Euler International Mathematical Institute

Abstracts

Geometry in the Large

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Contents

ABSTRACTS

Nikolay Abrosimov. *Explicit volume formula for a hyperbolic tetrahedron in terms of edge lengths*

The talk is based on our joint work with Vuong Huu Bao [1].

We consider a compact hyperbolic tetrahedron of a general type. It is a convex hull of four points called vertices in the hyperbolic space $\mathbb{H}^3$. It can be determined by the set of six edge lengths up to isometry. Denote by ℓ_{ij} the length of the edge connecting i-th and j-th vertices. We put θ_{ij} for the dihedral angle along the corresponding edge. For further considerations, we use the notion of edge matrix of the tetrahedron formed by hyperbolic cosines of its edge lengths.

An *Edge matrix* $E(T)$ is formed by hyperbolic cosines of the edge lengths and defined as follows

$$E(T) = \langle \mathrm{ch}\ell_{ij} \rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & \mathrm{ch}\ell_{12} & \mathrm{ch}\ell_{13} & \mathrm{ch}\ell_{14} \\ \mathrm{ch}\ell_{12} & 1 & \mathrm{ch}\ell_{23} & \mathrm{ch}\ell_{24} \\ \mathrm{ch}\ell_{13} & \mathrm{ch}\ell_{23} & 1 & \mathrm{ch}\ell_{34} \\ \mathrm{ch}\ell_{14} & \mathrm{ch}\ell_{24} & \mathrm{ch}\ell_{34} & 1 \end{pmatrix},$$

where $\ell_{ii} = 0$ and $\mathrm{ch}\ell_{ii} = 1$.

We establish necessary and sufficient conditions for the existence of a tetrahedron in $\mathbb{H}^3$.

Theorem 1. *A compact hyperbolic tetrahedron T with edge matrix E is exist if and only if the following inequalities hold*

$$\begin{aligned} &\text{(i)} && \ell_{13} + \ell_{23} \geqslant \ell_{12} \geqslant |\ell_{13} - \ell_{23}|, \\ &\text{(ii)} && \ell_{14} + \ell_{24} \geqslant \ell_{12} \geqslant |\ell_{14} - \ell_{24}|, \\ &\text{(iii)} && \ell_1 \leqslant \ell_{34} \leqslant \ell_2, \quad \textit{where} \quad ch\ell_1 = C - S, \quad ch\ell_2 = C + S \quad \textit{and} \\ &&& C = ch\ell_{13}\, ch\ell_{14} - csch^2\ell_{12}(ch\ell_{13}\, ch\ell_{12} - ch\ell_{23})(ch\ell_{14}\, ch\ell_{12} - ch\ell_{24}), \\ &&& S = csch^2\ell_{12}\sqrt{(ch\ell_{23} - ch(\ell_{13}+\ell_{12}))(ch\ell_{23} - ch(\ell_{13}-\ell_{12}))} \\ &&& \quad \times \sqrt{(ch\ell_{24} - ch(\ell_{14}+\ell_{12}))(ch\ell_{24} - ch(\ell_{14}-\ell_{12}))} \end{aligned}$$

Then we find relations between their dihedral angles and edge lengths in the form of a cosine rule.

Theorem 2. *Let E be the edge matrix of a compact hyperbolic tetrahedron T. Then the following conditions hold*

$$\begin{aligned} &\text{(i)} && c_{ii} > 0, \\ &\text{(ii)} && \det E < 0, \\ &\text{(iii)} && \cos\theta_{5-i,5-j} = \frac{-c_{ij}}{\sqrt{c_{ii} \cdot c_{jj}}}, \end{aligned}$$

where $i, j \in \{1,2,3,4\}$, $c_{ij} = (-1)^{i+j}E_{ij}$ is ij-cofactor of edge matrix E and $\theta_{5-i,5-j}$ is a dihedral angle along edge $\ell_{5-i,5-j}$ which is opposite to ℓ_{ij}.

Finally, we obtain exact integral formula expressing the volume of a hyperbolic tetrahedron in terms of the edge lengths.

Theorem 3. *Let T be a compact hyperbolic tetrahedron given by its edge matrix E and $c_{ij} = (-1)^{i+j}E_{ij}$ is ij-cofactor of E. We assume that all the edge lengths are fixed exept ℓ_{34} which varies. Then the volume $V = V(T)$ is given by the formula*

$$V = \frac{1}{2}\int\limits_{\ell_1}^{\ell_{34}} \Big[\frac{-t}{\sqrt{-\Delta^3}}\Big(\frac{c_{14}(c_{11}c_{23} - c_{12}c_{13})}{c_{11}} + \frac{c_{24}(c_{13}c_{22} - c_{12}c_{23})}{c_{22}}\Big) - \frac{sht}{\sqrt{-\Delta}}$$
$$\times \Big(\frac{\ell_{24}\, sh\ell_{24}\, c_{14} + \ell_{14}\, sh\ell_{23}\, c_{13}}{c_{11}} + \frac{\ell_{13}\, sh\ell_{13}\, c_{23} + \ell_{23}\, sh\ell_{14}\, c_{24}}{c_{22}} + \ell_{12}\, sh\ell_{12}\Big)\Big]dt,$$

where cofactors c_{ij} and edge matrix determinant $\Delta = \det E$ are functions in one variable ℓ_{34} denoted by t. The lower limit of integration ℓ_1 is defined by expression

$$ch\ell_1 = ch\ell_{13}\, ch\ell_{14} - csch^2\ell_{12}\Big[(ch\ell_{13}\, ch\ell_{12} - ch\ell_{23})(ch\ell_{14}\, ch\ell_{12} - ch\ell_{24}) - \sqrt{(ch\ell_{23} - ch(\ell_{13}+\ell_{12}))(ch\ell_{23} - ch(\ell_{13}-\ell_{12}))} \times \sqrt{(ch\ell_{24} - ch(\ell_{14}+\ell_{12}))(ch\ell_{24} - ch(\ell_{14}-\ell_{12}))}\Big].$$

The latter volume formula can be regarded as a new version of classical Sforza's formula [2] for the volume of a tetrahedron but in terms of the edge matrix instead of the Gram matrix.

This work was supported by the Ministry of Science and Higher Education of Russia (agreement No. 075-02-2020-1479/1).

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[1] Abrosimov, N.V. and Vuong B.: Explicit volume formula for a hyperbolic tetrahedron in terms of edge lengths. arXiv:2107.03004 [math.MG]

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Sergei Agapov. *On the construction of exact solutions in the problem of integrable geodesic flows*

In the talk various methods of integration of quasi-linear systems of PDEs arising in the problem of integrable geodesic flows on 2-surfaces will be described.

Victor Alexandrov. *Recognition of affine-equivalent polyhedra by their natural developments*

The classical Cauchy rigidity theorem for convex polytopes reads that if two convex polytopes have isometric developments then they are congruent. In other words, we can decide whether two polyhedra are isometric or not by using their developments only. We study a similar problem about whether it is possible, using only the natural developments of two convex polyhedra of Euclidean 3-space, to understand that these polyhedra are (or are not) affine-equivalent.

The talk is based on the e-print:

V. Alexandrov, *Recognition of affine-equivalent polyhedra by their natural developments,* arXiv:2106.13659.

Manuel Amann. *Maximal antipodal sets and the topology of generalised symmetric spaces*

After commenting on the topology of (generalised) symmetric spaces we prove several long-standing conjectures by Chen–Nagano on cohomological descriptions of the cardinalities of maximal antipodal sets in symmetric spaces. We actually extend these conjectures to the setting of generalised symmetric spaces of finite abelian p-groups and verify them mostly in this broader context drawing upon techniques from equivariant cohomology theory.

Valerii Berestovskii. *Solution to a generalization of one Topononogov's problem*

In the early 1970s, V.A.Toponogov proposed the following interesting problem.

Problem 1. At the closed upper half-plane of the Cartesian plane (x, y), there is defined a continuously differentiable real-valued function f such that $f(x, 0) \equiv 0$ and $|\frac{\partial f}{\partial y}| \leqslant |\frac{\partial f}{\partial x}|$ everywhere. Prove that $f \equiv 0$ everywhere.

The first positive solution to this problem was obtained in [1] as a corollary of

Theorem 1. *Let f be a continuously differentiable real-valued function on a closed half-space $\mathbb{R}^{n+1}_+ = \{(x,t) = (x_1,\dots,x_n,t) \in \mathbb{R}^{n+1} | t \geqslant 0\}$ such that $|\frac{\partial f}{\partial t}| \leqslant A\|\frac{\partial f}{\partial x} = (\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})\|$, where A is a nonnegative constant and*

$\|\cdot\|$ *is the standard Euclidean norm on* $\mathbb{R}^n$. *Then, for each point* $u^1 \in \mathbb{R}^{n+1}_+$, *there exists a point* $u^0 \in H \cap K_{u^1}$ *such that* $f(u^1) = f(u^0)$. *Here* $H = \{(x,t) \in \mathbb{R}^{n+1} | t = 0\}$, $K_{u^1} = \mathbb{R}^{n+1}_+ \cap (u^1 + (-C))$, $C = \{w \in \mathbb{R}^{n+1} | \|w\| \leqslant \sqrt{A^2+1} w_{n+1}\}$ *is pointed closed cone.*

In the notation and conditions of Theorem 1, it is equivalent to the following

Theorem 2. *If, for each point* $u \in \mathbb{R}^{n+1}_+$, *there exists a nonzero vector* $v \in C$ *such that* $df(u)(v) = 0$, *then, for each point* $u^1 \in \mathbb{R}^{n+1}_+$, *there is a point* $u^0 \in H \cap (u^1 + (-C))$ *such that* $f(u^1) = f(u^0)$.

Note that Theorems 1 and 2 are trivial if $A = 0$. For $A > 0$, we shall give a somewhat different interpretation of Theorem 2 and its natural generalization. To do this, we shall recall some necessary definitions from [5].

A *Lorentzian manifold* is a pair (M, g), where M is a C^∞-smooth manifold of dimension $n+1 \geqslant 2$ and g is a smooth symmetric tensor field of type $(0,2)$ (called a *Lorentzian metric*) on M with the signature $(+, \ldots, +, -)$. Then a nonzero vector $v \in TM$ is called *nonspacelike* (respectively, *timelike, isotropic, spacelike*) if $g(v,v) \leqslant 0$ (respectively, $g(v,v) < 0$, $g(v,v) = 0$, $g(v,v) > 0$). If a Lorentzian manifold (M,g) admits a (global) smooth timelike vector field X, then the manifold (M,g) is called *time-oriented by the field* X. Then any nonspacelike vector $v \in T_pM$, $p \in M$, is either *future-directed* or *past-directed* which means that $g(X(p), v) < 0$ (respectively, $g(X(p), v) > 0$). Further, the *causal future* $J^+(L)$ (respectively, the *causal past* $J^-(L)$) of a subset L in (M,g) is defined as the set of all points $q \in M$, for which there exists a continuous piecewise continuously differentiable curve $c = c(t)$, $t \in [a,b]$, with future directed (respectively, past directed) nonspacelike tangent vectors such that $c(a) \in L$, $c(b) = q$. We shall call shortly curves of both such types as *nonspacelike curves.* In the similar way, we define the *chronological future* $I^+(L)$ (respectively, the *chronological past* $I^-(L)$), if we require additionally that the tangent vectors of the curve c are timelike.

Definition 1. A *space-time* (M,g) is a connected Hausdorff C^∞-smooth Lorentzian manifold with a countable base of the topology τ and a time-orientation X. A space-time is called *globally hyperbolic* if the sets $I^+(p) \cap I^-(q)$, $p, q \in M$, constitute a base of τ and all sets $J^+(p) \cap J^-(q)$, $p, q \in M$, are compact.

The simplest example of a globally hyperbolic space-time is a *Minkowski space-time*, i.e. the manifold $M = \mathbb{R}^{n+1}$, $n+1 \geqslant 2$, with a Lorentzian metric g having constant components g_{ij} in the canonical coordinates $(x_1, \ldots, x_n, t)$ in $\mathbb{R}^{n+1}$, where $g_{ij} = 0$ for $i \neq j$, $g_{11} = \cdots = g_{nn} = 1, g_{(n+1)(n+1)} = -1$. The time-orientation is determined by the vector field X with the constant components $(0, \ldots, 0, 1)$ in the canonical coordinates in $\mathbb{R}^{n+1}$. Clearly, for $A > 0$, Theorem 2 is equivalent to

Theorem 3. *Let* f *be a continuously differentiable real-valued function on a closed half-space* $\mathbb{R}^{n+1}_+ = \{(x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} | t \geqslant 0\}$ *of the Minkowski space-time* (M,g), *and for each point* $u \in \mathbb{R}^{n+1}_+$, *there exists a nonzero nonspacelike vector* $v \in T_uM$ *such that* $df(u)(v) = 0$. *Then, for each point* $u^1 \in \mathbb{R}^{n+1}_+$, *there is a point* $u^0 \in H \cap J^-(u^1)$ *such that* $f(u^1) = f(u^0)$.

Definition 2. A nonspacelike curve $c = c(t)$, $t \in (a,b)$, is said to be *inextendible* if there are no limits $\lim_{t \to a^+} c(t)$ and $\lim_{t \to b^-} c(t)$.

Definition 3. A *Cauchy surface* in a space-time (M,g) of dimension $n+1$ is a closed subset $S \subset M$ that is an n-dimensional topological manifold intersected by any inextendible nonspacelike curve in (M,g) at exactly one point.

If S is any Cauchy surface in a space-time (M,g) then $M = J^+(S) \cup J^-(S)$.

Theorem 4, [6]. *A space-time is globally hyperbolic if and only if it admits a Cauchy surface.*

Theorem 5, [6]. *For any Cauchy surface* S *in a globally hyperbolic space-time* (M,g) *there exists a homeomorphism* $\psi : M \cong S \times \mathbb{R}$ *such that* $S_t = \psi^{-1}(S \times \{t\})$ *is a Cauchy surface for every* $t \in \mathbb{R}$.

In [7] the authors proved: any globally hyperbolic space-time admits a smooth Cauchy surface S and then as ψ in Theorem 5 one can take a diffeomorphism.

Theorem 6, [2]. *Let* f *be a continuously differentiable real-valued function on a globally hyperbolic space-time* (M,g) *with a Cauchy surface* S, *and for each point* $u \in M$, *there exists a nonzero nonspacelike vector* $v \in T_uM$ *such that* $df(u)(v) = 0$. *Then, for each point* $u^1 \in J^+(S)$ *(respectively,* $u^1 \in J^-(S)$*) there exists a point* $u^0 \in S \cap J^-(u^1)$ *(respectively,* $u^0 \in S \cap J^+(u^1)$*) such that* $f(u^1) = f(u^0)$.

Another example of globally hyperbolic space-time is the *de Sitter space-time of the first kind* [2–5], one-sheeted hyperboloid $S(R) : \Sigma_{k=1}^n x_k^2 - t^2 = R^2, R > 0$, in the Minkowski space-time $(\mathbb{R}^{n+1}, g)$, $n+1 \geqslant 3$, supplied with the Lorentzian metric G induced from $(\mathbb{R}^{n+1}, g)$. The time-orientation is defined by the tangent vector field Y to $S(R)$, orthogonal to all spacelike sections of the form $S(R,c) = S(R) \cap \{(x,c) \in \mathbb{R}^{n+1} | c \in \mathbb{R}\}$, whose last component in the ambient space $\mathbb{R}^{n+1}$ equals 1. The above sets $S(R,c)$ are smooth Cauchy surfaces in $(S(R), G)$. The isometry groups of Minkowski and de Sitter space-times act transitively on the set of timelike geodesics and on each such geodesic.

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Alexander A. Borisenko. *The generalizations of Cohn-Vossen−Toponogov splitting theorem*

It would be told about generalizations Cohn-Vossen−Toponogov splitting theorem in Euclidean, Lorentzian, Minkowski spaces. It would be proved some extrinsic analogues of these theorems.

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Dmitri Burago. *On some geometry problems involving PDEs and Dynamics*

This is a survey talk based on joint projects with D. Chen, S. Ivanov, Ya. Kurylev, M. Lassas, J. Lu, and, if time permits, with L. Polterovich, B. Kleiner at al. As the main themes, I plan Inverse Problems (Including boundary rigidity and the Inverse Gel'fand's Transform), presence of positive metric entropy in nearly integrable systems, discretization in PDEs and metric geometry. In an unlikely event some time is left, a bit of combinatorial group theory and Riemannian geometry. My intention is to try to give an accessible talk, sweeping technical details under the rug. Questions: welcome at any point.

Soumya Dey. *The liftable mapping class groups of regular cyclic covers*

We shall discuss about a symplectic criterion for a mapping class to be liftable under a regular cyclic cover of closed oriented surfaces, and some of its interesting applications.

References:

[1] Liftable mapping class groups of regular cyclic covers (with N. Agarwal, N. Dhanwani, K. Rajeevsarathy), to appear in *Houston Journal of Mathematics.* (2021) *arXiv:1911.05682.*

[2] My homepage: https://sites.google.com/site/soumyadeymathematics/

Jost-Hinrich Eschenburg. *From Milnor's Morse theory to the Thom isomorphism*

1. Introduction

Professor Toponogov's famous triangle comparison theorem was a starting point for global Riemannian geometry, in particular in connection to nonnegative sectional curvature. The sphere theorem of Berger and Klingenberg and the soul theorem of Gromoll and Cheeger are two classical applications.

The most prominent Riemannian manifolds of nonnegative sectional curvature are compact symmetric spaces. The indecomposable ones are particularly interesting, not only individually but also together for sufficiently large dimensions: As was pointed out in John Milnor's book on Morse theory [3], they make a strong contribution to algebraic topology. In fact, they are iterated *centrioles* in the classical compact groups which is fundamental for Bott periodicity.

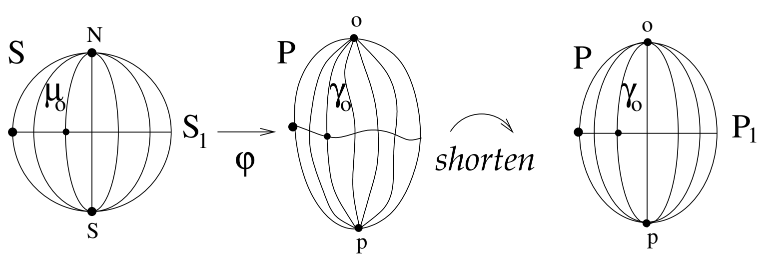
Like spheres, many other compact symmetric spaces have "poles", "meridians" (shortest geodesics between poles) and "equators", correctly "centrioles" (connected components of midpoint sets for meridians). Sometimes all maps from the k-sphere into such space sending poles to poles (at least in most homotopy classes) can be deformed onto maps sending meridians to meridians. The deformed maps are just geodesic suspensions over maps from the equator $\mathbb{S}^{k-1}$ into the centriole. This is the main construction step in [3]. Its iteration implies Bott's periodicity theorem for the orthogonal, unitary and symplectic groups. E.g. the 8th iterated centriole of SO_p is $SO_{p/16}$, hence $\pi_k(SO_\infty) \cong \pi_{k-8}(SO_\infty)$.

In a recent joined work with Bernhard Hanke [2] we extend Milnor's approach in two ways.

(1) We exhaust $\mathbb{S}^k$ by going all the way down until $\mathbb{S}^0$. Thus we deform each map $\mathbb{S}^k \to SO_p$ (say) into a normal form given by a k-fold geodesic suspension. This is the restriction of a linear map $\mathbb{R}^{k+1} \to \mathbb{R}^{p\times p}$ which is nothing else than a representation of the Clifford algebra $Cl(\mathbb{R}^k)$ on $\mathbb{R}^p$. Moreover, Mmps $\mathbb{S}^k \to SO_p$ can be viewed as clutching maps of vector bundles over $\mathbb{S}^{k+1}$, and the normal forms make them (generalized) Hopf bundles, cf. [1].

(2) Milnor's methods allow dependence on finitely many additional parameters, even locally. Thus we may replace the sphere $\mathbb{S}^k$ by certain sphere bundles over finite CW-complexes. In the language of topological K-theory these results imply classical Bott–Thom isomorphism theorems.

2. Milnor's construction



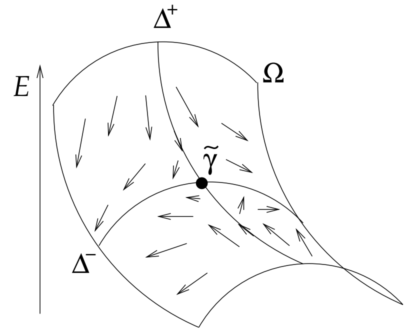
A sphere $\mathbb{S} = \mathbb{S}^k$ with $k \geqslant 1$ has poles (north and south poles N and S), meridians (shortest geodesics between the poles) and an equator $\mathbb{S}_1$ (midpoint sets of the meridians). The same notions apply for many other compact symmetric spaces. Recall that a *symmetric space* is a Riemannian manifold P with a *symmetry* at any point $o \in P$, an isometry $s_o \in I(P)$ with $s_o s_o = \mathrm{id}$ such that o as an isolated fixed point of s_o. Like in the case of spheres there is often another isolated fixed point p for s_o which means $s_o = s_p$. The shortest geodesics from o to p will be called *meridians* again, and the set of their midpoints is the *equator*. But now the "equator" may have several connected components, called *centrioles* P_1. As *reflective* submanifolds (components of the fixed point set of an isometric involution) they are totally geodesic in P, hence again symmetric.

"Sometimes" all maps $\varphi : \mathbb{S} \to P$ assigning poles to poles ($N \mapsto o$, $S \mapsto p$) can be deformed into maps φ_1 throwing meridians onto meridians, by simultaneously straightening the φ-images of the meridians. This deformation is *relative*: it preserves φ on those meridians μ_o whose image is already a meridian γ_o. The deformed map φ_1 is uniquely determined by its restriction to the equator $\mathbb{S}_1$: it is the *geodesic suspension* over $\varphi_1|_{\mathbb{S}_1}$ from the poles o and p. When $k \geqslant 2$, we have

$$\mathrm{Map}(\mathbb{S}, P)_* \simeq \mathrm{Map}(\mathbb{S}_1, P_1)_*$$

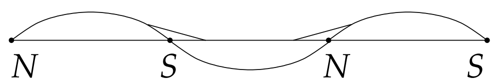
which implies $\pi_k(P) \cong \pi_{k-1}(P_{k-1})$. When P_1 has poles again, it has a centriole P_2 etc., and we can iterate: $\pi_k(P) = \pi_{k-j}(P_j)$. E.g. $P = SO_p$ with $16|p$ has at least 8 iterated centrioles for the poles $o = I$, $p = -I$. Moreover, $P_8 = SO_{p/16}$, thus $\pi_k(SO_\infty) = \pi_{k-8}(SO_\infty)$ which is Bott periodicity, the main result of [M].

When is "sometimes"? This is answered by Morse theory for the energy function E on the path space $\Omega = \Omega(P, o, p)$ of all paths $\omega : [0, 1] \to P$, $\omega(0) = o$, $\omega(1) = p$ where $E(\omega) = \int_0^1 |\omega'(t)^2| dt$.[1] More precisely, the deformation is given via the negative gradient flow of the energy, $-\nabla E$. Almost all points of Ω flow into the set $\Omega^{\min}$ of minima, which are the shortest geodesics from o to p. The only exceptions are those points which flow to some *saddle*, those in the domain of attraction Δ^+ of a non-minimal critical point, a non-minimal geodesic $\tilde\gamma$ from o to p.



If $\operatorname{codim} \Delta^+ = \dim \Delta^- = \operatorname{index} \tilde\gamma$ is large, those points can be easily avoided by any subset of dimension $< \operatorname{index} \tilde\gamma$. Then the map $\varphi : \mathbb{S}_1 \to \Omega$ can be deformed into $\varphi_1 : \mathbb{S}_1 \to \Omega^{\min}$.

Thus we have to make sure that the index of non-minimal geodesics is large. A model case is the sphere itself, $P = \mathbb{S}^p$.



A non-minimal geodesic from N to S must cover a great circle at least one and a half times, hence it meets two conjugate points, each with multiplicity $p - 1$, according to the number of perpendicular 2-spheres containing the geodesics. Thus the index is at least $2(p - 1)$, and it grows to ∞ with the dimension p. The method of counting the 2-spheres which contain the geodesic works in general.

In the case $P = SO_p$, the meridians are shortest geodesics from I to $-I$, in particular one-parameter subgroups, thus their midpoints square to $-I$, i.e. they form the set of *complex structures* J in SO_p; this set has two isometric connected components. More generally, a chain of iterated centrioles $SO_p \supset P_1 \supset P_2 \cdots \supset P_k$ corresponds to a *Clifford family* $J_1, \dots, J_k$ (a family of anticommuting complex structures in SO_p), and P_{k+1} is a component of the set of complex structures $J \in SO_p$ which anticommute with $J_1, \dots, J_k$. Here is the first period.

k	0	1	2	3	4
P_k	SO_{8q}	SO_{8q}/U_{4q}	U_{4q}/Sp_{2q}	$\mathbb{G}_q(\mathbb{H}^{2q})$	Sp_q
P_{k+4}	Sp_q	Sp_q/U_q	U_q/O_q	$\mathbb{G}_{q/2}(\mathbb{R}^q)$	$SO_{q/2}$
$\lvert\pi_1 \otimes \mathbb{Q}\rvert$	1	1	∞	1	1

$(*)$

3. Maps from $\mathbb{S}^{4m-1}$ to SO_p

Let $k = 4m - 1$ and $J_1, \dots, J_k \in SO_p$ a Clifford family (a Cl_k-module structure on $\mathbb{R}^p$) where $J_k = J_1 \cdots J_{k-1}$, and put $J_0 := \mathrm{id} = I$. Let $\mathrm{Map}_*(\mathbb{S}^{k-i}, P_i)$ denote the set of maps which throw a certain meridian

[1] We need to approximate continuous paths by H^1-curves or even better by geodesic polygons. If we assume an upper bound for the energy, we may also bound the number of vertices and work on a finite dimensional approximation of Ω.

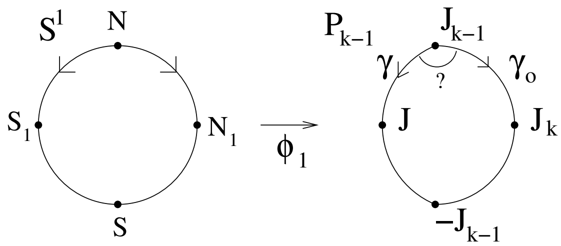
of $\mathbb{S}^{k-i}$ onto the shortest geodesic from J_i to $-J_i$ through J_{i+1} in P_i. Using the deformation process above we obtain a map

$$\theta : \mathrm{Map}_*(\mathbb{S}^1, P_{k-1}) \to \cdots \to \mathrm{Map}_*(\mathbb{S}^k, P_0)$$

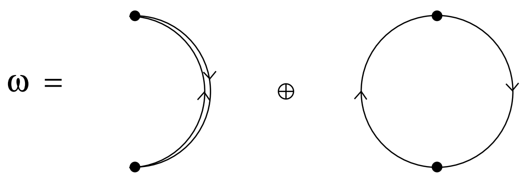
where $P_0 = SO_p$. We would like to complete this chain by an isomorphism

$$\mathrm{Map}_*(\mathbb{S}^0, P_k) \to \mathrm{Map}_*(\mathbb{S}^1, P_{k-1}), \quad \varphi \mapsto \varphi_1$$

straightening the φ-images of the two meridians of $\mathbb{S}^1$.



Let $\omega = \gamma * \gamma_o^{-1}$. A new feature of this case is that the homotopy class $[\omega] \in \pi_1(P_{k-1}) = \mathbb{Z}$ (cf. $(*)$) must be a given data. This is realized by the winding number of $\det \omega$ for the embedding $P_{k-1} \hookrightarrow U_{p/2}$, $J \mapsto JJ_{k-1}^{-1}$, using the complex structure $sfi = J_1 \cdots J_{k-2}$. Being shortest, γ has the form $\gamma(t)J_{k-1}^{-1} = \mathrm{diag}(e^{-\pi i t} I_r, e^{\pi sfit} I_s) \in U_{p/2}$ where $r + s = p/2$, while the fixed meridian γ_o has the form $\gamma_o(t)J_{k-1}^{-1} = e^{\pi sfit} I_{p/2}$. Thus we have a splitting $\mathbb{R}^p = \mathbb{C}^{p/2} = \mathbb{C}^r \oplus \mathbb{C}^s$ preserved by $J_1, \ldots, J_k$, and ω splits as (ω_0, ω_1) such that ω_0 runs to and fro between $\pm J_{k-1}|_{L_0}$ along the same path in $\mathbb{C}^r =: L_0$ while $\omega_1(t)J_{k-1}^{-1} = e^{2\pi sfit} I_s$ on $\mathbb{C}^s =: L_1$. So the picture of ω at the right part of the previous figure was wrong. Instead, its decomposition with respect to $\mathbb{R}^p = L_0 \oplus L_1$ looks as follows.



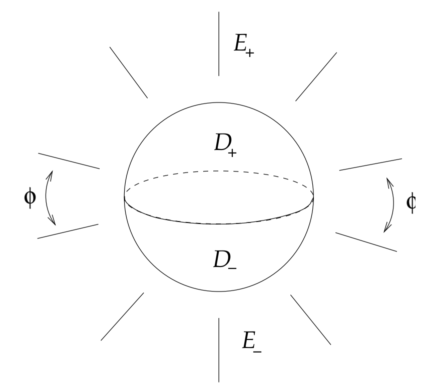
In a last step we homotop the component ω_0 canonically onto its initial value $J_{k-1}|_{L_0}$. Thus after this deformation we obtain $\tilde{\omega} = (J_{k-1}|_{L_0}, \omega_1)$, and the homotopy class $[\tilde{\omega}] = [\omega]$ is determined by $s = \dim L_1$. Composing with θ, i.e. applying iterated geodesic suspensions to $\tilde{\omega}$ in order to embed it into $\mathrm{Map}(\mathbb{S}^k, SO_p)_* := \{\varphi : \mathbb{S}^k \to SO_p : \varphi(N) = I\}$, we see that the 0-component is contractible to id_{L_0} while the 1-component is the restriction to $\mathbb{S}^k$ of the linear map $\mu : \mathbb{R}^{k+1} \to \mathrm{End}(L_1)$, $e_i \mapsto J_i|_{L_1}$, $e_0 \mapsto \mathrm{id}_{L_1}$. This restriction of the module multiplication $\mu : Cl_k \to \mathrm{End}(L_1)$ on a Cl_k-module L_1 to $\mathbb{S}^k \subset \mathbb{R}^k \oplus \mathbb{R} \cdot 1 \subset Cl_k$ will be called *Hopf map*. We also allow for trivial extensions and put $\mathrm{Hopf}(\mathbb{S}^k, SO_p)_s$ the maps which are Hopf maps on $L_1 \subset \mathbb{R}^p$ and $=$ id on the complement $L_0 = \mathbb{R}^p \ominus L_1$. Using the precise condition for high index of non-minimal geodesics in this case we have shown (recall $\pi_k(SO_p) = \pi_1(P_{k-1}) = \mathbb{Z}$)

Theorem 1.

$$\mathrm{Hopf}(\mathbb{S}^k, SO_p)_s \overset{\simeq_d}{\subset} \mathrm{Map}(\mathbb{S}^k, SO_p)_s$$

that is this inclusion is d-connected, provided that $s \geqslant p_k d$ and $p \geqslant 4s$ where p_k is the dimension of the irreducible Cl_k-representations.

4. Vector bundles over spheres and sphere bundles



Now we assume $n = 4m$ and put $k = n - 1$. An oriented vector bundle $\mathscr{E} \to \mathbb{S}^n$ can be decomposed into two trivial bundles E_+ and E_- living over the northern and southern hemispheres $D_\pm$. Along the common boundary, the equator $\mathbb{S}^k$, these bundle E_+ and E_- have to be identified by a map $\varphi : \mathbb{S}^k \to SO(E_+, E_-)$, the so called *clutching map*. We will write $\mathscr{E} = (E_+, \varphi, E_-)$.

E.g. let $J_1, \dots, J_n$ be a Clifford family on $L = \mathbb{R}^p$. Then the product $W = J_1 \cdots J_n$ is an involution which anticommutes with all J_i, thus $L = L^+ \oplus L^-$ with $W = \pm\mathrm{id}$ on $L^\pm$, and the Clifford module multiplication is a map $\mu : \mathbb{S}^k \to SO(L_+, L_-)$. Then $\mathscr{L} = (L^+, \mu, L^-)$ is called *Hopf bundle*. By the results of section 3 these give *all* bundles over $\mathbb{S}^k$:

Theorem 2 [1, 11.5]. *For sufficiently large p we have*

$$\widetilde{KO}(\mathbb{S}^n) \cong \mathscr{M}_{n-1}/r(\mathscr{M}_n)$$

where $\mathscr{M}_n$ denotes the set of equivalence classes of Cl_n-modules and $r : \mathscr{M}_n \to \mathscr{M}_{n-1}$ the restriction map.

Now we replace the n-sphere $\mathbb{S} = D_+ \cup D_-$ by an n-sphere bundle $\hat{V} = D_+V \cup D_-V = S(V \oplus \mathbb{R})$ (unit sphere bundle of $V \oplus \mathbb{R}$) where $V \to X$ is an oriented n-dimensional vector bundle over a finite CW-complex X, and $D_\pm V$ are two copies of the disk bundle $DV = \{v \in V : |v| \leqslant 1\}$. A vector bundles $\mathscr{E} \to \hat{V}$ can be written as a triple $\mathscr{E} = (E_+, \sigma, E_-)$ where $E_\pm$ are bundles over $D_\pm V \simeq X$ and $\sigma : SV \to SO(E_+, E_-)$ the clutching map.

E.g. let $\Lambda \to X$ be a $Cl(V)$-module bundle. Let $J_i = \mu(e_i)$ for any local oriented orthonormal basis $B = (e_1, \dots, e_n)$ of V. Then $W = J_1 \cdots J_n$ is independent of B, and as before it is fibrewise an involution anticommuting with all J_i. Again, $\Lambda = \Lambda^+ \oplus \Lambda^-$ (eigenspaces of W) and $\mu : SV \to SO(\Lambda^+, \Lambda^-)$. Then $\mathscr{L} = (\Lambda^+, \mu, \Lambda^-)$ is called the *Hopf bundle of* Λ. An example is the spinor bundle $\Lambda = \Sigma$ when V carries a spin structure; then $\mathscr{L} = \mathscr{S}$ is called the spinor-Hopf bundle.

Theorem 3. *Every vector bundle $\mathscr{E} \to \hat{V}$ is stably isomorphic to $E \oplus \mathscr{L}$ where $E \to X$ is any vector bundle over X and $\mathscr{L} = (\Lambda^+, \mu, \Lambda^-) \to \hat{V}$ a Hopf bundle. Moreover, (E, Λ) is unique up to stable isomorphisms.*

Remark. When V carries a spin structure, there is a vector bundle $E_1 \to X$ such that

$$\mathscr{L} = \begin{cases} E_1 \otimes \mathscr{S} & \text{when } m \text{ even,} \\ E_1 \otimes_{\mathbb{H}} \mathscr{S} & \text{when } m \text{ odd.} \end{cases}$$

In the first case, Theorem 3 can be viewed as a Thom isomorphism theorem for KO-theory with Thom class $[\mathscr{S}]$: the KO-theories over X and over the Thom space $X^V = \hat{V}/D_+V$ are isomorphic by multiplication with $[\mathscr{S}]$.

Idea. The fibre of $\mathscr{E}$ plays the rôle of $\mathbb{R}^p$ which carried a Clifford module structure. Similar we may assume now that there is a $Cl(V)$-module bundle $\Lambda = \Lambda^+ \oplus \Lambda^-$ with $E_+ = \Lambda^+$. In fact, we embed E_+ into a sufficiently large Λ^+, put $F = \Lambda^+ \ominus E_+$ and pass to $\tilde{\mathscr{E}} = F + \mathscr{E} = (F, \mathrm{id}, F) \oplus (E_+, \sigma, E_-)$ with $\tilde{E}_+ = \Lambda^+$. As before we obtain an invariant splitting $\Lambda = \Lambda_0 \oplus \Lambda_1$ such that $\mathscr{E}$ is stably isomorphic to $(\Lambda_0^+, \mathrm{id}, \Lambda_0^+) \oplus (\Lambda_1^+, \mu, \Lambda_1^-)$, using induction over the cells of X.

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Vladimir Golubyatnikov and Liliya Minushkina. *On stability of cycles of some block-linear dynamical systems*

Following [1,2], we study trajectories of 6D dynamical system which simulates functioning of one natural gene network:

$$\frac{dx_0}{dt} = L_0(x_5) - k_0 x_0; \quad \frac{dx_1}{dt} = \Gamma_1(x_0) - k_1 x_1; \quad \frac{dx_2}{dt} = L_2(x_1) - k_2 x_2;$$
$$\frac{dx_3}{dt} = \Gamma_3(x_2) - k_3 x_3; \quad \frac{dx_4}{dt} = L_4(x_3) - k_4 x_4; \quad \frac{dx_5}{dt} = \Gamma_5(x_4) - k_5 x_5. \tag{1}$$

Here and below $x_j + 1 \geqslant 0,\ j = 0, 1, \dots, 5;\ i = 0, 1, 2$. Positive coefficients k_j describe the rates of natural degradation of components of the gene network. Step functions L_{2i} and Γ_{2i+1} are defines as follows:

$L_{2i}(w) = a_{2i} - 1$ for $-1 \leqslant w \leqslant 0$; $L_{2i}(w) = -1$ for $w > 0$;

$\Gamma_{2i+1}(u) = a_{2i+1} - 1$ for $u > 0$; $\Gamma_{2i+1}(u) = -1$ for $-1 \leqslant u \leqslant 0$; $a_j > 1$.

Similar systems were studied in [2,3] as gene networks models. Problems of detection of cycles and other attractors in their phase portraits appear here naturally.

Let $Q^6 := \prod_{j=0}^{j=5}[-1, a_j - 1] \subset \mathbb{R}^6$. The planes $x_j = 0$ subdivide Q^6 to 64 smaller parallelepipeds. We call them *blocks*, and enumerate by binary multi-indices $\{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\}$:

$$\varepsilon_j = 0, \text{ if } x_j \leqslant 1 \text{ for all points of this block}; \quad \varepsilon_j = 1 \text{ otherwise.} \tag{2}$$

In each block of this decomposition, solutions of the system (1) can be expressed explicitly, see [2]. These trajectories are piecewise smooth and their vertices are located on the coordinate hyperplanes $x_j = 0$. Similar discretizations of phase portraits for other gene networks models were considered in [2,3,4]. Simple calculations show that

1. *Parallelepiped Q^6 is positively invariant domain of the system* (1);

2. *For any pair B_1, B_2 of adjacent blocks, trajectories of all points of their common 5-dimensional face $F = B_1 \cap B_2$ pass either from B_1 to B_2 or from B_2 to B_1.*

We denote transitions of trajectories from B to B' as $B \to B'$. So, the graph composed by boolean 6-dimensional cube with the vertices $\{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\}$ can be oriented according to directions of these arrows. Here $\varepsilon_j = 0$ or $\varepsilon_j = 1$, as in (2).

Definition. *The valence $V(B)$ of a block B is a number of its 5-dimensional faces such that trajectories of their points come out of B to its adjacent blocks.*

This definition can be formulated for analogous higher-dimensional systems. There are 12 one-valent blocks, 40 three-valent blocks, and 12 five-valent blocks in the decomposition of Q^6. It was shown in [2] that the system (1) has a cycle in Q^6 which is contained in the interior of the union W_1 of all the 1-valent blocks and follows the arrows of the diagram:

$$\begin{tikzcd}
\{110011\} \arrow[r] & \{010011\} \arrow[r] & \{000011\} \arrow[r] & \{001011\} \arrow[d] \\
\{110010\} \arrow[u] & & & \{001111\} \arrow[d] \\
\{110000\} \arrow[u] & & & \{001101\} \arrow[d] \\
\{110100\} \arrow[u] & \{111100\} \arrow[l] & \{101100\} \arrow[l] & \{001100\} \arrow[l]
\end{tikzcd} \tag{3}$$

Let F_0, F_1, etc. be the intersections of adjacent blocks in the diagram (3):
$F_0 = \{110011\} \cap \{010011\}$, where $x_0 = 0$; $F_1 = \{010011\} \cap \{000011\}$, where $x_1 = 0$; ...and $F_{12} = F_0$, where we have $x_0 = 0$, $x_1 > 0$, $x_2 < 0$, $x_3 < 0$, $x_4 > 0$, $x_5 > 0$. Let $\Phi : F_0 \to F_0$ be the composition of the shifts of the points of F_0 along their trajectories of (1), this is the Poincaré map of a cycle $C \subset W_1$, its existence was shown in [2]. So, W_1 is an invariant domain of the system (1).

Following [2,3], we reduce our studies of fixed points of the Poincaré map Φ to that of existence and uniqueness of a non-zero fixed point of the "normalized" Poincaré map $\widehat{\Phi} : K^5(u_1, u_2, u_3, u_4, u_5) \to K^5(u_1, u_2, u_3, u_4, u_5)$ of the unit 5D cube $K^5 = [0, 1]^5 \approx F_0$ into itself. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ be the coordinate function of $\widehat{\Phi}$. Note, that $\widehat{\Phi}(0, 0, 0, 0, 0) = (0, 0, 0, 0, 0)$, but this fixed point does not generate a cycle of (1).

Lemma 1. *The map $\widehat{\Phi}$ is monotonic with respect to natural partial order in K^5:*
$(u_1, u_2, u_3, u_4, u_5) \preceq (v_1, v_2, v_3, v_4, v_5)$ *if* $u_k \leqslant v_k$ *for all* k.

Lemma 2. *The first derivatives of all functions φ_j are positive, and all their second derivatives are negative.*
$$\left.\frac{d\varphi_1(u_1, 0, 0, 0, 0)}{du_1}\right|_{u_1=0} > 1.$$

Theorem. *If $a_j > k_j$, for all $j = 0, 1, 2, \ldots, 5$, then the dynamical system* (1) *has a unique cycle C in the invariant domain W_1. This cycle is stable.*

In a similar way, we have shown uniqueness and exponential stability of such a cycle for one 4-dimensional analogue of the system (1), see [3]. One very particular case $k_1 = k_2 = k_3 = k_4 = 1$ of that 4-dimensional system was studied earlier in [4].

The authors are indebted to V.V.Ivanov for useful discussions.

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Brendan Guilfoyle and Wilhelm Klingenberg. *Toponogov Conjecture on complete convex surfaces*

Let $P \hookrightarrow \mathbb{R}^3$ be complete, convex, C^2-regular and homeomorphic to a plane. Denote by $\kappa_i(p)$ denote the principal curvatures of P at p.

Toponogov Conjecture [1]. *For P as above we have* $\inf_{p\in P} |\kappa_1(p) - \kappa_2(p)| = 0$.

The Conjecture implies that either there exists an *umbilic point* $p \in P$ or that there exists a sequence $p_n \in P$ leaving any compact subset of P (and thereby leaving any compact subset of $\mathbb{R}^3$) with $\lim_{n\to\infty} \kappa_1(p_n) - \kappa_2(p_n) = 0$. The latter situation may designated an *umbilic point at infinity* of P.

Theorem (Guilfoyle-Klingenberg 2020 [2]. *Asssume that P is of regularity $C^{3,\alpha}$. Then the Toponogov Conjecture holds for P.*

The proof is indirect and associates to P a Riemann-Hilbert boundary problem for holomorphic discs in (TS^2, NP). Here, $NP \hookrightarrow TS^2$ is the family of lines normal to $P \hookrightarrow \mathbb{R}^3$, realized as a Lagrangian surface in the parameter space TS^2 of all oriented affine lines of $\mathbb{R}^3$. If the Toponogov Conjecture does not hold for P_0, then the Fredholm Index of the Riemann-Hilbert problem would be negative and there would *not exist* a holomorphic disc with boundary in (TS^2, NP_0). The details appeared in [3]. However by spacelike mean curvature flow with boundary as in [4], the authors prove that it *does exist*. This part of the argument makes reference to the canonical neutral Kähler metric of TS^2 which was introduced in [5]. The strategy of the proof is similar to that of the Caratheodory Conjecture [6] on the existence of two umbilics on compact convex surfaces homeomorphic to a sphere. While V.A. Toponogov was aware of the analogy between the two conjectures, one does not imply the other. The details of the proof given in [6] have been published in [3] and [4], except the required boundary estimates for mean curvature flow in (TS^2, NP_0), which are presently under review.

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Alexander Gutman and Larisa Kononenko. *Binary correspondences and an algorithm for solving an inverse problem of chemical kinetics in a nondegenerate case*

We consider a singularly perturbed system of ordinary differential equations which describes a process in chemical kinetics [1, 2]:

$$\dot{x}(t) = f\big(x(t), y(t), t, \varepsilon\big),$$
$$\varepsilon\,\dot{y}(t) = g\big(x(t), y(t), t, \varepsilon\big),$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $t \in \mathbb{R}$, ε is a small parameter, f, g are sufficiently smooth functions. Direct and inverse problems are stated for such a system for the case in which the right-hand sides are polynomials of arbitrary degree.

Binary correspondences are used for formalization of problems, their basic components, properties, and constructions [3, 4, 5]. It is shown how the inverse problem of chemical kinetics can be corrected and made more practical by means of the composition with a simple auxiliary problem which represents the relation between functions and finite sets of numerical parameters being measured. Formulas for the solution of the inverse problem are presented for the degenerate case $\varepsilon = 0$ and conditions of unique solvability are indicated for the corrected inverse problem.

An iteration algorithm is proposed for finding an approximate solution to the inverse problem for the nondegenerate case $\varepsilon \neq 0$. At each step of the algorithm, the solution of the inverse problem for the above-considered case $\varepsilon = 0$ is combined with the solution of the direct problem which is reduced to the proof of the existence and uniqueness of a solution in the case $\varepsilon \neq 0$. The conjecture is stated on convergence of the algorithm and an approach to its justification is developed which is based on the Banach fixed-point theorem.

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Olesya Khromova, Pavel Klepikov, and Eugene Rodionov. *Sectional Curvature and Ricci Solitons of Metric Lie Groups with the semisymmetric connection*

One of the important problem of Riemannian geometry is the task of establishing connections between the curvature and topology of a Riemannian manifold, and, in particular, a question of influence of the sign of sectional curvature on the topology of a Riemannian manifold. A number of theorems Riemannian geometry are well known in this direction: the Hadamard - Cartan theorem on a complete simply connected Riemannian manifold of non-positive sectional curvature, M. Gromov's theorem on the Riemannian manifold of non-negative Ricci curvature, A.D. Aleksandrov - V.A.Toponogov's comparison theorem aboute angles of triangle, sphere theorem, extremal theorems in Riemannian geometry, a number of other results (see [1,2]).

Riemannian manifolds of sign-definite sectional curvature have been studied by many mathematicians. This question is most studied in the homogeneous Riemannian case. In this direction, the classification of homogeneous Riemannian manifolds of positive sectional curvature and some results on homogeneous Riemannian manifolds of nonpositive sectional curvature were obtained by M.Berger, N.Wallach, L. Bergery, D.V. Alekseevsky, E.Heinze, L.Bergery and other mathematicians (see reviews [3,4]).

In this paper we investigate Riemannian manifolds whose metric connection is a connection with vectorial torsion, or semisymmetric connection. The Levi-Civita connection contains into this class of connections. Although the curvature tensor of these connections does not possess the symmetries of the Riemannian curvature tensor, it possible to determine a sectional curvature for semisymmetric connection. In this paper we study the sign of sectional curvature of a semisymmetric connection. As the main test example, we consider Lie groups with a left-invariant Riemannian metrics.

Let (M, g) be a Riemannian manifold, ∇^g be a Levi-Civita connection, ∇ be a semisymmetric connection which is defined by the formula $\nabla_X Y = \nabla^g_X Y + g(X, Y)V - g(V, Y)X$, where V is some fixed vector field, X, Y are arbitrary vector fields. Let us define the sectional curvature at the point $p \in M$ by the formula $K(X, Y) = R(X, Y, X, Y)$, where $R(X, Y, Z, U) = g(R(X, Y)Z, U)$ and $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ is a curvature tensor, $\{X, Y\}$ is an orthonormal basis.

Lets $\pi(X) = g(X, V)$. Then we have

Theorem 1. *Let (M, g) be a Riemannian manifold, ∇^g be a Levi-Civita connection, ∇ be a semisymmetric connection. The sectional curvatures $K^g(X, Y)$ and $K(X, Y)$ coincide if and only if $g(\nabla^g_Y V, Y) + g(\nabla^g_X V, X) - \pi(V) + \pi^2(X) + \pi^2(Y) = 0$.*

Further, we will assume that $(M, g) = (G, g)$ is a connected Lie group G with a left invariant Riemannian metric g, $\dim G \geqslant 3$. We will also assume that all vector fields, including field V, are left-invariant. Based on these assumptions, the following statements are true.

Theorem 2. *Let (G, g) be a connected Lie group G with a left-invariant Riemannian metric g, ∇ be a semisymmetric connection defined by a non-zero left-invariant vector field V. Then there always exists an orthonormal platform $\{X, Y\}$ for which $K(X, Y) \neq K^g(X, Y)$.*

Theorem 3. *Let (G, g) be a 3-dimensional Lie group G with a left invariant Riemannian metric g, ∇ be a semisymmetric connection. Then the following statements are true.*
(1) In the unimodular case, the condition $K > 0$ implies that the universal covering of the group G is isomorphic to $SU(2)$.
(2) In nonunimodular case there is a section of non positive sectional curvature.

Theorem 4. *Let (G, g)be a connected Lie group G with a left invariant Riemannian metric g, $\dim G \geqslant 3$ and ∇ be a semisymmetric connection. Then the following statements are true.*
(1) If the group G is commutative, then $K \in [-\|V\|^2, 0]$ (in the case of the Levi-Civita connection, the sectional curvature is equal to zero).
(2) If the group G is compact, $\mathrm{rk} G > 1$ and the metric g is bi-invariant, then $K_\sigma \leqslant 0$ for some two-dimensional direction σ (in the case of the Levi-Civita connection, the sectional curvature is non-negative). (3) If the group G is compact and the metric g is bi-invariant, then the condition $K > 0$ implies that the universal covering of the group G is isomorphic to $SU(2)$. Moreover, for such Lie groups, any connection with the condition $\|V\| < 1/2$ has positive sectional curvature, and the δ-pinching function of the sectional curvature takes the values $\delta(\|V\|) \in (0, 1]$.

Another interesting question for mathematicians and physicists (which was reected in the reviews by H. D. Cao, R. M. Aroyo - R. Lafuente, [5, 6]) is the question of Ricci solitons. Ricci solitons are the solution to the Ricci flow and are a natural generalization of Einstein metrics. This question more detail was studied in the case of trivial Ricci solitons, or Einstein metrics, as well as in the homogeneous Riemannian case. In this paper we investigate semisymmetric connections on three-dimensional Lie groups with the metric of an invariant Ricci soliton. A classification of these connections on three-dimensional Lie groups with left-invariant Riemannian metric of the Ricci soliton is obtained. It is proved that in this case there are nontrivial invariant semisymmetric connections. In addition, it is shown that there are nontrivial invariant Ricci solitons. Earlier L. Cerbo proved that on unimodular Lie groups with left-invariant Riemannian metric and Levi-Civita connection all invariant Ricci solitons are trivial [7]. In the non-unimodular case, a similar result up to dimension four was obtained by P.N. Klepikov and D.N. Oskorbin [8].

A metric g of a complete Riemannian manifold (M, g) is called a Ricci soliton if it satisfies the equation

$$r = \Lambda g + L_P g, \quad (1)$$

where r is the Ricci tensor of the g metric, $L_P g$ is the Lie derivative of the metric g in the direction of the complete differentiable vector field P, the constant $\Lambda \in \mathbb{R}$. If $M = G/H$ is a homogeneous space, then a homogeneous Riemannian metric satisfying (1) is called a homogeneous Ricci soliton, and if $M = G$ is a Lie group, and the field P is left-invariant than g is called an invariant Ricci soliton. Moreover, an invariant Ricci soliton is called trivial if $L_P g(Y, Z) = \tau \cdot g(Y, Z)$ for some $\tau \in \mathbb{R}$, and any $Y, Z \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G. A connection with vector torsion on a Riemannian manifold (M, g) is said to be trivial if the vector field V defining this connection is equal to zero.

Theorem 5. *Let (G, g, ∇) be a three-dimensional Lie group with a left-invariant Riemannian metric g and a semisymmetric connection ∇ different from the Levi-Civita connection. Then, among such Lie groups there are groups and semisymmetric connections on them that admit nontrivial invariant Ricci solitons.*

Remark. Theorem 5 gives an answer to L.Cerbo's conjecture about invariant Ricci solitons on metric Lie groups in the class of nontrivial semisymmetric connections.

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Pavel Klepikov. *Locally homogeneous (pseudo)Riemannian manifolds with restrictions on the Schouten–Weyl tensor*

The study of manifolds of constant Ricci curvature, or Einstein manifolds, is the subject of papers of many mathematicians (see, for example, the review [2]). Recently, various generalizations of Einstein manifolds have been studied, one of which is (pseudo)Riemannian manifolds with zero Schouten–Weyl tensor. In the case of constant scalar curvature, such manifolds are also known as Einstein-like manifolds (first considered by A. Gray in [1]) or as manifolds with zero divergence Weyl tensor (if the dimension of the manifold is greater than three).

Note that in the case of four-dimensional Lie groups with a left-invariant pseudo-Riemannian metric, the class of manifolds with the trivial Schouten–Weyl tensor is not completely classified. So, for example, G. Calvaruso and A. Zaeim in [3] classified Lie groups with the trivial Weyl tensor, and in [4, 5] the same authors obtained a classification of Einstein and Ricci-parallel metrics on four-dimensional pseudo-Riemannian Lie groups.

The classification of four-dimensional Lie groups with a left-invariant pseudo-Riemannian metric and zero Schouten–Weyl tensor, which are not conformally flat nor Ricci-parallel, are obtained in this paper, completes the classification of four-dimensional locally homogeneous (pseudo)Riemannian manifolds with zero Schouten–Weyl tensor. Note that the classification of four-dimensional Lie groups with a left-invariant Lorentzian metric was also obtained in the paper [6] using other methods.

Ricci solitons are another generalization of an Einstein manifolds, first considered by R. Hamilton in [7]. Homogeneous Ricci solitons have been studied by many mathematicians, but the classification of homogeneous Ricci solitons is known only in low dimensions and is not exhaustive (see [8, 9]). An important tool in the study of Ricci solitons is a algebraic Ricci solitons on Lie groups, which were first considered by J. Lauret (see [10]).

One of the main results of this paper is the proof of the following theorem.

Theorem. *Let (G, g) be a Lie group with the left-invariant (pseudo)Riemannian metric of a nontrivial algebraic Ricci soliton, and let the Schouten–Weyl tensor SW be trivial. Then the Ricci soliton is steady and the Ricci operator ρ is of Segre type $\{(1\dots 1\, 2\dots 2)\}$ and has only one eigenvalue, which equals 0; moreover, the number of 2's in the Segre type is necessarily nonzero.*

Pseudo-Riemannian manifolds with isotropic Schouten-Weyl tensor naturally arise when one study locally conformally homogeneous (pseudo) Riemannian manifolds [11]. Earlier, these manifolds were studied in [12, 13] in the case of three-dimensional Lie groups with a left-invariant Lorentzian metric. A complete classification of three-dimensional metric Lie groups whose Schouten–Weyl tensor is isotropic was obtained in this papers.

We have obtain a classification of four-dimensional locally homogeneous pseudo-Riemannian manifolds with a nontrivial isotropy subgroup and isotropic Schouten–Weyl tensor in this paper. This result continues the research begun in the three-dimensional case.

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Evgeniy Kornev. *Almost Paracomplex Structure on Six-dimensional Sphere*

It is well-known fact that six-dimensional sphere admits nonintegrable almost complex structure. In this talk we show how can obtain the nonintegrable almost paracomplex structure on six-dimensional sphere S^6. We consider a six-dimensional sphere as homogeneous space $G_2/\mathrm{SU}(3)$, where G_2 is the specific simple compact group of orthogonal transformations of $\mathbb{R}^7$, and $\mathrm{SU}(3)$ is the special unitary group for $\mathbb{C}^3$. An almost paracomplex structure on manifold M is a continuous field P of tangent spaces linear authomorphisms so that $P^2 = id$, where id is the identify operators field on M. Following the framework used in [1] we obtain the G_2-invariant nonintegrable almost paracomplex structure on S^6 by following way: Firstly, we build on S^6 the G_2-invariant degenerated skew-symmetric linear 3-form Ω with the radical (kernel) of rank 3. The orthogonal complement D to this radical also is the tangent subspaces distribution of rank 3 on S^6. This pair of distributions provide the splitting of tangent bundle $T(S^6)$ into Whitney sum of 3-form Ω radical and its orthogonal complement D, and both these distribution have a rank 3.

Any almost paracomplex structure on manifold M has only two eigen values ± 1, and can be specified by pair of tangent subspaces distribution of the same rank those considered as eigen subspaces distribution for eigen values ± 1. Let R be the 3-form Ω radical on S^6. Setting $P|_R = id$, $P|_D = -id$ we obtain the G_2-invariant almost paracomplex structure on S^6. In [1] it is proved that such paracomplex structure is nonintegrable.

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Nina Lebedeva. *Locally homogeneous C^0-Riemannian manifolds*

We show that locally homogeneous manifolds with continuous Riemannian metric tensor are smooth. This is joint work with Artem Nepechiy.

Philippe G. LeFloch. *Recent advances on Einstein's constraint equations: asymptotic localization and gravitational singularities*

I will present recent mathematical developments [1]–[3] on Einstein's field equations and, especially, Einstein's constraint equations. The Seed-to-Solution Method, which I introduced in collaboration with T.-C.

Nguyen (Paris), generates classes of asymptotically Euclidean, Einstein manifolds with prescribed asymptotics at infinity, and covers metrics with the weakest possible decay (having possibly infinite ADM mass) as well as metrics with the strongest possible decay (of Schwarzschild type). Motivated by Carlotto and Schoen's optimal localization problem, we formulate and solve an asymptotic localization problem by establishing sharp decay estimates for the Einstein operator. On the other hand, Einstein's constraint equations also play a central role in understanding spacetimes in the vicinity of singularity hypersurfaces. I will also discuss recent work in collaboration with B. Le Floch (ENS and Sorbonne, Paris) and G. Veneziano (College de France, Paris, and CERN, Geneva), concerning bouncing cosmological spacetimes.

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E-print https://arxiv.org/abs/1712.10045

Daniil Mamaev. *On Morse Index Retrieval*

Consider a smooth function f in a neighbourhood of the unit sphere S^{d-1}. Its smooth extensions inside the unit ball B^d generically have only non-degenerate critical points. S. Barannikov, following a question by V.I. Arnold, gave a lower bound for the number of critical points of such extensions making use of what later became known as Morse-Barannikov complexes [1]. We further elaborate on the topic and study the question of how different two extensions of f can be.

Let $\mathscr{M}_f$ be the set of the Morse functions $F\colon B^d \to \mathbb{R}$ that agree with f near S^{d-1} and have a unique critical point at the origin O. To each function $F \in \mathscr{M}_f$ there corresponds a number $\lambda_F = \lambda_F(O)$, the Morse index of O with respect to F. We are interested in the set $\Lambda_f = \{\lambda_F \mid F \in \mathscr{M}_f\} \subset [d+1] = \{0, \ldots, d\}$.

Sometimes Λ_f is just a singleton. For example, if $\operatorname{grad} f$ points inside the ball at a point of global maximum of the restriction of f to S^{d-1}, then O is the global maximum for any $F \in \mathscr{M}_f$ hence $\Lambda_f = \{n\}$. Another simple observation is that all the indices in Λ_f are of the same parity. Indeed, the restriction of $\operatorname{grad} F$ to S^{d-1} gives rise to a map $S^{d-1} \to S^{d-1}$, and the degree of this map is equal to $(-1)^{\mu_F+n}$.

The main result we present here is the following

Theorem. *Let $d \geqslant 2$ and $0 \leqslant \lambda \leqslant d-2$. Then there exists a smooth function f in a neighbourhood of S^{d-1} such that $\Lambda_f = \{\lambda, \lambda+2\}$.*

The proof consists of three parts. First, we find a Morse-Barannikov complex C that can be the one of such a function f. Secondly, employing the techniques of Cerf's theory [2], we construct two functions F_1 and F_2 such that $\lambda_{F_1} = \lambda$, $\lambda_{F_2} = \lambda + 2$, and their Morse-Barannikov complexes at S^{d-1} are equal to C. Lastly, we modify F_1 and F_2 in such a way that they agree near S^{d-1}.

If time permits, we will also discuss possible approaches to construct functions f with larger Λ_f and the difficulties that arise on the way. For instance, there are Morse-Barannikov complexes that could correspond to a function f with $\Lambda_f = [d+1] \cap 2\mathbb{Z}$, but the existence of the function itself is obstructed by higher-dimensional spaces of tunnelings between the critical points.

The talk is based on my master's thesis. I am deeply indebted to Gaiane Panina for posing the problem and supervising my research. This research is supported by the Russian Science Foundation under grant 21-11-00040.

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Alexander Mednykh. *Volume of polyhedra in spaces of constant curvature and its application to the knot theory*

We investigate the existence of hyperbolic, spherical or Euclidean structure on a class of polyhedra arising as fundamental sets for geometrical manifolds, orbifolds or cone manifolds. We present trigonometrical identities involving the lengths of singular geodesics and cone angles of such cone-manifolds. Then these identities are used to produce explicit integral formulae for the volume of cone manifolds whose underlying space is the three-dimensional sphere and singular set is a given knot or link.

Frank Morgan. *Isoperimetric Problems and Densities*

2000 years ago Zenodorus essentially proved that the circle is the least-perimeter way to enclose given area, the original isoperimetric theorem in the plane. Since then the topic and its generalizations have remained at the forefront throughout mathematics and its applications, now more than ever, especially since the crucial role of isoperimetric estimates in manifolds with density in Perelman's 2002 proof of the Poincaré Conjecture. We'll discuss some such and other related recent results and open questions, including the Convex Body Isoperimetric Conjecture and optimal tilings of hyperbolic surfaces.

Igor G. Nikolaev. *Aleksandrov's curvature of metric spaces via Cauchy-Schwarz inequality*

Let (M, g) be a Riemannian manifold, let TM denote its tangent bundle and $\xi = (x, v)\,, \zeta = (x + dx, v + dv) \in TM$. Translate the vector $v + dv$ (Levi-Civita) parallelly to the point x and denote the resulting vector by v'. Let $d\theta$ be the angle between v and v' at the point x. Then the Sasaki metric $d\sigma^2$ (introduced by Shigeo Sasaki in 1958) is defined by

$$d\sigma^2 = ds^2 + |v|^2\, d\theta^2.$$

In K-spaces $\mathbb{S}_K$ of constant curvature K, the formula for $d\theta$ is not difficult to derive in terms of the six distances between the "tails" and "heads" in M (via exp) of the vectors v and $v + dv$:

$$\begin{aligned}&\operatorname{cosq}_K(v, v + dv) = \cos(d\theta)\\ &= \frac{\cos\widehat{\kappa}b + \cos\widehat{\kappa}x\cos\widehat{\kappa}y}{\sin\widehat{\kappa}x\sin\widehat{\kappa}y} - \frac{(\cos\widehat{\kappa}x + \cos\widehat{\kappa}d)(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f)}{(1 + \cos\widehat{\kappa}a)\sin\widehat{\kappa}x\sin\widehat{\kappa}y},\end{aligned}$$

where $\widehat{\kappa} = \sqrt{K}$ if $K > 0$, $\widehat{\kappa} = i\sqrt{-K}$ if $K < 0$ and obvious adjustments should be made for $K = 0$. Here, $x = |v|\,, y = |v + dv|\,$, a and b are the distances between "tails" and "heads", f is the distance from the "head" of v to the "tail" of $v + dv$ and g is the distance from the "head" of $v + dv$ to the "tail" of v (assuming that v, dx and dv are sufficiently small when $K > 0$). In $\mathbb{S}_K$, the inequality $|\operatorname{cosq}_K(v, v + dv)| = |\cos d\theta| \leqslant 1$ obviously holds, which is equivalent to the Cauchy-Schwarz inequality. In a general metric space, $|\operatorname{cosq}_K|$ can be greater than one and even arbitrarily large. We consider the Cauchy-Schwarz inequality K-property in a general metrics space in a weaker form: always $\operatorname{cosq}_K \leqslant 1$ or always $\operatorname{cosq}_K \geqslant -1$. In previous papers, we used the notion of cosq to construct the Sasaki semimetric, which generates the "extended" Sasaki metric, to give a solution of Aleksandrov's problem on metric characterization of C^m-Riemannian manifolds for $m = 2, 3, \ldots$. In a series of joint papers with I.D. Berg, we studied the relation between the Cauchy-Schwarz inequality property and curvature of geodesically connected metric spaces. We prove that a geodesically connected metric space (of sufficiently small diameter if $K > 0$) is Aleksandrov's space of curvature $\leqslant K$ if and only if the Cauchy-Schwarz inequality K-property holds. In addition, we provide some rigidity theorems. We derive from our results a complete solution of Gromov's curvature problem in the context of Aleksandrov spaces of curvature bounded above.

Yurii Nikonorov and Yulia Nikonorova. *On one property of a planar curve whose convex hull covers a given convex figure*

This talk is based on the paper [3], which is devoted to the proof of the following conjecture by A. Akopyan and V. Vysotsky:

Conjecture ([1]). *Let γ be a curve such that its convex hull covers a planar convex figure K. Then* $\operatorname{length}(\gamma) \geqslant \operatorname{per}(K) - \operatorname{diam}(K)$.

It should be noted that this conjecture was confirmed earlier in the case when γ is passing through all extreme points of K (see Theorem 7 in [1]). The proof for the general case was obtained in [3].

We identify the Euclidean plane with $\mathbb{R}^2$ supplied with the standard Euclidean metric d, where $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$. For any subset $A \subset \mathbb{R}^2$, $\operatorname{co}(A)$ means the convex hull of A. For every points $B, C \in \mathbb{R}^2$, $[B,C]$ denotes the line segment between these points.

A convex (planar) figure is any compact convex subset of $\mathbb{R}^2$. We shall denote by $\operatorname{per}(K)$ and $\operatorname{bd}(K)$ respectively the perimeter and the boundary of a convex figure K. Note that the perimeter of any line segment (i.e. a degenerate convex figure) is assumed to be equal to its double length. Note also that the diameter $\operatorname{diam}(K) := \max\{d(x,y)\,|\,x,y \in K\}$ of a convex figure K coincides with the maximal distance between two parallel support lines of K. Recall that an extreme point of K is a point in K which does not lie in any open line segment joining two points of K. The set of extreme points of K will be denoted by $\operatorname{ext}(K)$. It is well-known that $\operatorname{ext}(K)$ is closed and $K = \operatorname{co}(\operatorname{ext}(K))$ for any convex figure $K \subset \mathbb{R}^2$.

A planar curve γ is the image of a continuous mapping $\varphi : [a,b] \subset \mathbb{R} \mapsto \mathbb{R}^2$. From now on we will call planar curves simply *curves* for brevity, since no other curves are considered in this talk. As usually, the length of γ is defined as $\operatorname{length}(\gamma) := \sup\{\sum_{i=1}^m d(\varphi(t_{i-1}),\varphi(t_i))\}$, where the supremum is taken over all finite increasing sequences $a = i_0 < i_1 < \cdots < i_{m-1} < i_m = b$ that lie in the interval $[a,b]$. A curve γ is called *rectifiable* if $\operatorname{length}(\gamma) < \infty$.

We call a curve $\gamma \subset \mathbb{R}^2$ *convex* (*closed convex*) if it is a closed connected subset of the boundary (respectively, the whole boundary) of the convex hull $\operatorname{co}(\gamma)$ of γ.

Let us consider the following

Example. *Suppose that the boundary* $\operatorname{bd}(K)$ *of a convex figure* K *is the union of a line segment* $[A,B]$ *and a convex curve* γ *with the endpoints* A *and* B. *Then* $K \subset \operatorname{co}(\gamma)$ *and* $\operatorname{length}(\gamma) = \operatorname{per}(K) - d(A,B)$. *Moreover,* $\operatorname{length}(\gamma) = \operatorname{per}(K) - \operatorname{diam}(K)$ *if and only if* $d(A,B) = \operatorname{diam}(K)$.

The main result is the following

Theorem ([3]). *For a given convex figure* K *and for any planar curve* γ *with the property* $K \subset \operatorname{co}(\gamma)$, *the inequality*

$$\operatorname{length}(\gamma) \geqslant \operatorname{per}(K) - \operatorname{diam}(K) \tag{1}$$

holds. Moreover, this inequality becomes an equality if and only if γ *is a convex curve,* $\operatorname{bd}(K) = \gamma \cup [A,B]$, *and* $\operatorname{diam}(K) = d(A,B)$, *where* A *and* B *are the endpoints of* γ.

Since obviously $\operatorname{per}(K) \geqslant 2\operatorname{diam}(K)$, the inequality (1) immediately implies the following widely known inequality: $\operatorname{length}(\gamma) \geqslant \frac{1}{2}\operatorname{per}(K)$, see e.g. [2].

The strategy of the proof of the above theorem in [3] is as follows. We fix a convex figure $K \subset \mathbb{R}^2$. Then we prove the existence of a curve γ_0 of minimal length among all curves γ satisfying the condition $K \subset \operatorname{co}(\gamma)$. After that we prove the inequality $\operatorname{length}(\gamma_0) \geqslant \operatorname{per}(K) - \operatorname{diam}(K)$ and study all possible cases of the equality $\operatorname{length}(\gamma_0) = \operatorname{per}(K) - \operatorname{diam}(K)$, where γ_0 is an arbitrary curve of minimal length among all curves γ satisfying the condition $K \subset \operatorname{co}(\gamma)$.

It should be noted also the following well-known result (it could be proved using the Crofton formula, see e.g. pp. 594–595 in [1]), which is an important tool in the proof of the main result:

Proposition. *Let* $\varphi : [c,d] \to \mathbb{R}$ *be a parametric continuous curve with* $\varphi(c) = \varphi(d)$. *Then the length of the curve* $\gamma = \{\varphi(t)\,|\,t \in [c,d]\}$ *is greater or equal to* $\operatorname{per}(\operatorname{co}(\gamma))$. *Moreover, the equality holds if and only if* γ *is closed convex curve.*

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Dmitry Oskorbin and Eugene Rodionov. *Conformal Killing vector fields on two-symmetric Lorentzian manifolds*

In this work we study the conformal Killing equation on 2-symmetric Lorentzian manifolds. The Lorentzian manifolds we consider in this article are the generalized Cahen − Wallach spaces, which are convinient to use because of the coordinate system they have. Using this coordinates we describe the general solution of the conformal Killing equation on locally indecomposable 2-symmetric Lorentzian manifolds, which are generalized Cahen − Wallach spaces, as was proved by A.S. Galaev and D.V. Alekseevsky [1, 2]. Finally, we give explicit description of all possible dimensions of the algebra of conformal Killing fields on 2-symmetric Lorentzian manifolds of small dimensions.

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Gaiane Panina. *Colored Tverberg theorems for prime powers*

By a joint work with Duško Jojić and Rade T. Živaljević

The colored Tverberg theorem of Blagojeviȁ, Matschke, and Ziegler provides optimal bounds for the colored Tverberg problem, under the condition that the number of intersecting rainbow simplices is a prime number.

One of our principal new ideas is to replace the ambient simplex Δ, used in the original Tverberg theorem, by an "abridged simplex" of smaller dimension, and to compensate for this reduction by allowing vertices to repeatedly appear a controlled number of times in different rainbow simplices.

Following this strategy we obtain an extension of the afore mentioned result to an optimal colored Tverberg theorem for multisets of colored points, which is valid for each prime power $r = p^k$, and reduces to the result for $k = 1$.

Configuration spaces, used in the proof, are combinatorial pseudomanifolds which can be represented as multiple chessboard complexes. Our main topological tool is the Eilenberg–Krasnoselskii theory of degrees of equivariant maps for non-free actions.

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Conrad Plaut. *Discrete Methods in Geometry and Topology*

My interest in discrete methods arose when I was a graduate student and postdoc around 1990. This interest was connected with two problems: a purely metric proof of Toponogov's Theorem (i.e. local Alexandrov comparisons to global ones) and embedding local groups in global groups inspired by a paper of Fukaya that my advisor asked me to read. The first project was successful ([5,6,8]) and the second was not, at least initially. (Although it did lead me to discover ([7]) an error in Jacoby's 1957 proof ([3]) of the local version of Hilbert's Fifth Problem, which was then correctly solved more than 15 years later by Goldbring ([4]).) Eventually, working with Valera Berestovskii, the second problem evolved ([1,9,2]) into discrete homotopy theory, which produces covering spaces that are "universal covering maps at a given scale". These methods can be used to produce generalized universal covers for exotic spaces ([1,2,12]) as well as finiteness theorems for Riemannian manifolds ([10]) and spectra related to the length spectrum ([11]). After giving an overview of these ideas, I will introduce "weakly chained" spaces ([12]), which can be defined in a single paragraph using only the definition of metric space, and discuss their connection to a conjecture that, in its name, also invokes Toponogov.

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Vladimir Rovenski. *Extrinsic Geometry of Foliations*

Extrinsic geometry deals with properties of submanifolds and leaves of foliations, which depend on the second fundamental form. Although the Riemannian curvature belongs to the intrinsic geometry, a special part called the mixed curvature is also part of the extrinsic geometry of foliations. One of the goals of the talk is to argue that the study of the mixed curvature is a fundamental problem of extrinsic geometry of foliations. At the beginning of the talk, we focus on the concept of mixed curvature related to Toponogov's conjecture (1979) on foliations with positive mixed sectional curvature, see [1, 2, 3].

The mixed scalar curvature (an averaged mixed sectional curvature) is associated with several problems in the extrinsic geometry of foliations: integral formulas and splitting of manifolds; the mixed Einstein–Hilbert action; prescribing the mixed curvature, see survey in [4]. Integral formulas in Riemannian geometry can be viewed as "conservation laws" of quantities when the metric changes. Integral formulas for foliations provide obstructions for existence of foliations or their compact leaves with given geometry. In the second part of the talk, we discuss recent results on integral formulas with the mixed scalar curvature for regular and singular foliations and distributions on Riemannian and metric-affine spaces, see [5, 6, 7, 8, 9, 10].

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Idzhad Sabitov. *Bendable surfaces without visible deformations of their shape*

The bending of a bendable surface is called bending into itself (or bending of sliding) if the trajectory of any its point remains on the surface itself. We consider the bendings of sliding of cylindrical and conical surfaces and prove that they are nontrivial except only the cases when the surfaces are rotational ones.

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I.Kh. Sabitov, *Mathematical Notes, 2021, 110:1, 119 –130.*
DOI https://doi.org/10.4213/mzm13037.
The English version Mathematical Notes, 2021, 110:1, 130 –138.

Vladimir A. Sharafutdinov *Some theorems and conjectures by Victor Toponogov*

The talk is addressed first of all young participants. This is a short survey of Toponogov's results and of their further developments. I am not sure all such developments are covered by the talk. I will be grateful for any addition and comment.

The bibliography list below is the important part of the talk. It is included into the abstract for the audience convenience since I will refer to the list many times during the talk.

Outline

1. Toponogov's comparison theorem
2. Extremal theorems
   - Manifolds of maximal diameter
   - Manifolds of maximal injectivity radius
3. Open manifolds of positive (non-negative) curvature
   - orispaces
   - absence of closed geodesics
   - the estimate of injectivity radius
   - splitting theorem
   - Gromoll – Meyer's theorem
4. CAT manifolds
5. Some problems unsolved by Victor
   - Problem of the Euler characteristic
   - Burov's problem
   - big Cheeger – Gromoll's problem
   - Ferus' theorem
   - the Carathéodory problem

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Wilderich Tuschmann. *Spaces and moduli spaces of nonnegatively curved riemannian metrics*

Consider a smooth manifold with a Riemannian metric satisfying some sort of curvature constraint like, for example, positive scalar curvature, non-negative Ricci or negative sectional curvature, being Einstein, Kähler, Sasaki, etc. A natural question to study is then what the space of all such metrics does look like. Moreover, one can also pose this question for corresponding moduli spaces of metrics, i.e., quotients of the former by (suitable subgroups of) the diffeomorphism group of the manifold, acting by pulling back metrics.

These spaces are customarily equipped with the topology of smooth convergence on compact subsets and the quotient topology, respectively, and their topological properties then provide the right means to measure 'how many' different metrics and geometries the given manifold actually does exhibit; but one can topologize and view those also in very different manners.

In my talk, I will report on some general results and open questions about spaces and moduli spaces of metrics with a focus on non-negative Ricci or sectional curvature as well as other lower curvature bounds on closed and open manifolds, and, in particular, also discuss broader non-traditional approaches from metric geometry and analysis to these objects and topics inspired by Victor Toponogov's work.

Andrei Vesnin. *Right-angled hyperbolic knots and links*

A polyhedron in a hyperbolic space $\mathbb{H}^3$ is said to be *right-angled* if all its dihedral angels equal $\pi/2$, and *ideal* if all its vertices belong to $\partial\mathbb{H}^3$. The simplest ideal right-angled polyhedron in $\mathbb{H}^3$ is a regular octahedron. Let G be a group generated by reflections in faces of an ideal right-angled polyhedron. The method presented in [1] admits to construct orientable and non-orientable hyperbolic 3-manifolds whose fundamental groups are torsion-free subgroups of G of index four.

We will discuss volumes and other properties of ideal right-angled polyhedra and their applications to study of right-angled knots [2] and maximal volume of hyperbolic polyhedra with given combinatorial structure [3].

By [4], if P is an ideal right-angled hyperbolic polyhedron with $V \geqslant 9$ vertices then

$$\mathrm{vol}(P) \leqslant (V-5) \cdot \frac{v_8}{2},$$

and the equality holds if and only if $V = 9$. Here $v_8 = 8\Lambda(\pi/4)$, and

$$\Lambda(x) = -\int_0^x \log|2\sin t|dt$$

is the Lobachevskii function. To fifteen decimal places, v_8 is 3.663862376708876.

A knot or link in the 3-sphere is said to be *right-angled* if its complement is a hyperbolic 3-manifold which can be decomposed into ideal right-angled polyhedra. Among well-known examples of right-angled links are the 2-component Whitehead link and the 3-component Borromean rights. The Whitehead link complement can be glued from one regular octahedron, as well as the Borromean rings complement can be glued from two regular octahedra. In [2] there was formulated the conjecture that there does not exist a right-angled knot. The volumes of ideal right-angled polyhedra with at most 23 faces were enumerated in [5]. This enumeration admits to verify the conjecture for knots with small number of crossings.

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Dmitri Volkov. *Yuri Alexandrovich Volkov and his work on the theory of convex bodies*

In his relatively brief life (he died of cancer before reaching the age of 51), Yuri Alexandrovich Volkov succeeded in doing a great deal in mathematics and geometry. Not only did he leave a deep trace in several branches of geometry, but he also initiated several new concepts and methods, This is an attempt to give a survey of Volkov's remarkable work and its influence for today. Obviously, we could not cover everything. This survey concentrates on two topics: variational methods in theory convex polyhedra and stability theorems in problems of Minkowski and Weyl

Volkov Yuri Alexandrovich, was born on September 2, 1930 in the city of Kazan. His parents were Nina Antonovna Volkova, a doctor, and Mitrofanov Alexander Ivanovich, an agronomist. He graduated from a secondary school in Kazan in 1947 and entered the Department of Mathematics and Mechanics of Leningrad University where he was an outstanding student. He chose geometry as his speciality, his supervisor being A.D.Alexandrov. He completed his undergraduate course in 1952. He completed PhD thesis at Leningrad State University under the supervision of A. D. Alexandrov and successfully defended it in 1955. In his thesis, Volkov used the variational method to prove the existence of a polyhedron with a pre-specified development (see [1, 2]). Unfortunately, thesis defence rules at the time did not require the results described in the thesis to be published, and as such Volkov's proof was only known to the participants of Alexandrov's geometry seminar.

The main idea behind the proof is very simple. It is clear that any convex polyhedron can be represented as a partion in tetrahedra with a common apex at one of the vertices of the polyhedron. Conversely, if certain conditions are imposed on the set of tetrahedra, then it can be assembled into a convex polyhedron. From the pre-specified surface net(development), we will construct a solid net satisfying all of these conditions, except maybe one: the angles at each internal edge, the sum of the dihedral angles of the tetrahedra meeting at said edge, will be no less than 2π, the quantity necessary for gluing to be possible. It will turn out, however, that if the angle at even one edge exceeds 2π, then this edge can be shortened, obtaining a net of the same form. This is based on the fact that shortening one edge can only increase the angles at all the other edges, which will thus not go below 2π either. One naturally expects that the net with the shortest total internal edge length will have all angles equal to 2π and will thus be assemblable into the desired polyhedron.

After his thesis defence, Volkov concentrated on the stability problem in Minkowski's theorem, which he solved in 1959 ([4], p. 218). The proof of the existence of a cap with a pre-specified development was only published in 1960 ([6]); Volkov wrote that he would examine the general (convex polyhedron) case in his next article. As such, he changed his approach from the one he used for his PhD thesis (where he presents a proof of the polyhedron case first, then sketches the proof for the cap case).

In the early 1960s, Volkov discovered that his method of abstract polyhedra yielded results in the Weyl–Cohn-Vossen problem (estimating the change in external distances on convex surfaces in response to changes in internal distances). In 1962, he presented his results on the Cohn-Vossen problem in a geometry seminar ([4], p. 223). The proof of Volkov's bounds on the changes in distances did not rely on the proof of A. V. Pogorelov's famous theorem stating that two closed isometric surfaces are equal, so Volkov proved that theorem as well. He had thus obtained by 1962 new proofs of the fundamental theorems of the theory of convex surfaces. But the results were not yet published, and he had to decide what to publish first. It is likely for this reason that Volkov's PhD thesis results were of less importance for him.

Volkov published his results on Minkowski's problem [7] in 1963 and on the Cohn-Vossen problem [8, 9] in 1968 and conducted a brilliant defence of his doctorate thesis. Only in 1968, however, did he suggest using the method of abstract polyhedra to prove S. P. Olovianishnikov's theorem on the existence of an infinite polyhedron [11] to graduate student Yelena Gavrilovnna Podgornova. She defended her PhD thesis, containing a detailed proof of Olovianishnikov's theorem, in 1972. Based on Volkov and Podgornova's research, the article [10] was written, in which Alexandrov and Olovianishnikov's theorems are proved in full detail. This article was meant to be published in LSU's Vestnik, but due to the long waiting times they decided to publish it in the Works of Taskent Pedagogical Institute, where Podgornova worked after finishing her postgraduate studies. Unfortunately, [10] was practically inaccessible to geometry researchers ([12], p. 37). Volkov did not publish any more of his work on the theory of polyhedra in his lifetime. As such, his results were at the time known only to a narrow circle of geometers in the USSR.

The situation only began to improve after the publication of an English translation of Alexandrov's book [3]. V. A. Zalgaller, who organized the entire translation project, included [6, 9] as an addendum, listing [10] only in the references. Unfortunately, in the footnotes ([3], p. 492), it is written that the second article on the polyhedron existence problem was never published. This article is, of course, [10]. Almost at the same time as Alexandrov's book was translated, two new articles appeared [13, 14], in which Volkov's variational approach is also applied to the proof of Alexandrov's theorem ([15], p. 334). Worth noting is that [13] gives an algorithm for constructing a convex polyhedron from its development.

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Bao Vuong. *Volume of some polyhedra in three-dimensional spaces of constant curvature*

After the Mostow-Prasad rigidity theorem [1], each geometrical invariant of a hyperbolic 3-manifold is a topological invariant. The most important geometric invariants are the volume. Spherical, Euclidean, and hyperbolic 3-manifolds can be constructed by gluing together convex polyhedra in $X = \mathbb{S}^3, \mathbb{E}^3$, or $\mathbb{H}^3$ along their faces. The volume of the resulting manifold is the volume of the gluing polyhedra. We consider antiprism and antipodal octahedron in spherical, Euclidean, and hyperbolic spaces. We prove necessary and sufficient conditions for the existence of these polyhedron in each space. The relations between dihedral angles and edge lengths are founded in a form of cosine theorem. Explicit integral volume formulas are obtained for the polyhedra.

This is our joint work with Nicolay Abrosimov.

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