F(M)$.

It can be proven following theorems.

Theorem 6. *Let (M, F) — foliated manifold where M — is a smooth, connected and finite-dimensional manifold. Then the group $Diff^0_F(M)$ is a topological group with $F-$compact open topology.*

Theorem 7. *Let all leaves of foliated manifold (M, F) are closed subsets of M. Then the group $Diff^0_F(M)$ is closed subset of $Diff_F(M)$ in $F-$compact open topology.*

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Amos Nevo. *The Shannon–McMillan–Breiman theorem for Rokhlin entropy in actions of general groups*

(Based on joint work with F. Pogorzelski (Leipzig University))

In recent years, the classical theory of entropy for a measure-preserving dynamical system has been revolutionized by ground-breaking work initiated by Lewis Bowen and by Brandon Seward. Two distinct definitions of entropy were proposed, which apply to actions of very general groups, including non-amenable ones. The first, initiated by Bowen in 2008, applies to sofic groups, a very large class of groups which generalizes the class of amenable groups. It was termed sofic entropy by Bowen, and is based on the elaborate auxiliary machinery provided by the sofic structure. A completely different definition, initiated by Seward in 2014, applies to all countable groups whatsoever. This definition is simpler and makes no appeal to auxiliary structures, and is very natural and direct. Its motivation is a classical theorem of Rokhlin which gives an alternative characterization of the Kolmogorov–Sinai entropy of a a measure-preserving transformation, and Seward has thus termed this invariant Rokhlin entropy.

We will start with a very brief account of Seward's definition of Rokhlin entropy and some of its properties, and then describe our own recently developed approach to the construction of orbital Rokhlin entropy for probability-measure-preserving free actions of all countable groups. This construction is motivated by Seward's definition and Danilenko's notion of orbital entropy (which he developed in the amenable case). We will then formulate our main result, namely that orbital Rokhlin entropy actually coincides with Rokhlin entropy and satisfies a natural version of the Shannon–McMillan–Breiman pointwise convergence theorem. We will demonstrate the dynamical significance of the entropy equipartition result for finite partitions that this convergence theorem entails, and its relation to the boundary of the group in the case of actions of free non-Abelian groups.

Vladimir Nezhinskii. *Rational graphs*

The work deals with the spatial graph theory. In the talk, analogues of rational knots will be defined and classified (up to isotopy).

Mikhail Ovchinnikov. *On classification of nonorientable 3-manifolds of small complexity*

The talk is devoted to the problem of enumeration of nonorientable compact 3-manifolds in increasing order of their complexity.

Let M be a compact 3-manifold and P be a 2-polyhedron lying in M. P is called *spine* of M if $M \backslash P \approx \partial M \times (0, 1]$, either $\approx \mathrm{Int} B^3$. A spine is called *simple spine* if the link of each its singular point is homeomorphic to the circle with diameter, either to the circle with three radii. The minimal number of the second type singular points over all simple spines of a given 3-manifold M is called *complexity* $c(M)$. There are some results on enumeration of orientable compact 3-manifolds and closed nonorientable 3-manifolds in increasing order of their complexity ([1,2]).

We consider nonorientable 3-manifolds with arbitrary boundary. The set of 26 simple spines with two second type singular points which contains all minimal spines of complexity two nonorientable 3-manifolds has been generated by computer program. As a result of a detailed consideration we have the most natural presentations for 3-manifolds of the set. In the most complexive cases manifolds are described by pairs of the kind (direct product of the Mobius band and interval, proper arc). We get the corresponding manifold by removing a regular neighbourhood of the arc. For each manifold a 2-polyhedron lying in the 3-space is constructed which naturally shows the embedding of the spine in appropriate simple nonorientable 3-manifold ("thicken" Mobius band in the last example). In particular some pairs of homeomorphic manifolds have been recognized.

The work is supported by RFBR grant No.17-01-00690

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Makoto Ozawa. *Multibranched surfaces in 3-manifolds*

A multibranched manifold is a second countable Hausdorff space that is locally homeomorphic to multibranched Euclidean space. In this talk, we concentrate compact 2-dimensional multibranched manifolds (multibranched surfaces) embedded in 3-manifolds. We give a necessary and sufficient condition for a multibranched surface to be embedded in some closed orientable 3-manifold. Then we can define the genus of a multibranched surface in virtue of Heegaard genera of 3-manifolds, and show an inequality between the genus, the number of branch loci and regions. We determine whether two multibranched surfaces have same neighborhood by means of local moves. Similarly to the graph minor, we also introduce a minor on multibranched surfaces, and consider the obstruction set for the set of multibranched surfaces embedded in the 3-sphere. This talk is a survey including recent joint works with Kazufumi Eto, Shosaku Matsuzaki, Mario Eudave-Munoz, Kai Ishihara, Yuya Koda, Koya Shimokawa.

Multibranched manifold

A *multibranched manifold* is a second countable Hausdorff space that is locally homeomorphic to multibranched Euclidean space.

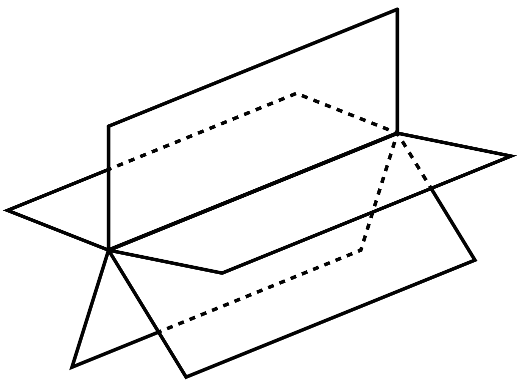

FIGURE 1. multibranched Euclidean space (The quotient space obtained from i copies of $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geqslant 0\}$ by identifying with their boundaries $\partial\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n = 0\}$ via identify mappings.)

Degree and wrapping number

Any multibranched surface X can be constructed from a closed 1-manifold $B_X = B_1 \cup \cdots \cup B_m$ (*brach loci*) and a compact 2-manifold with boundary $S_X = S_1 \cup \cdots \cup S_n$ (*regions*) by identifying via a covering map $f : \partial S_X \to B_X$. The *degree* of a branch locus B_i is defined as $deg(B_i) = d$ if $f|_{f^{-1}(B_i)} : f^{-1}(B_i) \to B_i$ is a d-fold covering ($d > 2$). For each component C of ∂S_X, the *wrapping number* of C is defined as $wrap(C) = w$ if $f|_C$ is a w-fold covering.

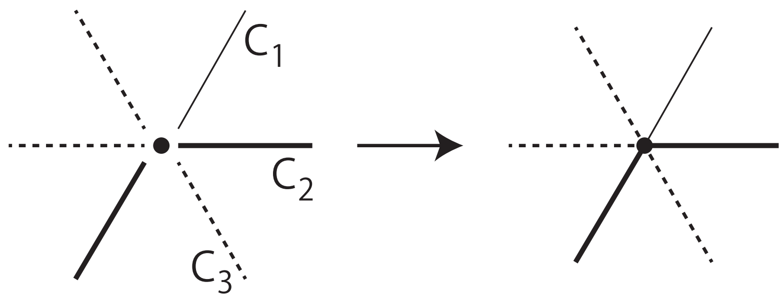

FIGURE 2. $deg(B_i) = 6$, $wrap(C_i) = i$

We say that a multibranched surface X is *regular* if for each branch locus B_i of X, $deg(B_i) = k \times wrap(C)$ for some $k \in \mathbb{N}$ and any component C of $f^{-1}(B_i)$.

Embeddability in $\mathbb{R}^4$ and 3-manifolds

By the Menger–Nöbeling theorem, any finite 2-dimensional CW complex can be embedded into the 5-dimensional Euclidian space $\mathbb{R}^5$. Furthermore, we have

PROPOSITION 1 ([5]). *Any multibranched surface can be embedded in the 4-dimensional Euclidean space* $\mathbb{R}^4$.

PROPOSITION 2 ([5]). *A multibranched surface can be embedded in some closed orientable 3-manifold if and only if it is regular.*

The following is a fundamental problem.

PROBLEM 3. For a given multibranched surface, determine whether it can be embedded in the 3-sphere S^3 (or equivalently $\mathbb{R}^3$).

Genus – filtration via 3-manifolds

It is known that any closed orientable 3-manifold M has a decomposition into two handlebodies V_1 and V_2, where $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2$. The *Heegaard genus* $g(M)$ of M is defined as the minimal genus among all such decompositions. For a regular multibranched surface X, we define the *genus* of X as the minimal Heegaard genus $g(M)$ among all closed orientable 3-manifold M, where X can be embedded in M.

THEOREM 4 ([5]). *If a regular multibranched surface X has m branch loci and n regions, then*

$$g(X) \leqslant m + n.$$

Moreover, if the wrapping number of each branch locus is 1*, then*

$$g(X) \leqslant n.$$

For a graph G, we obtain a regular multibranched surface by taking a product with S^1, that is, for each vertex v_i of G, $v_i \times S^1$ forms a loop and for each edge e_j of G, $e_j \times S^1$ forms an annulus.

THEOREM 5 ([3]). $g(G \times S^1) \leqslant 2g(G)$.

When $g(G) = 0$, namely, G is planar, $G \times S^1$ can be embedded in the 3-sphere S^3 and hence $g(G \times S^1) = 0$. Thus Theorem 5 is best possible when $g(G) = 0$.

When $g(G) \geqslant 1$, G has a K_5 or $K_{3,3}$ minor. In the following theorem, we determine the genus of $K_5 \times S^1$ and $K_{3,3} \times S^1$, and therefore Theorem 5 is best possible when $g(G) = 1$.

THEOREM 6 ([3]). $g(K_5 \times S^1) = 2$ *and* $g(K_{3,3} \times S^1) = 2$.

REMARK 7. Theorem 5 is now in progress. We have almost approached the equality in Theorem 5.

IX-move and XI-move

Let A be an annulus region of X whose boundary consists of two branch loci, where at least one branch locus B_i has the wrapping number 1, or a Möbius-band region whose boundary has the wrapping number 1. An *IX-move* along A is an operation shrinking A into the core circle. An *XI-move* is a reverse operation of an IX-move.

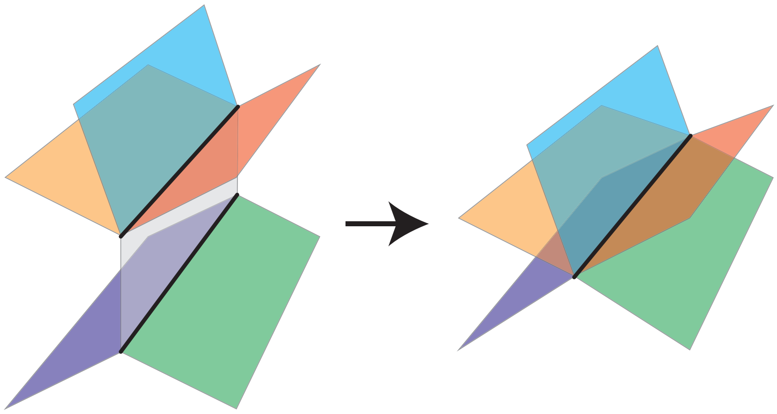

FIGURE 3. IX-move ([4])

Neighborhood equivalence

Here, we assume that a multibranched surface is regular, does not have disk regions, and the degree $deg(B_i)$ is greater than 2 for each branch locus B_i.

THEOREM 8 ([4]). *Let X, X' be multibranched surfaces in an orientable 3-manifold M. If $N(X)$ is isotopic to $N(X')$ in M, then X is transformed into X' by a finite sequence of IX-moves, XI-moves and isotopies.*

Minor

Here, we allow the degree $deg(B_i)$ of a branch locus B_i to be 1 or 2. We denote by $\mathscr{M}$ the set of all regular multibranched surfaces (modulo homeomorphism). For $X, Y \in \mathscr{M}$, we write $X < Y$ if X is obtained by removing a region of Y, or X is obtained from Y by an IX-move. We define an equivalence relation $\sim$ on $\mathscr{M}$ as follows: if $X < Y$ and $Y < X$, then $X \sim Y$. We define a partial order $\prec$ on $\mathscr{M}/\sim$ as follows. Let X, $Y \in \mathscr{M}$. We denote $[X] \prec [Y]$ if there exists a finite sequence $X_1, \ldots, X_n \in \mathscr{M}$ such that $X_1 \sim X$, $X_n \sim Y$ and $X_1 < \cdots < X_n$.

Obstruction set

A multibranched surface class $[X]$ is called a *minor* of a multibranched surface class $[Y]$ if $[X] \prec [Y]$. In particular, $[X]$ is called a *proper minor* of $[Y]$ if $[X] \prec [Y]$ and $[Y] \nprec [X]$. A subset $\mathscr{P}$ of $\mathscr{M}/\sim$ is said to be *minor closed* if for any $[X] \in \mathscr{P}$, every minor of $[X]$ belongs to $\mathscr{P}$. For a minor closed set $\mathscr{P}$, we define the *obstruction set* $\Omega(\mathscr{P})$ as follows:

$$\Omega(\mathscr{P}) = \{[X] \in \mathscr{M}/\sim \mid [X] \notin \mathscr{P},\ \text{Every proper minor of } [X] \text{ belongs to } \mathscr{P}.\}$$

The set of multibranched surfaces embeddable into S^3, denoted by $\mathscr{P}_{S^3}$, is minor closed. As a 2-dimensional version of Kuratowski's and Wagner's theorems, we consider the next problem.

PROBLEM 9. Characterize the obstruction set $\Omega(\mathscr{P}_{S^3})$.

THEOREM 10 ([3], [1], [5]). *The following multibranched surfaces belong to $\Omega(\mathscr{P}_{S^3})$.*

- *non-orientable closed surfaces*
- *$K_5 \times S^1$ and $K_{3,3} \times S^1$ [3]*
- *X_1, X_2, X_3 given in [1]*
- *$X_g(p_1, p_2, \ldots, p_n)$ given in [5]*

A partial order on multibranched surfaces

In this subsection, we restrict multibranched surfaces to the set $\mathscr{X}$ of all connected compact multibranched surfaces X embedded in a closed orientable 3-manifold M satisfying the following conditions: X is maximally spread (that is, applied XI-moves to X as much as possible), essential in M, has no open disk sector, no branch of degree 1 or 2.

We define a binary relation $\leqslant$ over $\mathscr{X}$ as follows.

Definition 11. For $X, Y \in \mathscr{X}$, we denote $X \leqslant Y$ if

(1) there exists an isotopy of Y in M so that $Y \subset N(X)$ and $B_Y \subset N(B_X)$, and
(2) there exists no essential annulus in $N(X) - Y$.

For equivalence classes $[X], [Y] \in \mathscr{X}/\sim$, we define a binary relation $\preceq$ over $\mathscr{X}/\sim$ so that $[X] \preceq [Y]$ if $X \leqslant Y$.

Theorem 12 ([6]). *The relation $\preceq$ is well-defined on $\mathscr{X}/\sim$ and $(\mathscr{X}/\sim; \preceq)$ is a partially ordered set.*

Theorem 13 ([6]). *For equivalence classes $[X], [Y] \in \mathscr{X}/\sim$, if $[X] \preceq [Y]$ and $[X] \neq [Y]$, then either B_Y is toroidal or S_Y is cylindrical.*

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Burak Özbağci. *Genus one Lefschetz fibrations on disk cotangent bundles of surfaces*

We describe a Lefschetz fibration of genus one on the disk cotangent bundle of any closed orientable surface S. As a corollary, we obtain an explicit genus one open book decomposition adapted to the canonical contact structure on the unit cotangent bundle of S.

Taras Panov. *A geometric view on SU-bordism*

(Based on joint work with Zhi Lu, Ivan Limonchenko and Georgy Chernykh)

The development of algebraic topology in the 1960 culminated in the description of the special unitary bordism ring. Most leading topologists of the time contributed to this result, which combined the classical geometric methods of Conner–Floyd, Wall and Stong with the Adams–Novikov spectral sequence and formal group law techniques that emerged after the fundamental 1967 work of Novikov. Thanks to toric topology, a new geometric approach to calculations with SU-bordism has emerged, which is based on representing generators of the SU-bordism ring and other important SU-bordism classes by quasitoric manifolds and Calabi–Yau hypersurfaces in toric varieties.

Seonjeong Park. *Torus orbit closures in Richardson varieties*

(Based on joint work with Eunjeong Lee and Mikiya Masuda)

The flag variety $\mathscr{F}\ell_n$ is a smooth projective variety consisting of chains $(\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ of subspaces of $\mathbb{C}^n$ with $\dim_{\mathbb{C}} V_i = i$. Then the standard action of $\mathbb{T} = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ induces a natural action of $\mathbb{T}$ on $\mathscr{F}\ell_n$. For v and w in the symmetric group $\mathfrak{S}_n$ with $v \leqslant w$ in Bruhat order, the Richardson variety X_w^v is defined to be the intersection of the Schubert variety X_w and the opposite Schubert variety $w_0 X_{w_0 v}$, and it is an irreducible $\mathbb{T}$-invariant subvariety of $\mathscr{F}\ell_n$. In general, Richardson varieties are not a toric variety. In this talk, we give some combinatorial interpretation of the generic torus orbit closures in Richardson varieties. We also show that if Richardson variety is itself a smooth toric variety, then it is a Bott tower.

Garik Petrosyan. *On the boundary value periodic problem for a semilinear differential inclusion of fractional order with delay*

In the present paper, for a semilinear fractional order functional differential inclusion in a separable Banach space E of the form

$$ {}^{C}D^{q}x(t) \in Ax(t) + F(t, x_t), \ t \in [0, T], \tag{1} $$

we consider the problem of existence of mild solutions to this inclusion satisfying the following periodic boundary value condition

$$ x(0) = x(T). \tag{2} $$

The symbol ${}^{C}D^{q}$ denotes the Caputo fractional derivative of order $q \in (0, 1)$, x_t prehistory of the function until $t \in [0, T]$, that is $x_t(s) = x(t + s), s \in [-h, 0], 0 < h < T$. Everywhere in the sequel we suppose that the linear operator A satisfies condition

- (A) $A : D(A) \subseteq E \to E$ is a linear closed (not necessarily bounded) operator generating a bounded C_0–semigroup $\{U(t)\}_{t\geqslant 0}$ of linear operators in E.

We will assume that the multivalued nonlinearity $F : [0, T] \times C([-h, 0]; E) \to Kv(E)$, where $Kv(E)$ - a collection of all nonempty compact convex subsets of E, obeys the following conditions:

- $(F1)$ for each $\xi \in C([-h, 0]; E)$ the multifunction $F(\cdot, \xi) : [0, T] \to Kv(E)$ admits a measurable selection;
- $(F2)$ for a.e. $t \in [0, T]$ the multimap $F(t, \cdot) : E \to Kv(E)$ is upper semicontinuous;
- $(F3)$ there exists a function $\alpha \in L^{\infty}_{+}([0, T])$ such that

$$ \|F(t, x_t)\|_E \leqslant \alpha(t)(1 + \|x_t\|_{C([-h,0];E)}) \text{ for a.e. } t \in [0, T], $$

- $(F4)$ there exists a function $\mu \in L^{\infty}([0, T])$ such that for each bounded set $\Delta \subset C([-h, 0]; E)$ we have:

$$ \chi(F(t, \Delta)) \leqslant \mu(t)\varphi(\Delta), $$

  for a.e. $t \in [0, T]$, where $\varphi(\Delta) = \sup_{s\in[-h,0]} \chi(\Delta(s))$, χ is the Hausdorff MNC in E, $\Delta(s) = \{y(s) : y \in \Delta\}$.

Theorem 1. *Under conditions* (A), $(F1) - (F4)$, *suppose, additionally that*

- $(A1)$ *the semigroup U is exponentially decreasing in the sense that*

$$ \|U(t)\| \leqslant e^{-\eta t}, \quad t \geqslant 0 $$

  for some $\eta > 0$.

If

$$ \frac{k}{\eta} < 1, \tag{3} $$

where $k = \max\{\|\alpha\|_\infty, \|\mu\|_\infty\}$, then problem (1)-(2) *has a solution.*

The work is supported by the Ministry of Education and Science of the Russian Federation in the frameworks of the project part of the state work quota (Project No 1.3464.2017/4.6) and by RFBR and MOST according to the research project No 17-51-52022.

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Sergei Pilyugin. *Approximate and exact dynamics in group actions*

Let Φ be a uniformly continuous action of a finitely generated group G on a metric space.

The shadowing property of Φ means that, given an approximate trajectory, we can find an exact trajectory close to it. The inverse shadowing property of Φ means that, given a family of approximate trajectories (generated by a so-called approximate method), for any fixed exact trajectory of Φ, we can find a member of this family that is close to this fixed trajectory.

The Reductive Shadowing Theorem (RST) states that if the action of a one-dimensional subgroup of G is topologically Anosov (i.e., it has the shadowing property and is expansive), then the action Φ is topologically Anosov as well (and hence, Φ has the shadowing property).

The first RST was proved in [1] for the groups $\mathbb{Z}^p$; later it was generalized to the case of virtually nilpotent groups [2]. At the same time, it was shown in [2] that the RST is not valid, for example, for the Baumslag–Solitar groups $\mathrm{BS}(1,n)$ with $n > 1$.

It is shown in [3] that an analog of the RST for the case of inverse shadowing (with "topologically Anosov" replaced by the so-called "Tube Condition") is also valid for virtually nilpotent groups.

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Grigory Polotovskiy. *V. A. Rokhlin and D. A. Gudkov against the background of the 16th Hilbert problem (on the Rokhlin-Gudkov's correspondence in 1971–1982)*

The story is about the period of the mathematical biography of Vladimir Abramovich Rokhlin connected with its occupations topology of real algebraic varieties and about his friendship and cooperation with the Nizhni Novgorod mathematician Dmitry Andreevich Gudkov. The talk is based on the correspondence of D. A. Gudkov with V. A. Rokhlin in 1971–1982 from Gudkov's archive. In total 15 letters of V. A. Rokhlin and 8 letters of D. A. Gudkov[1] are kept. This correspondence is completely published in [1], pp. 191–208.

The 70th — 80th years of the last century can be characterized as the period of "Storm and Drang" in the field of the first part of the 16th Hilbert problem. The start of this period was laid by Gudkov's classification of nonsingular curves of degree 6 (1969) and by his conjecture ("the Gudkov congruence") about the congruence modulo 8 for some topological characteristic of M-curves of even degree, and the followed proofs of "half" of this congruence (i.e. modulo 4) by V. I. Arnold (1971) and in full (modulo 8) by V. A. Rokhlin (1972).

I will list some main subjects touched on in the specified correspondence.

1) A discussion of structure and content of the survey [2] on which D. A. Gudkov began to work not later than 1970[2]. In particular, V. A. Rokhlin periodically informed D. A. Gudkov about new versions of the proof of the Gudkov congruence.

2) In turn, V. A. Rokhlin wrote (the letter of 14.11.1971): "Of course, I will be grateful for any information on your examples. Can you send me your articles or even your survey for me to study the subject more thoroughly?" And Dmitry Andreevich shared in the letters all information known to him, in particular, about plane curves of odd degree, spatial curves and algebraic surfaces in $\mathbb{R}P^3$ (for example, in letters of 24.04.1972 and 11.06.1972) and bibliographic data (in the letter of 1973). V. A. Rokhlin wrote in the letter of 15.01.1974: "I looked for information on curves of degrees 8, 10, 12 in the compositions sent by you, but found almost nothing. Is there any more or less full table of the constructed curves?" According to this inquiry of V. A. Rokhlin and at the request of D. A. Gudkov, such table was done, sent to V. A. Rokhlin and then published in [3].

3) Different scientific-organizational questions. In particular, the topic of the development of topological education in Gorky was discussed. As a result, Gudkov with the help and support of Rokhlin's students, O. Ya. Viro and V. M. Kharlamov, created an obligatory course of topology at the Nizhni Novgorod (Gorky) University, based on records of lectures of V. A. Rokhlin. Also the situation connected with simultaneous emergence of papers [4] and [5] containing close results was discussed.

Unfortunately, friendly relations between V. A. Rokhlin and D. A. Gudkov were mainly epistolary. In letters, they invited each other for a visit repeatedly, but their health problems interfered with that. So they saw each other seldom. I can specify only three meetings, among them — a visit of V. A. Rokhlin and his wife A. A. Gurevich to Gorky in the 70th.

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[1] D. A. Gudkov kept draft copies or copies of many letters.

[2] D. A. Gudkov also consulted much with V. I. Arnold and O. A. Oleynik about this survey.

Correspondence and friendship between V. A. Rokhlin and D. A. Gudkov quickly extended to their students: O. Ya. Viro, V. M. Kharlamov, T. Fiedler and V. I. Zvonilov in Leningrad–St. Petersburg and G. A. Utkin, A. B. Korchagin, E. I. Shustin and G. M. Polotovskiy in Gorky–Nizhni Novgorod. This remarkable circumstance led to regular information exchange and exchange of results, long before they were published. This considerably accelerated the development of research on topology of real algebraic varieties.

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Theodore Popelensky. *On certain new results on the Steenrod algebra* mod p

For the Steenrod algebra mod $p, p > 2$, three additive bases are well known. Namely, basis of admissible monomials, Milnor basis, and P^t_s-basis. To be more precise, there is a family of P^t_s-bases. We develop certain new additive bases in the Steenrod algebra mod p. We investigate which pairs of bases has triangular transition matrix. Finally we plant to discuss certain applications.

Clement Radu Popescu. *Resonance varieties. Definition and results*

Falk introduced the resonance varieties in the context of complex hyperplane arrangements in 1997. Since then these cohomology jump loci have been studied by several authors: Cohen–Suciu, Libgober–Yuzvinsky, Falk–Yuzvinsky, Papadima–Suciu. They were generalised by Dimca–Papadima–Suciu. I will present results obtained in collaboration with Berceanu, Măcinic, Papadima and Suciu.

The results are obtained over the field of complex numbers $\mathbb{C}$.

Let $A = (A^\bullet, d)$ be a commutative differential graded algebra (for short cdga) and $\mathfrak{g}$ be a Lie algebra. On the tensor product $A \otimes \mathfrak{g}$ we define a graded differential Lie algebra structure (for short dgla) with the Lie bracket $[\alpha \otimes x, \beta \otimes y] = \alpha\beta \otimes [x, y]$ and differential $\partial(\alpha \otimes x) = d\alpha \otimes x$.

We define the set of flat connections as $\mathscr{F}(A, \mathfrak{g}) = \{\omega \in A^1 \otimes g \mid \partial\omega + \frac{1}{2}[\omega, \omega] = 0\}$. If the cdga A is connected (i.e. $A^0 = \mathbb{C} \cdot 1$) and $\mathfrak{g} = \mathbb{C}$, then $\mathscr{F}(A, \mathbb{C}) = H^1(A)$.

For $\mathscr{F}(A, \mathfrak{g})$ there exists a filtration which contain informations about the algebra A. Given a connected cdga A which is also of finite $q-$type ($A^{\leqslant q}$ is finite dimensional) and a linear representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$, the tensor product $A \otimes V$ becomes a cochain complex with the *covariant derivative* $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$. More precisely, for $\omega \in \mathscr{F}(A, \mathfrak{g})$ define $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega\colon A^i \otimes V \to A^{i+1} \otimes V$. Explicitly, if $\omega = \sum_i \alpha_i \otimes a_i$, with $\alpha_i \in A^1$ and $a_i \in \mathfrak{g}$ then $d_\omega(\beta \otimes v) = d\beta \otimes v + \sum_i \alpha_i\beta \otimes \theta(a_i)v$. For $\omega \in \mathscr{F}(A, \mathfrak{g})$, $d^2_\omega = 0$ and we get the *Aomoto cochain complex*:

$$(A \otimes V, d_\omega)\colon\ A^0 \otimes V \xrightarrow{d_\omega} A^1 \otimes V \xrightarrow{d_\omega} A^2 \otimes V \xrightarrow{d_\omega} \cdots$$

The *relative resonace varieties* in degree $i \geqslant 0$ and depth $r \geqslant 0$, with respect to the representation θ is the set

$$\mathscr{R}^i_r(A, \theta) = \{\omega \in \mathscr{F}(A, \mathfrak{g}) \mid \dim_{\mathbb{C}} H^i(A \otimes V, d_\omega) \geqslant r\}$$

If $\mathfrak{g}$ and V are finite dimensional then the relative resonance varieties are Zariski closed in $\mathscr{F}(A, \mathfrak{g})$. These form a filtration of the set of flat connections. The simplest case is that of resonance varieties $\mathscr{R}^i_r(A)$ for which the Lie algebra $\mathfrak{g} = \mathbb{C}$ and $\theta = \mathrm{id}_{\mathbb{C}}$.

Suppose that A is 1-finite and $\mathfrak{g}$ is finite dimensional. Consider $\mathscr{F}^1(A, \mathfrak{g}) \subseteq \mathscr{F}(A, \mathfrak{g})$ consisting of tensors $\eta \otimes g$ for which $d\eta = 0$. For a representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ consider $\Pi(A, \theta) \subseteq \mathscr{F}^1(A, \mathfrak{g})$ the subvariety of the tensors which satisfy also the equality $\det(\theta(g)) = 0$. For the case $\mathfrak{g} = \mathbb{C}$, $\mathscr{F}^1(A, \mathbb{C}) = \mathscr{F}(A, \mathbb{C})$ and $\Pi(A, \theta) = \{0\}$.

In general we obtained:

THEOREM 1. *Let $\omega = \eta \otimes g$ be an arbitrary element of $\mathscr{F}^1(A, \mathfrak{g})$. Then ω belongs to $\mathscr{R}_1^k(A, \theta)$ if and only if there is an eigenvalue λ of $\theta(g)$ such that $\lambda\eta$ belongs to $\mathscr{R}_1^k(A)$. Moreover,*

$$\Pi(A, \theta) \subseteq \bigcap_{k: H^k(A) \neq 0} \mathscr{R}_1^k(A, \theta).$$

Let A be a connected and 1-finite cdga for which the variety $\mathscr{R}_1^1(A)$ decomposes as a finite union of linear subspaces of A^1, and let θ be a finite dimensional representation of a finite dimensional Lie algebra $\mathfrak{g}$.

We get the following result:

THEOREM 2. *Suppose $\mathscr{R}_1^1(A) = \bigcup_{C \in \mathscr{C}} C$, a finite union of linear subspaces. For each $C \in \mathscr{C}$, let A_C denote the sub-cdga of the truncation $A^{\leqslant 2}$ defined by $A_C^1 = C$ and $A_C^2 = A^2$. Then, for any Lie algebra $\mathfrak{g}$,*

$$\mathscr{F}(A, \mathfrak{g}) \supseteq \mathscr{F}^1(A, \mathfrak{g}) \cup \bigcup_{0 \neq C \in \mathscr{C}} \mathscr{F}(A_C, \mathfrak{g}), \tag{1}$$

where each $\mathscr{F}(A_C, \mathfrak{g})$ is Zariski-closed in $\mathscr{F}(A, \mathfrak{g})$. Moreover, if A has zero differential, A^1 is non-zero, and $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sol}_2$, then (1) *holds as an equality, and, for any θ,*

$$\mathscr{R}_1^1(A, \theta) = \Pi(A, \theta) \cup \bigcup_{0 \neq C \in \mathscr{C}} \mathscr{F}(A_C, \mathfrak{g}). \tag{2}$$

For a 1-formal 1-finite space X and the cdga $A = (H^{\bullet}(X, \mathbb{C}), d = 0)$ it is known that the resonance variety, $\mathscr{R}_1^1(A)$ is a finite union of linear subspaces of A^1.

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a finite simple graph with vertex set V and edge set E, and π_Γ the right-angled Artin group associated to Γ. An example of a space with the above mentioned properties (1-formal, 1-finite space) is the classifying space X_Γ of π_Γ. For the computation of the resonance variety $\mathscr{R}_1^1(A)$, the model of the space is the cdga $A_\Gamma = (H^{\bullet}(\pi_\Gamma, \mathbb{C}), d = 0)$. For this we can apply Theorem 2 and we have obtain the following irreducible decomposition:

$$\mathscr{F}(A_\Gamma, \mathfrak{sl}_2) = \bigcup_{\mathsf{W} \subseteq \mathsf{V}} S_{\mathsf{W}}$$

where W runs through the subsets of the vertex set of Γ, maximal with respect to an order $\preccurlyeq$ defined in terms of the connected components of the induced subgraph Γ_{W}. S_{W} is a certain combinatorially defined, closed subvariety of $\mathbb{C}^{\mathsf{W}} \otimes \mathfrak{sl}_2$.

For an arbitrary Lie algebra $\mathfrak{g}$, we show that the variety $\mathscr{F}(A_\Gamma, \mathfrak{g})$ contains the union of the subvarieties S_{W}, defined in a similar manner as above. If $\mathfrak{g}$ is semisimple and different from $\mathfrak{sl}_2$ we show by example that this containment can be strict.

Consider now X to be a *quasi-projective manifold* (irreducible, smooth, quasi-projective variety), and assume $b_1(X) > 0$ ($b_1(X)$ is the first Betti number of X). In this case, the irreducible decomposition of $\mathscr{R}_1^1(A)$, for a suitable Gysin model A of X, can be described in geometric terms. The irreducible components are all linear, and they are indexed by a finite list, denoted $\mathscr{E}_X$, of equivalence classes of regular "admissible" maps $f\colon X \to S$, where the quasi-projective manifolds S are 1-dimensional, and have negative Euler characteristic.

In the 1-formal situation, we use again Theorem 2 to find explicit (global) irreducible decompositions for $\mathscr{F}(H^{\bullet}(X, \mathbb{C}), \mathfrak{g})$ and $\mathscr{R}_1^1(H^{\bullet}(X, \mathbb{C}), \theta)$ in the case when $\mathfrak{g} = \mathfrak{sl}_2$, solely in terms of the set $\mathscr{E}_X$ and of the representation θ.

In the (much more delicate) general case, we showed that $\mathscr{F}(A, \mathfrak{g}) = \mathscr{F}^1(A, \mathfrak{g})$ and $\mathscr{R}_1^1(A, \theta) = \Pi(A, \theta)$, for an arbitrary Gysin model A of X, and for a representation θ of $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sol}_2$, provided that $\mathscr{R}_1^1(X) = \{0\}$.

Consider now a particular case of a quasi-projective manifold. Let Γ be a finite simple graph with cardinality n vertex set V and edge set E. The *partial configuration space* of type Γ on a space Σ is

$$F(\Sigma, \Gamma) = \{z \in \Sigma^{\mathsf{V}} \mid z_i \neq z_j, \quad \text{for all} \quad ij \in \mathsf{E}\}. \tag{3}$$

When $\Gamma = K_n$, the complete graph with n vertices, $F(\Sigma, \Gamma)$ is the classical ordered configuration space of n distinct points in Σ. Consider $\Sigma = \Sigma_g$ to be a compact genus g Riemann surface with partial configuration space denoted $F(g, \Gamma)$. For this quasi-projective manifold $X = F(g, \Gamma)$ there is a specific Gysin model A of X. For this data we obtained the irreducible decompositions of $\mathscr{F}(A, \mathfrak{g})$ (where $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sol}_2$) and of $\mathscr{R}_1^1(A, \theta)$ for any linear representation $\theta : \mathfrak{g} \mapsto \mathfrak{gl}(V)$, ($\mathfrak{g}$ as before).

Nigel Ray. *Partially ordered sets in algebraic topology*

Partially ordered sets and lattices have played an influential rôle in the development of analytic topology over many decades, but their relationship with homotopy theory and algebraic topology may be less well known to a general audience. I shall try to address this situation by focusing on two or three types of example.

Firstly I shall consider certain posets of subgroups of a given group, and introduce Quillen's applications of their associated homotopy type to algebraic properties of the group; these include a conjecture from 1978 which appears still to be open. Secondly, I shall discuss the poset of compactifications of a locally compact Hausdorff space, with special reference to cases involving configuration spaces of unordered pairs of particles in a Riemannian manifold, and their associated laws of collision. Thirdly, if time allows, I shall refer briefly to Bousfield's lattice of spectra in stable homotopy theory, and its relationship to Ravenel's unresolved 1977 Telescope Conjecture. My aim is to provide an overview, rather than technical details, and I hope to make the talk entertaining and accessible to non-experts.

Daniil Rudenko. *Non-Euclidean tetrahedra and rational elliptic surfaces*

I will explain how to construct a rational elliptic surface out of every non-Euclidean tetrahedra. This surface "remembers" the trigonometry of the tetrahedron: the length of edges, dihedral angles and the volume can be naturally computed in terms of the surface. The main property of this construction is self-duality: the surfaces obtained from the tetrahedron and its dual coincide. This leads to some unexpected relations between angles and edges of the tetrahedron. For instance, the cross-ratio of the exponents of the spherical angles coincides with the cross-ratio of the exponents of the perimeters of its faces. The construction is based on relating mixed Hodge structures, associated to the tetrahedron and the corresponding surface.

Valery Ryzhikov. *Multiple mixing, and weakly homoclinic groups of measure-preserving actions*

Rokhlin's well-known problem on multiple mixing has remained unsolved since the publising of [1]. However we know that a theoretical counterexample has to have an absolutely continuous component in spectrum (B. Host) and infinite rank (S. Kalikow, V. Ryzhikov), it prefers to be far from actions of algebraic nature (B. Marcus, M. Ratner et al), see [2].

One can add that the actions that do not have multiple mixing property cannot have the ergodic homoclinic group. The homoclinic group $H(T)$ of a measure-preserving transformation T was introduced by M. Gordin [3], who noticed that the ergodicity of $H(T)$ implies the mixing property of T. In connection with this we use the following more general definition, extending the homoclinic invariants to non-mixing systems.

Let Φ be a set of automorphisms of a probability space (X, μ). We denote by $WH(\Phi)$ the (weakly homoclinic) group of all transformations S such that S is homoclinic with respect to any mixing sequence $\varphi_j \in \Phi$:

$$\varphi_j^{-1} S \varphi_j \to Id,\ j \to \infty,$$

as

$$\mu(A \cap \varphi_j B) \to \mu(A)\mu(B)$$

for all measurable sets A, B.

Lemma 1. *The ergodicity of the group $WH(\Phi)$ implies the weak multiple mixing property of Φ: if $\varphi_{m,j}$ $(1 \leqslant m \leqslant k)$ and $\varphi_{m',j}^{-1}\varphi_{m,j}$ $(1 \leqslant m' < m \leqslant k\ \varphi_j)$ are mixing sequences, then*

$$\mu(A \cap \varphi_{1,j} A_1 \cap \dots \varphi_{k,j} A_k) \to \mu(A)\mu(A_1)\dots\mu(A_k),\ j \to \infty,$$

holds for any measurable sets $A, A_1, \dots, A_k$.

The following theorem implies the results on multiple mixing for Gaussian dynamical systems and Poisson suspensions from [4], [5].

Theorem 2. *The weakly homoclinic groups of the classical Gaussian action of the orthogonal group $O(\infty)$ and the weakly homoclinic group of Poisson action of the infinite-automorphism group are both ergodic. So these actions have weak multiple mixing property.*

As a rule, the problem of determining the ergodicity of weakly homoclinic groups is not simple. It is open even for generic Z-actions. There are some open theoretical questions as well. Can an action with the ergodic (weakly) homoclinic group have a proper factor with the trivial (weakly) homoclinic group? More generally: is an action with the ergodic (weakly) homoclinic group disjoint from any action with the trivial (weakly) homoclinic group?

It's natural to begin the study from K-actions. To show the following assertion, we apply the results of D. Ornstein on isomorphism of Bernoulli transformations and T. Austin's theorem on weak Pinsker property of K-automorphisms.

Theorem 3. *K-automorphisms have the ergodic (weakly) homoclinic group.*

Remark 4. In connection with the proof of this theorem the following question arises: *given K-automorphism T and some its Bernoulli factor $\mathscr{B}$, can T be splitting into two independent factors such that one of them is a factor of $\mathscr{B}$?*

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Takashi Sato. *GKM-theoretical description of the double coinvariant rings of pseudo-reflection groups*

Let G be a Lie group, T a maximal torus of G, and W the Weyl group of G. A maximal torus T acts on the flag manifold G/T by the left multiplication. The Weyl group W acts on T and $H^*(BT)$ naturally. The equivariant cohomology ring of G/T with rational coefficients is the quotient ring of the tensor product of two copies of the polynomial ring $H^*(BT)$ and its ideal is generated by the difference of each copies of W-invariants polynomials. The equivariant cohomology rings of a manifold with a good torus action are described in terms of one-dimensional orbits and fixed points. This is called the GKM theory. In the case of flag manifolds, their equivariant cohomology rings are described by the data of the Weyl groups and the root systems.

I generalize the GKM theory from Weyl groups to pseudo-reflection groups. The double coinvariant ring of a pseudo-reflection group is a quotient ring of the tensor product of two copies of the ring of polynomial functions on the vector space on which the pseudo-reflection group acts. It is an analogue of the equivariant cohomology rings of flag varieties. There is a natural homomorphism from the double coinvariant ring to the direct product of copies of the ring of polynomial functions, and McDaniel shows that its image is described in terms of pseudo-reflections analogously to the GKM theory.

I will introduce his work and give other description which is more suitable for combinatorics.

Khurshid Sharipov. *Second-order differential invariants of submersions*

In this paper, we study the second-order differential invariants of submersions with respect to the group of conformal transformations.

The study of differential invariants of submersions has been the subject of numerous studies [3-6]. In [6] second and third order differential invariants of submersions are found with respect to conform transformations.

Second-order differential invariants of submersions of Euclidean spaces with respect to the group of motions are studied in the papers [3,4].

Let G be a Lie group of transformation of a Riemannian manifold M. If the group G is a $k-$ dimensional Lie group, then it has k infinitesimal generators (vector fields).

Definition 1. The function $I(p)$ on M is called the invariant of the transformation group G, if $I(p) = I(gp)$ for each element of $g \in G$, $p \in M$.

Let M, B be smooth manifolds and $p \in M$. Let $f, g : M \to B$ be smooth maps satisfying the condition $f(p) = g(p) = q$.

1) f has a first order contact with g at a point p if $(df)_p = (dg)_p$ as $T_pM \to T_pB$. Mappings.

2) f has a touch of $k-$ th order with g at a point p if the mapping $(df) : TM \to TB$ has a touch of order $(k-1)$ with the mapping (dg) at each point of T_pM. This fact is written as follows: $f \sim_k g$ at p ($k-$ positive number).

Denote by $J^k(M,B)_{p,q}$ - sets of equivalence classes with respect to "$\sim_k$ at p" in the space of mappings $f : M \to B$, that satisfy the condition $f(p) = q$.

We set $J^k(M,B) = \bigcup_{(p,q)\in M\times B} J^k(M,B)_{p,q}$. It is known that this set is a smooth manifold of dimension $n + m\sum_{i=0}^{k} C_{n+i-1}^{i}$ [2].

Definition 2. The manifold $J^k(M,B)$ is called the space of $k-$jets.

The action of the group G on M generates some action of the group on $J^k(M,B)$. This action is called the $k-$th extension of the action of the group G on $J^k(M,B)$. The infinitesimal generators of the $k-$th continuation of the group G on $J^k(M,B)$ are the $k-$th extensions of the infinitesimal generators of the group G.

Definition 3. The function $I \in J^k(M,B)$ is called a differential invariant of order k of the group G, if it is preserved under the action of the $k-$th continuation G on $J^k(M,B)$, i.e. $g^k(I) = I$ for any $g^k \in G^k$.

The following theorem [1] is known.

THEOREM 4. *The function I is an invariant of order k of a transformation group G only and only if it is the first integral of the infinitesimal generator of G^k.*

Now consider a conformal vector field in three-dimensional Euclidean space:

$$X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \tag{1}$$

The flow of this vector field generates the group of conformal mappings. Let us find invariant functions and invariant sets of this group. Now consider the submersion $\varphi : R^3 \to R, \varphi(x_1, x_2, x_3) = f(x_1, x_2) - x_3$. The level surfaces of this submersion are regular surfaces given by the explicit function $L_c = \{(x_1, x_2, x_3) \in R^3 : x_3 = f(x_1, x_2) - c\}$. Find the second-order differential invariants of this submersion (invariants of level surface) with respect to conformal transformations generated by the flow of a vector field (1). To do this, we find the continuation of the flow of a conformal vector field (1) in $J^2(x_1, x_2, x_3, p_1, p_2, p_{11}, p_{12}, p_{22})$, where $p_1 = \frac{\partial x_3}{\partial x_1}$, $p_2 = \frac{\partial x_3}{\partial x_2}$, $p_{11} = \frac{\partial^2 x_3}{\partial x_1^2}$, $p_{12} = \frac{\partial^2 x_3}{\partial x_1 \partial x_2}$, $p_{22} = \frac{\partial^2 x_3}{\partial x_2 \partial x_2}$.

Let the flow of a vector field (1) translate the point (x_1, x_2, x_3). To the point (x_1', x_2', x_3'). Let us find the transformation formulas for the derivatives under this conformal transformation.

The extension of a vector field (1) in $J^2(x_1, x_2, x_3, p_1, p_2, p_{11}, p_{12}, p_{22})$, has the following form:

$$X^2 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - p_{11} \frac{\partial}{\partial p_{11}} - p_{12} \frac{\partial}{\partial p_{12}} - p_{22} \frac{\partial}{\partial p_{22}} \tag{2}$$

The directions in the tangent plane where the curvature takes its maximum and minimum values are always perpendicular, if k_1 does not equal k_2, and are called principal directions.

THEOREM 5. *Under the transformations generated by the flow of the vector field* (1)*, the principal directions of the level surfaces of the submersion are transferred to the principal directions.*

Let k_1, k_2 be principal curvatures of level surface the submersion $\varphi : R^3 \to R, \varphi(x_1, x_2, x_3) = f(x_1, x_2) - x_3$.

THEOREM 6. *The ratio $\frac{k_1}{k_2}$ of principal curvatures of level surface of the submersion is second-order differential invariant of the group of conformal transformations generated by the flow of a vector field* (1).

EXAMPLE 7. Consider the submersion $\varphi : R^3 \to R$, defined by the formula

$$\varphi(x_1, x_2, x_3) = \frac{1}{2}\left(x_1^2 + x_2^2\right) - x_3. \tag{3}$$

The level surfaces of this submersion are elliptic paraboloids.

The principal directions of an elliptic paraboloid are defined by the quadratic equation:

$$x_1 x_2 \lambda^2 + (x_2^2 - x_1^2)\lambda - x_1 x_2 = 0.$$

From here we find the main directions of the elliptic paraboloid $\overrightarrow{a} = (-x_2, x_1)$ and $\overrightarrow{b} = (x_1, x_2)$, they are known to be orthogonal.

Principal curvatures of an elliptic paraboloid are calculated by the formulas

$$k_1 = \frac{1}{(1 + x_1^2 + x_2^2)^{\frac{1}{2}}}, \quad k_2 = \frac{1}{(1 + x_1^2 + x_2^2)^{\frac{3}{2}}}. \tag{4}$$

Their ratio $\frac{k_1}{k_2} = 1 + x_1^2 + x_2^2$ does not change under the conformal transformations generated by the flow of the vector field (1).

Recall that the direction at a point on a surface is called asymptotic if the normal curvature in this direction is zero.

THEOREM 8. *Under conformal transformations generated by the flow of the vector field* (1)*, the asymptotic direction of the level surfaces of the submersion is transferred to the asymptotic direction.*

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Evgeny Shchepin. *Leibniz differential and Non-standard Calculus*

Leibniz believed that the numerical line, representing the variable x, divided into infinitely-small segments of the fixed infinitely small length, which he denoted dx and called the differential of the variable x. Leibniz notation for the integral, based on this view, they played an outstanding role in the development of mathematics and physics.

After the "expulsion" of infinitesimal from the analysis completed by Weierstrass, notation of Leibniz was preserved, although mathematicians they were attached to the other, "contravariant", meaning. Rehabilitation of infinitesimal by Robinson (the non-Standard analysis) was not quite adequate to the Leibniz Notations. The report will propose a "covariant" approach to the introduction of a definite infinitesimal differential, which, on the one hand is much easier the approach of Robinson and, on the other — better fits Leibniz and physicists.

Within the framework of this approach, new natural definitions of integral, differential forms and distributions (generalized functions) significantly expand the scope of these concepts. The notion of an integral based on the Leibniz differential turns out to be the most powerful (equivalent to Kurzweil–Henstock, see [1]) and ideally corresponds to the language of physicists.

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Eugenii Shustin. *Around Rokhlin's question*

The talk is concerned with the embedded topology of smooth real algebraic curves on smooth real algebraic surfaces. This topic is included into the generally understood Hilbert's 16th problem (first part). In a series of papers in 70's, V. A. Rokhlin made fundamental contributions to the Hilbert's 16th problem and posed several questions which initiated new research directions related to the real algebraic geometry. We address one of Rokhlin's questions and discuss its consequences.

Let $C \subset \mathbb{P}^2$ be a smooth real algebraic curve. A connected component of $\mathbb{R}C$ is called oval if it is null-homologous. Two ovals are ordered if one of them is inside the disc bounded by the other oval. A nest is a linearly ordered sequence of ovals. Two nests are called disjoint, if no oval of one of them is comparable with an oval of the other nest. Given two disjoint nests N_1, N_2, their total length is bounded from above as follows:

$$l(N_1) + l(N_2) \leqslant \frac{\deg C}{2} \ .$$

This comes from Bézout's theorem applied to C and an auxiliary real straight line through two points inside the deepest ovals in N_1 and N_2. More generally, if $N_1, ..., N_c$ are pairwise disjoint nests of C and there exists an auxiliary real curve of degree m having a real connected component passing through c points inside the deepest ovals of the given nests, then (cf. [8], Formula (12))

$$l(N_1) + ... + l(N_c) \leqslant \frac{m}{2} \deg C \ .$$

In [8] Rokhlin poses a question: *What is the maximal number $c(m)$ such that through any $c(m)$ real generic points in $\mathbb{P}^2$, one can trace a real plane curve of degree m with the real point set having a unique one-dimensional component?* He also mentions the bound $c(m) \geqslant 3m - 1$ (for the proof see [2], Proposition 4.7.2).

Since real rational curves have at most one one-dimensional component of the real point set, the following question naturally arises:

Question (Rokhlin-Kharlamov). *Is it true that through any configuration of $3m - 1$ real generic points in $\mathbb{P}^2$ one can trace a real rational curve of degree m?*

J.-Y. Welschinger [11] discovered a signed count of real rational pseudo-holomorphic curves in real rational symplectic four-folds, which does not depend on the choice of the point constraints (that is, in fact, an open Gromov-Witten invariant). Thus, an affirmative answer to the above question would follow from the non-vanishing of the corresponding Welschinger invariant $W_0(\mathbb{P}^2, m)$, and this is indeed so:

Theorem 1 (Mikhalkin [7], Itenberg-Kharlamov-Sh. [3]). *For any integer $m \geqslant 1$, one has $W_0(\mathbb{P}^2, m) > 0$. In particular, through any generic configuration of $3m - 1$ real points in $\mathbb{P}^2$ one can trace a real rational curve of degree m.*

Moreover, this result can be extended to a wide range of real del Pezzo surfaces:

Theorem 2 (Itenberg-Kharlamov-Sh. [5], Brugallé [1], Sh. [10]). *For any real del Pezzo surface X of degree $\geqslant 2$ with a nonempty, connected real part $\mathbb{R}X$ and any conjugation-invariant big and nef divisor class $D \in \mathrm{Pic}(X)$, one has $W_0(X, D) > 0$. In particular, through any generic configuration of $-DK_X - 1$ points in $\mathbb{R}X$, one can trace a real rational curve $C \in |D|$.*

It happens that, whenever one has the positivity of Welschinger invariants, it yields their asymptotic behavior comparable with that for the genus zero Gromov-Witten invariants GW_0:

Theorem 3 (Itenberg-Kharlamov-Sh. [5], Sh. [10]). *Under hypotheses of Theorem 2, one has*

$$\lim_{n\to\infty} \frac{\log W_0(X, nD)}{n \log n} = \lim_{n\to\infty} \frac{\log GW_0(X, nD)}{n \log n} = -DK_X \ .$$

The key ingredient behind Theorems 2 and 3 is a real version of the Caporaso-Harris type formula (equivalently, symplectic sum formula) developed in [4], [5], [1], [10].

For the multi-component real del Pezzo surfaces, an analogue of the Rokhlin-Kharlamov question can be reduced to the study of Welschinger invariants of positive genera introduced in [9], which count real curves having precisely one one-dimensional component in each connected component of the real surface. Examples of Bézout type restrictions to the topology of real algebraic curves on real del Pezzo surfaces, based on the non-vanishing of the invariants of [9], can be found in [6].

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Arkadiy Skopenkov. *Analogue of Whitney trick for eliminating multiple intersections*

The Whitney trick for cancelling *double* intersections is one of the main tools in the topology of manifolds. Analogues of the Whitney trick for *multiple* intersections were 'in the air' since 1960s. However, only in this century they were stated, proved and applied to obtain interesting results, most notably by Mabillard and Wagner [2], see a survey [4]. I shall describe simplifications and extensions of their construction which allowed to extend the multiple Whitney trick to codimension 2 [1] and to the case when general position multiple intersections have positive dimension [3], [5].

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Mikhail Skopenkov. *Surfaces containing two circles through each point*

(This is a joint work R. Krasauskas)

Motivated by potential applications in architecture, we find all analytic surfaces in 3-dimensional Euclidean space such that through each point of the surface one can draw two transversal circular arcs fully contained in the surface. The search for such surfaces traces back to the works of Darboux from XIXth century. We prove that such a surface is an image of a subset of one of the following sets under some composition of inversions:

- the set $\{p + q : p \in P, q \in Q\}$, where P and Q are two circles in 3-dimensional Euclidean space;
- the set $\{2[pxq]/|p + q|^2 : p \in P, q \in Q\}$, where P and Q are two circles in the unit 2-dimensional sphere;
- the set $\{(x, y, z) : A(x, y, z, x^2 + y^2 + z^2) = 0\}$, where A is a polynomial in $R[x, y, z, t]$ of degree 2 or 1.

The proof uses a new factorization technique for quaternionic polynomials. A substantial part of the talk is elementary and is accessible for high school students.

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Gregory Soifer. *Discreteness of deformations of co-compact discrete subgroups*

(Based on joint work with G. Margulis)

In late 50th Selberg proved local rigidity of co-compact discrete subgroup Γ of $G = SL_n(\mathbb{R}), n \geqslant 3$ [5]. One of the ingredient in his proof of this result is the statement that, after a small perturbation in G, the subgroup Γ remains co-compact and discrete. While compactness can be easily proved the proof of discreteness is quite complicated. Selberg's proof of discreteness is based on the analysis of fundamental domains for the action of Γ on the symmetric space associated with G. He conjectured that the statement still true for a group acting on symmetric space with a compact fundamental domain. This conjecture was proved by A. Weil. He proved discreteness of a small perturbation of a co-compact discrete subgroup Γ of any connected Lie group G [6]. Weil's proof is very different from Selberg's proof and is based on the analysis of coverings of G/Γ by (small) open sets and the corresponding coverings of G.

The purpose of our work is to simplify and generalize Weil's proof and to prove

Theorem 1. *Let $\tilde{X}$ be a locally compact simply connected metric space. Let Γ be a subgroup of $\mathbf{Isom}\tilde{X}$. Suppose that Γ acts properly discontinuously and freely on $\tilde{X}$ such that the space $\Gamma \diagup \tilde{X}$ is compact. Then there exist a neighbourhood U of the inclusion $\Gamma \hookrightarrow \mathbf{Isom}\tilde{X}$ such that for every $\varphi \in U$ the group $\varphi(\Gamma)$ acts properly discontinuously on $\tilde{X}$.*

While we use the basic Weil's construction we prove the discreteness of small deformations of a discrete co-compact subgroup of isometries of a locally compact metric space under some natural restrictions.
Weil's theorem [6] was generalized in [1], [2]. We would like to note that proofs in these papers are based on the study of a fundamental domain. This is not required in our proof based on the Weil construction. Recently Weil's theorem was extended to uniform lattices of locally compact groups [3].

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Grigory Solomadin. *Monodromy in weight graphs and its applications to torus actions*

A non-singular projective variety $X^n \subset \mathbb{C}P^N$ is toric iff the corresponding homogeneous ideal $\mathbb{I}(X)$ is prime and *unital binomial*, i.e. is generated by elements of the form $\underline{x}^{\underline{u}} \pm \underline{x}^{\underline{v}}$, where $\underline{x}^{\underline{u}} := x_1^{u_1} \cdots x_N^{u_N}$ are monomials. There is an algorithm [2] to check the (unital) binomiality of a given ideal using computer evaluations.

Consider an effective algebraic torus action $(\mathbb{C}^\times)^k : X^n$. Assume that the connected component $\mathrm{Aut}^0 X$ of the group $\mathrm{Aut}X$ of algebraic automorphisms is an algebraic group. Then all maximal tori are conjugate in $\mathrm{Aut}^0 X$ and have the same dimension (called the *rank* of $\mathrm{Aut}X$) by the Cartan's theorem. Hence, there is a maximal (w.r.t. the extension) torus action extending the given one. In this way, the existence of an effective $(\mathbb{C}^\times)^n$-action on X reduces to the existence of an extension of a given torus action to an effective $(\mathbb{C}^\times)^n$-action on X.

A simple observation is that the rank of $\mathrm{Aut}X$ provides the (best possible) upper estimate on the dimension of the effecive torus action on X. For example, the automorphism group $\mathrm{Aut}Gr(k,n)$ of the complex Grassmanian $Gr(k,n)$ of k-planes in $\mathbb{C}^n$ is well-known [1] to be the algebraic group of rank $n-1$. Hence, the natural $(\mathbb{C}^\times)^{n-1}$-action on $Gr(k,n)$ is maximal.

An interesting (seemingly open) question is to determine the dimension of the maximal effective (algebraic) torus action on the generic hypersurface $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$ of bidegree $(1,1)$, called the Minor hypersurface. There is the natural effective $(\mathbb{C}^\times)^k$-action on $H_{i,j}$ with isolated fixed points, where $k = \max\{i,j\}$. It is in fact maximal, which follows from

Theorem 1. (i) *The subgroup G of $\mathrm{Aut}(\mathbb{C}P^i \times \mathbb{C}P^j)$ leaving $H_{i,j}$ invariant, coincides with $\mathrm{Aut}H_{i,j}$;*
(ii) *One has $\mathrm{rkAut}H_{i,j} = \max\{i,j\}$.*

In order to prove part (i) of Theorem 1, one needs to consider the projective embedding of $H_{i,j}$ corresponding to the very ample anticanonical sheaf of Fano variety $H_{i,j}$. The part (ii) follows from the observation that the group G may be expressed explicitly as a certain $\mathbb{C}^\times$-bundle over $\mathbb{P}GL_{i+1} \times \mathbb{P}GL_{j-i}$ for $i > j$, or the semidirect product $\mathbb{P}GL_{i+1} \rtimes \mathbb{Z}_2$ for $i = j$.

If $(\mathbb{C}^\times)^k : X^n$ is a GKM-variety, then the upper estimate on the dimension of the torus extending the given effective $(\mathbb{C}^\times)^k$-action is applicable [3]. In this estimate the notion of a GKM-graph and a connection on it plays an essential role.

We propose the necessary condition for the effective $(\mathbb{C}^\times)^k$-action on a non-singular projective variety X^n with isolated fixed points (having $\mathrm{Aut}^0 X$ an algebraic group) to be extendible to an effective $(\mathbb{C}^\times)^n$-action on X. This condition is formulated in terms of the weight (hyper)graph and a connection of $(\mathbb{C}^\times)^k : X^n$. Those generalize the notions of GKM-theory, since the action $(\mathbb{C}^\times)^k : X^n$ is not required to be GKM. The discussed condition provides a useful method to decide whether a given non-singular projective variety is not toric. The main idea of this method is to recognize invariant subgraphs of the weight (hyper)graph w.r.t. the connection (of $(\mathbb{C}^\times)^k : X^n$), and to compare them with faces of a (supposedly existing, i.e. if X is toric) simple moment polytope of a projective toric variety.

As an application of this method, we describe all toric varieties among some particular (non-generic) non-singular hypersurfaces in the Cartesian product of two toric varieties, related to the problem of representatives in the unitary bordism ring [4]. Namely, those are the generalized Buchstaber-Ray variety $BR_{i,j} \subset BF_i \times \mathbb{C}P^j$, $i > j$, and Ray variety $R_{i,j} \subset BF_i \times BF_j$, where BF_n is a bounded flag manifold (a particular tower of $\mathbb{C}P^1$-bundles) of dimension n.

Theorem 2. (i) *$BR_{i,j}$ is a toric variety iff $i \leqslant j$ or $i > j = 0, 1$;*
(ii) *$R_{i,j}$ is a toric variety iff $\min\{i,j\} = 0, 1$ or $i = j = 2$.*

We also compute the integral cohomology rings of these hypersurfaces.

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Vladimir Subbotin. *Some classes of polyhedra with rhombic and deltoidal vertices*

The first known extension of the class of regular (Platonic) polyhedra are semi-regular (Archimedean) polyhedra. Currently, there are many papers expanding and generalizing the class of regular polyhedra in the case of nonconvex polyhedra, polyhedra in multidimensional and non-Euclidean spaces.

One of the generalizations of regular polyhedra is presented in the authors work [1]. In this paper, the symmetry properties of the elements of a polyhedron are put in the basis: all polyhedrons *strongly symmetric with respect to the rotation of the faces* (class FS) are found. Let's recall the definition of this class.

A closed convex polyhedron in E^3 is called strongly symmetric with respect to the rotation of the faces if each face has an rotation axis of the polyhedron that is perpendicular to this face and intersects its relative interior.

Note that the condition on the rotation axis of the faces can be relaxed [1]: the class FS is also obtained if we require that each face F has a rotation axis, which is the rotation axis of the star of the face F. The star of the face F (vertex V) is understood as the collection of all faces that have at least one common vertex with the face F (vertex V).

Thus, the global condition on the rotation axis can be replaced by the local axisymmetry of the stars of the faces.

Faces whose stars are locally axisymmetric will be called *locally symmetric*.

Among the FS-polytopes there are seven, each of which is not combinatorially equivalent to a regular or semi-regular (Archimedean) polyhedron.

In this paper, we introduce one classes of closed convex polyhedra in E^3: class RDS with symmetric rhombic or deltoidal vertices and locally symmetric faces. Proof of the completeness of the found list of RDS-polyhedra is given.

The vertex V of a polyhedron will be called *n-rhombic* (*n-deltoidal*) if its star consists of n equal rhombuses (deltoids) having a common vertex V and converging at this vertex either with acute or obtuse angles.

If the n-rhombic (*n-deltoidal*) vertex is located on a nontrivial rotation axis order n of polyhedron, then we will call such a vertex *symmetric*.

Theorem 1. *The following types of polyhedra completely describes the RDS class:*
1) two infinite bipyramid series with two n-deltoid vertices, $n = 3, 5, 6, 7, \ldots$;
2) thirty six combinatorially different types of polyhedra with deltoid vertices;
3) nine combinatorially different types of polyhedra with rhombic vertices;
4) two combinatorially different 150-polyhedra containing both deltoid and rhombic vertices.

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Dennis Sullivan. *Revelation and Mystery*

This is about two aspects of the research of Vladimir Abramovich Rokhlin [VAR] that both helped and "hindered" my own. Firstly, it is not difficult to emerge from graduate school in Princeton as a world class expert in one specific area and be essentially ignorant in all others. This happened to me in a specific subject dominated by Sergei Novikov [SN] and my thesis advisor Bill Browder [BB]: the classification of higher dimensional simply connected manifolds. The signature additivity theorem by VAR and SN turned out to be the essential tool for a complete understanding of all invariants of the piecewise linear [PL] manifolds in this class. Then using the technique of SN from his proof of the topological invariance of Pontryagin classes one could almost show the complete set of PL invariants were actually topologically invariant. This was a kind of holy grail at the time [middle 60's], but its acquisition was protected by a slight twist in the invariants related to an earlier theorem of VAR about the signature mod 16 of certain PL or smooth four manifolds. In 1969-70 it was determined by Kirby and Siebenman that this little twist of VAR's mod 16 theorem is actually the block of granite upon which rests the entire difference between the PL and topological theory of manifolds in higher dimensions. Yet, there is still a mystery about this Rokhlin invariant that persists to this time.

Secondly, the revelation related to VAR's research refers to ergodic theory, Lebesgue measure theory and dynamical systems: topics of total ignorance after my fascinating but specific education at Princeton. A new challenge [60's & 70's] was the problem of Ahlfors to show the Poincare limit set of a finitely generated discrete group of holomorphic transformations of the Riemann sphere was either the entire sphere or had Lebesgue measure zero. Armed with VAR's monograph treating separable Lebesgue spaces in a completely revealing way, one could attack from first principles the nature of the countable equivalence relation on the Riemann sphere presented by the above problem. The clarity of VAR's presentation of generating partitions and the idea of Lebesgue density points revealed some simple universal dichotomies about all examples. These could be shown to be inadequate to solve the Ahlfors problem as stated using all known available techniques to prove ergodicity unless further geometric information could be obtained. However, the first principles based on VAR's monograph plus the known technique of deformation due to Ahlfors Bers [AB], lead naturally to an angular ergodic theorem, which actually solved the real question motivating Ahlfors problem, showing there an analogue of Mostow rigidity for these groups relative to the Poincare limit set. These VAR type ideas plus the AB deformation idea could be transferred to the subject of holomorphic iteration [Fatou Julia theory] on the Riemann sphere with considerable success. Yet the analogous rigidity relative to the Julia limit set suitably formulated is still open. It is a main problem for forty years now. It seems to require a new revealing treatise extending that of VAR to treat all together: Lebesgue spaces, dynamics and holomorphic functions obtained by high iterates of a single rational function on the Riemann sphere.

András Szűcs. *Geometry versus algebra in homology theory and cobordism theory of singular maps*

By Thom's theorem any Z_2-homology class in any space X can be represented by a continuous map of a manifold. It remains an open question, how nice can we choose the map? (if the space X is a manifold.) We answer this question in the negative:

THEOREM 1. *a) For any $k > 1$ there is a manifold of dimension $4k$ and a k-dimensional cohomology class that can not be realized by an immersion.*

b) Any finite set τ of multisingularities is insufficient to realize every k-dimensional cohomology class in any manifold by a map with multisingularities only from the set τ.

The proof of b) is based on the construction of classifying spaces of cobordisms of singular maps. For each such classifying space there is a spectral sequence converging to the homotopy groups of the classifying spaces of cobordisms of singular maps that can be considered as the geometric version of the algebraic chain complex formed from the singularities introduced by Vassiljev. The differentials encode the adjacencies of the singularity strata. In an interesting special case these differentials are expressed through the multiplicative structure of the RING of stable homotopy groups of spheres $\oplus_{n=1}^{\infty}\pi^s(n)$.

Sergey Tikhonov. *Group actions: mixing, spectra, generic properties*

(Based on joint work with A. M. Stepin)

The talk will consist of two parts.

1. Survey of results obtained by partisipants Anosov seminar on the topics going back to V. A. Rokhlin.

2. Our recent contributions as follows:

a. Diagonalization of log-integrable operator-valued $\mathbb{R}$-cocycles;

b. New examples of smooth-integrable G-invariant Hamiltonian systems with compact homogenious phace spaces

$$T^*\left(G/\Gamma\right)$$

and positive topological entropy;

c. We have a classification of mixing $\mathbb{Z}^2$-actions T such that

$$\mu\left(T^{g_i}A_1 \cap T^{h_i}A_2 \cap A_3\right) \to 0,$$

as $i \to \infty$ for some sets $\left\{A_j \mid \mu\left(A_j\right) = \frac{1}{2}\right\}_{j=1,2,3}$ and increasing sequences $\{g_i\}, \{h_i\}, \{g_i - h_i\} \subset \mathbb{Z}^2$;

d. It is shown that a generic mixing actions can be approximated in the leash metric [1] by actions with a sufficiently wide set of limit operators. The results are used to prove that some properties of mixing transformations are generic. For example, generic $\mathbb{Z}^d$-actions have a multiple mixing.

e. We get examples of mixing transformations with arbitrary spectral multiplicity function $\mathscr{M}(T)$ from [2]; for instance, $\mathscr{M}(T)$ can be of the form $\{p, q, pq\}$, $\{p, q, r, pq, qr, pr, pqr\}$.

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Maria Trnkova. *Spun triangulations of closed hyperbolic 3-manifolds*

(This is a joint work with Feng Luo and Matthias Goerner)

W. Thurston showed that every hyperbolic 3-manifold can be decomposed into a set of ideal tetrahedra (and also simple closed geodesics in the case of closed manifolds). Unfortunately such a decomposition sometimes produces overlapping tetrahedra, a so called non-geometric ideal triangulation. There is a conjecture that every cusped hyperbolic 3-manifold admits a geometric ideal triangulation. We are interested in ideal triangulations of closed hyperbolic 3-manifolds. SnapPy's computations provide several examples of non-geometric ideal triangulations of closed manifolds. The smallest of them is known as Vol3. In this talk we show that Vol3 does not have any geometric ideal triangulation of small complexity. The main techniques that we use are Thurston's Dehn surgery theorem, Dehn parental test and a gluing variety.

Alexey Tuzhilin. *Gromov–Hausdorff distances to simplexes and some applications*

(Based on joint work with A. O. Ivanov)

In the talk we present some formulas for Gromov–Hausdorff distances from a bounded metric space to simplexes, i.e., to metric spaces such that all their non-zero distances are equal to each other. Then we apply the formulas to calculate the edges lengths of minimal spanning trees; to solve a generalized Borsuk problem concerning possibility to partition a metric space into a given number of parts having smaller diameters; to calculate clique cover number and chromatic number of a graph.

Introduction

The Gromov–Hausdorff distances from bounded metric spaces to so-called simplexes, i.e., the metric spaces having just one non-zero distance, play important roles in different mathematical problems. In [1] and [2] they were used to prove triviality of the isometry group of the Gromov–Hausdorff space. In [3] it was observed the relation between the distances and the lengths of minimum spanning tree edges. Recently we found a few more applications: for solving a generalized Borsuk problem [4], and to calculating the clique cover number and the chromatic number of a graph [5]. All that will be discussed in the talk.

Preliminaries and Main Results

For an arbitrary set X, we denote by $\#X$ its *cardinality*. If X is an metric space, then the distance between its points x and y is denoted by $|xy|$. If $A, B \subset X$ are non-empty, then we put $|AB| = \inf\{|ab| : a \in A,\, b \in B\}$. For each point $x \in X$ and a number $r > 0$ we denote by $U_r(x)$ the open ball with center x and radius r; for any non-empty $A \subset X$ and $r > 0$ we put $U_r(A) = \cup_{a\in A} U_r(a)$.

For non-empty A, $B \subset X$ we define the *Hausdorff distance* as

$$d_H(A,B) = \inf\{r > 0 : A \subset U_r(B) \;\&\; B \subset U_r(A)\}.$$

For metric spaces X and Y, a triple (X', Y', Z) consisting of a metric space Z and its subsets X' and Y' isometric to X and Y, respectively, is called a *realization of* (X, Y). The *Gromov–Hausdorff distance* $d_{GH}(X, Y)$ *between* X *and* Y is the infimum of real numbers r such that there exists a realization (X', Y', Z) of (X, Y) with $d_H(X', Y') \leqslant r$. It is well-known [8] that the d_{GH} restricted to the family of isometry classes of compact metric spaces is a metric.

Let $m > 0$ be a cardinal number and $\lambda > 0$ a real number. Denote by $\lambda\Delta_m$ the metric space of cardinality m such that all its non-zero distances equal λ. We call this space a *simplex of cardinality* m.

For any metric space X, the value $\operatorname{diam} X = \sup\{|xy| : x, y \in X\}$ is called the *diameter* of X. If $\operatorname{diam} X < \infty$, then X is called *bounded*.

For an arbitrary set X and a cardinal number $0 < m \leqslant \#X$ we denote by $\mathscr{D}_m(X)$ the set of all partitions of X into m non-empty parts. Then, for a metric space X and each $D = \{X_i\}_{i\in I} \in \mathscr{D}_m(X)$ we put

$$\operatorname{diam} D = \sup_{i\in I} \operatorname{diam} X_i,\;\; \alpha(D) = \inf\{|X_iX_j| : i \neq j\},\;\; \alpha_m(X) = \sup_{D\in\mathscr{D}_m(X)} \alpha(D).$$

Theorem 1 ([7]). *Let X be an arbitrary bounded metric space, m a cardinal number, and λ a positive real number. Then*

- *for $m > \#X$ it holds $2d_{GH}(\lambda\Delta_m, X) = \max\{\lambda, \operatorname{diam} X - \lambda\}$;*
- *for $0 < m \leqslant \#X$ it holds*

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathscr{D}_m(X)} \max\{\operatorname{diam} D,\, \lambda - \alpha(D),\, \operatorname{diam} X - \lambda\}.$$

Corollary 2. *Let X be an arbitrary bounded metric space, $0 < m \leqslant \#X$ a cardinal number, and λ a positive real number. Then*

(1) *for $\lambda \geqslant 2 \operatorname{diam} X$ we have $2d_{GH}(\lambda\Delta_m, X) = \lambda - \alpha_m(D)$;*
(2) *for $0 < \lambda < \operatorname{diam} X$ we have*

$$\big(2d_{GH}(\lambda\Delta_m, X) = \operatorname{diam} X\big) \;\Leftrightarrow\; \big(\operatorname{diam} D = \operatorname{diam} X \text{ for all } D \in \mathscr{D}_m(X)\big).$$

Let $G = (V, E)$ be an arbitrary graph with the vertices set V and the edges set E. Suppose that V is a metric space. Then for such G the *length* $|e|$ of each its *edge* $e = vw$ is defined as the distance $|vw|$; the *length* $|G|$ of the *graph* G is the sum of its edges lengths.

Let X be a finite metric space. The shortest tree of the form (X, E) is called a *minimum spanning tree on* X. We denote its length by $\operatorname{mst}(X)$, and the set of all minimum spanning trees on X by $\operatorname{MST}(X)$. Evidently, $\operatorname{MST}(X) \neq \varnothing$. For $G \in \operatorname{MST}(X)$ we denote by $\sigma(G)$ the vector, whose coordinates are the lengths of edges of the tree G, written in the descending order.

PROPOSITION 3. *For any $G_1, G_2 \in \operatorname{MST}(X)$ we have $\sigma(G_1) = \sigma(G_2)$.*

DEFINITION 4. For any finite metric space X, we denote by $\sigma(X)$ the vector $\sigma(G)$ for an arbitrary $G \in \operatorname{MST}(X)$.

THEOREM 5 ([3], [6]). *If X is a finite metric space, $\#X = n$, $\sigma(X) = (\sigma_1, \ldots, \sigma_{n-1})$, $m \in \mathbb{N}$, $2 \leqslant m \leqslant n$, and $\lambda \geqslant 2 \operatorname{diam} X$, then*

$$\sigma_{m-1} = \alpha_m(X) = \lambda - 2d_{GH}(\lambda\Delta_m, X).$$

The *classical Borsuk Problem* asks for how many parts one needs to cut a bounded subset of the Euclidean space to obtain pieces of smaller diameters. In 1933 Borsuk conjectured that any bounded subset of $\mathbb{R}^n$ can be cut into $n + 1$ subsets of smaller diameters. This conjecture was proved by Hadwiger for convex subsets with convex smooth boundaries (see [9] and [10]), but in 1993 it was disproved in general case by Kahn and Kalai [11].

Let us first generalize the problem for arbitrary bounded metric spaces and any partitions (not only finite). For a bounded metric space X, a cardinal number $0 < m \leqslant \#X$, and $D = \{X_i\}_{i \in I} \in \mathscr{D}_m(X)$, we say that D is a partition into parts of *strictly smaller diameters* if there exists $\varepsilon > 0$ such that $\operatorname{diam} X_i \leqslant \operatorname{diam} X - \varepsilon$ for all $i \in I$. By *Generalized Borsuk Problem* we mean the following: *Is it possible to partition a given bounded metric space X into m parts of strictly smaller diameters.*

THEOREM 6. *Let X be a bounded metric space and $0 < m \leqslant \#X$ a cardinal number. Choose an arbitrary $0 < \lambda < \operatorname{diam} X$, then X can be partitioned into m parts of strictly smaller diameters iff $2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam} X$.*

For a cardinal number $n > 0$ we denote by $\mathscr{M}_n$ the set of isometry classes of bounded metric spaces of cardinality at most n, endowed with the Gromov–Hausdorff distance. For $X \in \mathscr{M}_n$ and $r \geqslant 0$ let $S_r(X) = \{Y \in \mathscr{M}_n : d_{GH}(X, Y) = r\}$ be the sphere with center X and radius r.

COROLLARY 7. *Let $d > 0$ be a real number and $0 < m \leqslant n$ cardinal numbers. Choose an arbitrary $0 < \lambda < d$, then $S_{d/2}(\Delta_1) \cap S_{d/2}(\lambda\Delta_m)$ consists exactly of all metric spaces from $\mathscr{M}_n$, whose diameters are equal to d and that cannot be partitioned into m parts of strictly smaller diameters.*

A subgraph of an arbitrary simple graph G is called *clique*, if any its two vertices are connected by an edge (such subgraph is a complete graph itself). Let us note that each single-vertex subgraph is also a clique. The family of all cliques in a graph G forms a cover of the graph G vertices set. The least possible number of cliques forming a cover of the vertices set of a graph G is called the *clique cover number of* G and is often denoted by $\theta(G)$.

Let $G = (V, E)$ be a finite graph. Fix two real numbers $0 < a < b \leqslant 2a$ and define a metric on V as follows: the distance between adjacent vertices of G equals a, and non-adjacent vertices of G equals b.

OBSERVATION 8. *The space V can be partitioned into m subsets of strictly smaller diameters iff $\theta(G) = m = \#V$, and cannot iff $m < \theta(G)$.*

COROLLARY 9 ([5]). *Let m be the greatest positive integer with $2d_{GH}(a\Delta_m, V) = b$, then $\theta(G) = m + 1$ (if there is no such m, then we put $m = 0$).*

A *chromatic number* of a simple graph G is the smallest numbers of colors to get an admissible coloring: adjacent vertices have different colors. The chromatic number of G is sometimes denoted by $\gamma(G)$. The *dual graph* G' to G is the graph with the same vertices set, and with edges which join all non-adjacent vertices in G, and only them. It is well-known that $\gamma(G) = \theta(G')$.

Let $G = (V, E)$ be a finite graph. Fix two real numbers $0 < a < b \leqslant 2a$ and define a metric on V as follows: the distance between adjacent vertices of G equals b, and non-adjacent vertices of G equals a.

COROLLARY 10 ([5]). *Let m be the greatest positive integer with $2d_{GH}(a\Delta_m, V) = b$, then $\gamma(G) = m + 1$ (if there is no such m, then we put $m = 0$).*

REMARK 11. In [5] we obtained exact formulas for Gromov–Hausdorff distances between two-distance metrics spaces and simplexes.

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Victor Vassiliev. *On the homology of spaces of equivariant maps*

I will describe a method of calculating the cohomology groups of spaces of continuous maps $X \to Y$, where X is $n-$dimensional and Y is $n-$connected, and, more generally, of the spaces of such maps equivariant under the free finite group action on X and Y. For the main promotional example, I give the explicit calculation of rational cohomology groups of spaces of odd or even maps $S^n \to S^m$, $n < m$, or, which is the same, of the stable homology groups of spaces of non-resultant homogeneous polynomial maps $R^{n+1} \to R^{m+1}$. A couple of unsolved questions in equivariant homotopy theory will be formulated.

Anatoly Vershik. *V. A. Rokhlin — an outstanding mathematician and a person of extraordinary fate*

Short mathematical biography and walk of life of Vladimir Rokhlin.

Vladimir Vershinin. *Surfaces, braids, homotopy groups of spheres and Lie algebras*

(Based on the joint works with V. Bardakov, Jingyan Li, R. Mikhailov and Jie Wu)

We consider general surfaces: compact, possibly with punctures and boundary components. The only condition is that the fundamental group of the surface should be finitely generated. The fundamental group of a configuration space of a surface is the braid group of the surface. We consider in particular Brunnian braids, that is the braids which become trivial after deleting of any strand. We describe Brunnian braids of the projective plane and of the sphere with the help of homotopy groups of 2-sphere. The Cohen braids are the generalization of the Brunnian ones. We describe also the Lie algebras of pure braids on surfaces and the Lie algebras connected with Brunnian braids.

Yakov Veryovkin. *Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups*

(This is a joint work with Taras Evgenievich Panov)

We construct and study polyhedral product models for classifying spaces of right-angled Artin and Coxeter groups, general graph product groups and their commutator subgroups. By way of application, we give a criterion of freeness for the commutator subgroup of a graph product group, and provide an explicit minimal set of generators for the commutator subgroup of a right-angled Coxeter group.

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Oleg Viro. *Vladimir Abramovich Rokhlin and the topology of real algebraic varieties*

Barak Weiss. *Horocycle flow on the moduli space of translation surfaces*

For the horocycle flow
on surfaces with translation structures
we would like to know
all possible orbit-closures.
For homogeneous flows, indeed,
such fantastic results were achieved,
and if here the situation were the same
then any orbit-closure would be tame.
Showing that a classification holds
in this geometric setting
was a long term research goal of McMullen, Mirzakhani and Eskin.
But with Chaika and Smillie we show
that the answer to naive conjectures is ”no!”
Some orbit closures are bizarre creatures,
with spiky, spindly, fractal-like features.

Benjamin Weiss. *Recent results on the Rokhlin Lemma*

Many years ago Don Ornstein and I showed how to prove a version of the fundamental Rokhlin Lemma for a wide class of locally compact amenable groups that includes all countable amenable groups. This was based on a purely group theoretic construction of a quasi-tiling of the group by finitely many approximately invariant sets. In recent years this group theoretic construction was improved by Tomasz Downarowicz, Dawid Huczek and Guohua Zhang to show that exact tilings can also be found. It has been known for a long time that using towers with two different heights one can eliminate the small measure set inherent in the classic Rokhlin lemma. Using these exact group tilings Clinton T. Conley, Steve C. Jackson, David Kerr, Andrew S. Marks, Brandon Seward and Robin D. Tucker-Drob showed how to extend this (with a finite number of tiles) for free actions of any countable amenable group. I will give a survey of these new results.

Oyku Yurttas. *Geometric intersection of curves on non-orientable surfaces*

(This is joint work with Ferihe Atalan and Mehmetcik Pamuk)

In this talk we describe a coordinate system which provides an explicit bijection between the set of multicurves on a non-orientable surface $N_{k,n}$ of genus k with n punctures and one boundary component and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k)\setminus\{0\}$, and calculate the geometric intersection number of a given multicurve with so-called elementary curves, taking as input their coordinates. This coordinate system is an analog of the Dynnikov Coordinate system on the finitely punctured disk which has a wide range of dynamical and combinatorial applications.

This work is supported by TUBITAK (Project Number MFAG-117282).

Daniele Zuddas. *Branched coverings of* CP^2 *and other basic 4-manifolds*

(This is a joint work with Riccardo Piergallini)

Let M and N be compact PL oriented connected 4-manifolds. We are interested to the following question: is there a branched covering $p\colon M \to N$?

In our paper [13] we answer this question when M is an arbitrary closed oriented PL 4-manifold and N is one of the following manifolds: CP^2, $\overline{CP}^2$, $S^2 \times S^2$, $S^2 \tilde{\times} S^2$ or $S^3 \times S^1$. More generally, we consider also the cases when N is one of the followings connected sums: $\#_m CP^2 \#_n \overline{CP}^2$, $\#_n(S^2 \times S^2)$ and $\#_n(S^3 \times S^1)$. Here $\overline{CP}^2$ denotes CP^2 with the opposite orientation, and $S^2 \tilde{\times} S^2$ denotes the total space of the non-trivial S^2-bundle over S^2.

Moreover, the above question was answered by Piergallini [11] when the base N is a sphere: every closed oriented PL 4-manifold M is a branched covering of S^4.

Branched coverings of CP^2 arise in algebraic geometry: every smooth algebraic surface $S \subset CP^n$ admits a holomorphic branched covering $S \to CP^2$. Moreover, Auroux [1] extended this result to all closed symplectic 4-manifolds M, proving that that they are realizable as symplectic branched coverings of CP^2.

Suppose now that a PL (or smooth) closed oriented connected 4-manifold M admits a PL (or smooth) branched covering $p\colon M \to CP^2$. Let $CP^1 \subset CP^2$ be a projective line, which can be assumed to intersect the branch set of p transversally up to isotopy. Then the preimage $F = p^{-1}(CP^1)$ is a closed oriented surface in M, and the square $[F]^2$ of its homology class coincides with the degree of p. Therefore, $b_2^+(M) \geqslant 1$. This inequality represents an obstruction for a 4-manifold M to be a branched covering of CP^2.

In fact, the non-vanishing of $b_2^+(M)$ is a necessary and sufficient condition for the existence of a branched covering $p\colon M \to CP^2$. This is one of the main results of this note (Theorem 1).

We briefly recall the notion of branched covering, in order to introduce some terminology (see [2], [3] or [6] for more details).

A map $p\colon M \to N$ between compact oriented PL n-manifolds is called a *branched covering* if it is a non-degenerate orientation preserving PL map with the following properties: 1) there is an $(n-2)$-dimensional polyhedral subspace $B_p \subset N$, the *branch set* of p, such that the restriction $p_| : M - p^{-1}(B_p) \to N - B_p$ is an ordinary covering of finite *degree* $d(p)$ (we assume B_p to be minimal with respect to this property); 2) in the bounded case, $p^{-1}(\partial N) = \partial M$ and p preserves the product structure of a collar of the boundaries (which implies that the restriction to the boundary $p_| : \partial M \to \partial N$ is a branched covering of the same degree of p).

Moreover, p is called *simple* if the monodromy of the above mentioned ordinary covering sends any meridian around B_p to a transposition. In this case, also the restriction to the boundary $p_| : \partial M \to \partial N$ is simple.

For a closed oriented connected 4-manifold M we denote by $b_i(M)$ the i-th Betti number of M, namely the dimension of $H_i(M;\mathbb{R})$, and by $b_2^+(M)$ (resp. $b_2^-(M)$) the maximal dimension of a subspace of $H_2(M;\mathbb{R})$ where the intersection form β_M of M is positive (resp. negative) definite.

Theorem 1. *Let M be a closed connected oriented PL 4-manifold. Then, there exists a branched covering $p\colon M \to N$ with:*

a $N = CP^2 \Leftrightarrow b_2^+(M) \geqslant 1$;
b $N = \overline{CP}^2 \Leftrightarrow b_2^-(M) \geqslant 1$;
c $N = S^2 \tilde{\times} S^2 \Leftrightarrow b_2^+(M) \geqslant 1$ *and* $b_2^-(M) \geqslant 1$;
d $N = S^2 \times S^2 \Leftrightarrow b_2^+(M) \geqslant 1$ *and* $b_2^-(M) \geqslant 1$;
e $N = S^3 \times S^1 \Leftrightarrow b_1(M) \geqslant 1$.

In all cases, we can assume that p is a simple branched covering of degree $d \leqslant 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N. Moreover, B_p can be desingularized to become embedded in N, with the following estimates for the degree d: $d \leqslant 5$ in cases a and b for $b_2(M) \geqslant 2$ and β_M odd, case c for β_M odd, case d for β_M even, and case e; $d \leqslant 6$ in cases a and b for $b_2(M) \geqslant 2$ and β_M even, case c for β_M even, and case d for β_M odd; $d \leqslant 9$ in cases a and b for $b_2(M) = 1$.

We also mention the following generalization, which we state without insisting on the upper bounds for the degree for having a non-singular branch surface (in general, the cusps of the branch set of a simple branched covering can be removed in opposite pairs if $d(p) \geqslant 4$, while the nodes can be removed in pairs if $d(p) \geqslant 5$, see Iori and Piergallini [9]).

Theorem 2. *Let M be a closed connected oriented PL 4-manifold and let m and n be non-negative integers. Then, there exists a branched covering $p\colon M \to N$ with:*

a $N = \#_m CP^2 \#_n \overline{CP}^2 \Leftrightarrow b_2^+(M) \geqslant m$ *and* $b_2^-(M) \geqslant n$;
b $N = \#_n(S^2 \times S^2) \Leftrightarrow b_2^+(M) \geqslant n$ *and* $b_2^-(M) \geqslant n$;
c $N = \#_n(S^3 \times S^1) \Leftrightarrow \pi_1(M)$ *admits a free group of rank n as a quotient.*

In all cases, we can assume that p is a simple branched covering of degree $d \leqslant 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N.

Remark 3. As a consequence of Theorem 2 *a* and *b*, we obtain some simply connected 4-manifolds N admitting a simple branched covering $p\colon T^4 \to N$. Namely, they are $\#_m CP^2 \#_n \overline{CP}^2$ and $\#_n(S^2 \times S^2)$ for any $m \leqslant 3$ and $n \leqslant 3$. This extends the previous result by Rickman [15] concerning the case when N is $\#_2(S^2 \times S^2)$. All such manifolds N are *quasiregularly elliptic* (see Bonk and Heinonen [4] for the definition), since the composition of the universal covering of T^4 with p is a *quasiregular map* $R^4 \to N$. The question of which closed simply connected manifolds are quasiregularly elliptic was posed by Gromov in [7], [8]. According to Prywes [14], $b_2(M) \leqslant 6$ for any closed connected orientable quasiregularly elliptic 4-manifold M, in particular $\#_n(S^2 \times S^2)$ is not quasiregularly elliptic for $n \geqslant 4$. Hence our result implies a sharp answer to the Gromov question for such connected sums, while the cases of $\#_m CP^2 \#_n \overline{CP}^2$ with $m+n \leqslant 6$ and $\max(m,n) \geqslant 4$ remain still open.

The proofs of our results make use of an extension theorem for branched coverings, which was proved in our paper [12]. This theorem can be stated, in the special case we actually need, as follows (see Theorem 1.2 in [12]).

Definition 4. Let M be a closed oriented 3-manifold. A simple branched covering $q\colon M \to S^3$ is said to be *ribbon fillable* if it can be extended to a simple branched covering $\tilde{q}\colon W \to B^4$ whose branch set $B_{\tilde{q}}$ is a ribbon surface in B^4. In this case, it immediately follows that $M = \partial W$, $B_q = \partial B_{\tilde{q}}$ is a link, and $d(q) = d(\tilde{q})$.

Theorem 5 ([12]). *Let W be a compact connected oriented PL 4-manifold with boundary, and let $p\colon \partial W \to \partial B^4 = S^3$ be a simple d-fold covering, branched over a link $L \subset S^3$, with $d \geqslant 4$. If p is ribbon fillable, then it can be extended to a simple d-fold branched covering $\tilde{p}\colon W \to B^4$. Moreover, we can assume that the branch set $B_{\tilde{p}} \subset B^4$ is a properly self-transversally immersed PL locally flat surface in B^4, with $\partial B_{\tilde{p}} = L$. If $d \geqslant 5$, we can assume $B_{\tilde{p}}$ an embedded surface.*

Remark 6. in Theorem 5 the ribbon filling of p need not be defined on the given manifold W, but in a certain 4-manifold W' with the same boundary. On the other hand, the branch surface of $\tilde{p}$ may not be ribbon.

For brevity we only outline the proof of Theorem 1 a, and we refer the reader to [13] for a detailed proof of this and the other statements.

Proof of Theorem 1 a. That $b_2^+(M) \geqslant 1$ is a necessary condition has been observed above.

For the converse, assume that $b_2^+(M) \geqslant 1$. We start considering a homology class $\alpha \in H_2(M;\mathbb{Z})$ such that $\alpha^2 = 4$. Its existence follows from classical results for the intersection form of a smooth closed oriented 4-manifold (Donaldson's diagonalization theorem [5] for definite forms and the Serre classification of symmetric bilinear indefinite unimodular forms for the other cases [16], [10]).

Let $S \subset M$ be a closed oriented connected surface of genus g that represents α. Then, there is a 2-fold branched covering $q\colon S \to CP^1 \cong S^2$. We can stabilize q up to degree 4, obtaining a simple 4-fold branched covering $q' : S \to CP^1$.

Now take a fiberwise extension $p_1 : T_S \to T_{CP^1}$ between tubular neighborhoods T_S and T_{CP^1} of S and of CP^1, respectively. We make use of the disk bundle structure of tubular neighborhoods. Such an extension exists because $[S]^2 = d(q')$. Let $X = M - \operatorname{Int} T_s$ and $Y = CP^2 - \operatorname{Int} T_{CP^1}$, and observe that $Y \cong B^4$.

Now, the crucial point is that the restriction $r = p_{1|} : \partial T_S \to \partial T_{CP^1} \cong S^3$ is ribbon fillable. This follows from the fact that q' is branched over $2g+6$ points in the sphere, $2g+2$ of which have monodromy (1 2), while the remaining four points have monodromies (2 3), (2 3) and (3 4), (3 4), with respect to a suitable Hurwitz system. Hence, they can be paired with the same monodromy.

The circle bundle $\partial T_{CP^1} \to CP^1$ is the Hopf fibration $S^3 \to S^2$, hence r is branched over some fibers of the Hopf fibration, precisely the fibers over the branch points of q'. These fibers in pairs bound disjoint twisted annuli, which are the preimages, by the Hopf fibration, of $g+3$ disjoint arcs in CP^1 connecting two branch points of q' with the same monodromy as above. By pushing the interiors of such annuli inside B^4, we get the labeled ribbon surface providing the ribbon filling of r.

Now we can apply Theorem 5 to extend r over the ball Y as a simple branched covering $p_2\colon X \to Y$. Finally, the desired simple 4-fold branched covering is

$$p = p_1 \cup_\partial p_2 : M \to CP^2.$$

□

Remark 7. The above results hold also in the C^∞-category. Indeed the techniques involved in the proofs can be easily adapted to fit in a smooth environment. It is a standard fact that the PL and the C^∞-categories contain the same objects in dimension four.

By following similar ideas, we can obtain branched covering representation theorems for submanifolds. We state only the case of CP^2.

Theorem 8. *Let M be closed oriented 4-manifold and let $S \subset M$ be a locally flat connected oriented surface. If $d = [S]^2 \geqslant 4$, then there is a simple d-fold branched covering $p\colon M \to CP^2$ such that CP^1 intersects B_p transversally and $S = p^{-1}(CP^1)$.*

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Victor Zvonilov. *Maximally inflected real trigonal curves*

Let $\pi : \Sigma \to B$ be a real geometrically ruled surface with the exceptional section E, $E^2 = -d < 0$.

A *real trigonal curve* is a reduced real curve $C \subset \Sigma$ disjoint from E and such that the restriction $\pi : C \to B$ is of degree 3. In an affine chart on Σ, such a curve C is defined by the equation $y^3 + b(x)y + w(x) = 0$, where b and w are certain sections. A curve C is *maximally inflected* if C is nonsingular and all zeros of the discriminant $d(x) = 4b^3 + 27w^2$ are real.

A nonsingular real algebraic curve is *of type I* (*of type II*) if the set of its real points divides (not divides) the set of its complex points.

In [1], it was given a description of maximally inflected trigonal curves of type I in terms of the combinatorics of sufficiently simple graphs and, in the case $B = \mathbb{C}P^1$, it was obtained a complete classification of such curves.

In this talk, we extend the results of [1] to maximally inflected trigonal curves of type II.

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