O.A. Ladyzhenskaya centennial conference on PDE's

Book of abstracts

St. Petersburg, 2022

St. Petersburg Steklov
Institute of Mathematics

St. Petersburg Department
of Steklov Mathematical Institute
of Russian Academy of Sciences

O.A. Ladyzhenskaya centennial conference on PDE's

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ABSTRACTS

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Keynote speakers

Dynamics of concentrated vorticities in 2d and 3d Euler flows

Manuel Del Pino

University of Bath, UK

A classical problem that traces back to Helmholtz and Kirchhoff is the understanding of the dynamics of solutions to the Euler equations of an inviscid incompressible fluid when the vorticity of the solution is initially concentrated near isolated points in 2d or vortex lines in 3d. We discuss some recent results on these solutions' existence and asymptotic behavior. We describe, with precise asymptotics, interacting vortices, and traveling helices. We rigorously establish the law of motion of "leapfrogging vortex rings", initially conjectured by Helmholtz in 1858.

Generic regularity in obstacle problems

Alessio Figalli

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The classical obstacle problem consists of finding the equilibrium position of an elastic membrane whose boundary is held fixed and which is constrained to lie above a given obstacle. By classical results of Caffarelli, the free boundary is smooth outside a set of singular points. Explicit examples show that the singular set could be, in general, as large as the regular set. In a recent paper with Ros-Oton and Serra we show that, generically, the singular set has codimension 3 inside the free boundary, solving a conjecture of Schaeffer in dimension $n \leq 4$. The aim of this talk is to give an overview of these results.

References

[1] A. Figalli, X. Ros-Oton and J. Serra, Generic regularity of free boundaries for the obstacle problem, *Publ. Math. Inst. Hautes Etudes Sci.* **132** (2020), 181–292.

Hopf, Caccioppoli and Schauder, reloaded

Guiseppe Rosario Mingione

University of Parma, Italy

So called Schauder estimates are in fact a contribution, at various stages, of Hopf, Caccioppoli and Schauder, between the end of the 20s and the beginning of the 30s. Later on, they were extended, with various degrees of precision, to nonlinear uniformly elliptic equations.

I will present the solution to the longstanding open problems of proving estimates of such kind in the nonuniformly elliptic case and for minima of non-differentiable functionals (again considered in the nonuniformly elliptic case).

From joint work with Cristiana De Filippis.

On a construction of spatially periodic weak solutions to a Stefan problem and nonstandard estimates for the heat equation

Evgeny Yu. Panov

Yaroslav-the-Wise Novgorod State University, Russia

In order to demonstrate that the decay property of spatially periodic weak solutions to nonlinear diffusion equation

$$u_t = g(u)_{xx}, \quad (t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}, \tag{1}$$

may fail after small finite perturbations of initial data, we consider the case of Stefan problem when $g(u) = \max(0,u)$ and construct an even x-periodic solution with period 5, which has the following structure on the fundamental segment $|x| \le 5/2$: $u(t,x) > 0$ in the domain $|x| < r(t) \doteq 2 - e^{-\alpha t}$, $\alpha > 0$, and $u(0,x) = \phi(x) \in C_0^\infty((-1,1))$, $\phi(-x) = \phi(x)$; $u(t,x) = \psi(x) \le 0$ if $r(t) < |x| < 5/2$, and $\psi(x) = 0$ for $2 < |x| < 5/2$. A weak solution $u = u(t,x) \in L^\infty(\Pi)$ of equation (1) satisfies (1) in $\mathcal{D}'(\Pi)$. This implies the following Rankine-Hugoniot type relations on discontinuity curves $x = x(t) = \pm r(t)$:

$$[g(u)] \doteq g(u(t,x(t)+)) - g(u(t,x(t)-)) = 0, \quad -[u]x'(t) - [g(u)_x] = 0. \tag{2}$$

It follows from (2) that $u(t,r(t)-) = 0$ and that $\psi(x)x' = u_x(t,x-)$, $x = r(t)$. The latter relation allows to determine the function $\psi(x) = u_x(t,r(t)-)/r'(t) \le 0$, $|x| = r(t) \in [1,2)$. In the domain $|x| < r(t)$ the function $u(t,x)$ has to be a solution of the following initial boundary value problem for the heat equation

$$u_t = u_{xx}, \quad u(t,\pm r(t)) = 0, \quad u(0,x) = \phi(x).$$

Using the change $y = x/r(t)$ and the classical results [1], we find that this problem has a unique smooth solution $u(t,x)$. We establish a priory estimates, which guarantee the exponential decay of $u(t,\cdot)$ and $u_x(t,\cdot)$ as $t \to +\infty$. We also show that $u_x(t,r(t)) = o(r'(t))$ for sufficiently small $\alpha > 0$. This implies the boundedness of $\psi(x)$ and the correctness of our construction. Notice that $u(t,x) \rightrightarrows 0 = \frac{1}{5}\int_{|x|<5/2} u(0,x)dx$, as $t \to \infty$, which is relevant with general results [2]. The small perturbation $u(0,x) - \varepsilon\chi_{[2,3]}(x)$, $\varepsilon > 0$, of initial data yields the solution equaled $-\varepsilon$ in the half-strip $t > 0$, $x \in [2,3]$, and the decay property is no longer valid.

This work was supported by the RSF grant no. 22-21-00344.

References

[1] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and Quasi-Linear Equations of Parabolic Type*, AMS, Providence, RI, 1968.

[2] E.Yu. Panov, Decay of periodic entropy solutions to degenerate nonlinear parabolic equations, *J. Differential Equations* **269** (2020), no.1, 862–891.

On the radiative transfer equations

Olivier Pironneau

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In continuum mechanics radiative transfer is modelled by an integro-differential system of equations with a time variable, 3 space variables and two angular variables. It is numerically very challenging, especially that applications to climate and combustion require frequency dependent coefficients and coupling with the Navier-Stokes equations. Nevertheless, by using a formulation due to Chandrasekhar, existence and uniqueness can be established as a limit of an increasing sequence with very few constraints on the coefficients. The same property leads to a fast numerical algorithm with a ray-tracing type step and a convolution step accelerated by the H-matrix algorithm. Numerical results will be shown for the earth atmosphere and for the effect of sunlight on Boussinesq's instabilities in a lake.In continuum mechanics radiative transfer is modelled by an integro-differential system of equations with a time variable, 3 space variables and two angular variables. It is numerically very challenging, especially that applications to climate and combustion require frequency dependent coefficients and coupling with the Navier-Stokes equations. Nevertheless, by using a formulation due to Chandrasekhar, existence and uniqueness can be established as a limit of an increasing sequence with very few constraints on the coefficients. The same property leads to a fast numerical algorithm with a ray-tracing type step and a convolution step accelerated by the H-matrix algorithm. Numerical results will be shown for the earth atmosphere and for the effect of sunlight on Boussinesq's instabilities in a lake.

On obstacle problems for non coercive linear operators

José Francisco Rodrigues

University of Lisbon, Portugal

In a joint work with L. Boccardo and G. R. Cirmi, the existence and uniqueness of solution to the one and the two obstacles problems associated with a linear elliptic operator of second order, which is non coercive due to the presence of a general convection term with sharp integrability condition.

We show that the operator is weakly T-monotone and, as a consequence, we establish the Lewy–Stampacchia dual estimates and we study the comparison and the continuous dependence of the solutions as the obstacles vary.

As an application, we prove also the existence of solutions for a class of non coercive implicit obstacle problems.

The singular set in the Stefan problem

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The Stefan problem, dating back to the XIXth century, is probably the most classical and important free boundary problem. The regularity of free boundaries in the Stefan problem was developed in the groundbreaking paper (Caffarelli, Acta Math. 1977). The main result therein establishes that the free boundary is C^∞ in space and time, outside a certain set of singular points.

The fine understanding of singularities is of central importance in a number of areas related to nonlinear PDEs and Geometric Analysis. In particular, a major question in such context is to establish estimates for the size of the singular set. The goal of this talk is to present new results in this direction for the Stefan problem in $\mathbb{R}^3$. This is a joint work with A. Figalli and J. Serra.

What we know, and what we don't about second order elliptic and parabolic equations

Mikhail V. Safonov

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We discuss the general properties of solutions, such as Hölder regularity and Harnack inequalities for second order elliptic and parabolic equations, both in the divergence and non-divergence forms. Note that in the case of bounded coefficients, one can get rid of lower order terms by introducing the additional space variable. However, the ellipticity or parabolicity of the equation obtained by this procedure depends on the bounds for lower order coefficients, so that this method does not work for equations with singular coefficients. Moreover, so-called elliptic type Harnack inequalities in a parabolic cylinder $Q_T := (0,T) \times \Omega$ require Ω to be bounded and solutions to vanish on the whole lateral side $S_T := (0,T) \times \partial\Omega$. By introducing an auxiliary vitiable, the "new" domain Ω becomes unbounded, and additional arguments are needed even in the case of bounded lower order coefficients. We address this issue with special attention to unbounded coefficients by combining results and methods in [1] with techniques of other researchers.

References

[1] O.A. Ladyzhenskaya and N.N. Ural'tseva, A survey of results on the solvability of boundary value problems for uniformly elliptic and parabolic second-order quasilinear equations having unbounded singularities, *Russian Math. Surveys* **41** (1986), no.5, 1–31.

Minimizers of a variational problem for nematic liquid crystals with variable degree of orientation in two dimensions

Itai Shafrir[1], Dmitry Golovaty[2]

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We study the asymptotic behavior, when $k \to \infty$, of the minimizers of the energy

$$G_k(u) = \int_\Omega \Big((k-1)|\nabla |u||^2 + |\nabla u|^2 \Big),$$

over the class of maps $u \in H^1(\Omega, \mathbb{R}^2)$ satisfying the boundary condition $u = g$ on $\partial\Omega$, where Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$ and $g : \partial\Omega \to S^1$. The motivation comes from a simplified version of Ericksen model for nematic liquid crystals. We will present similarities and differences with respect to the analog problem for the Ginzburg-Landau energy.

Homogenization of hyperbolic equations with periodic coefficients

Tatiana A. Suslina

St. Petersburg State University, Russia

The talk is based on joint works with Mark Dorodnyi. I will give a survey of the results [1, 2, 3, 4, 5] on homogenization of hyperbolic equations in $\mathbb{R}^d$.

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a strongly elliptic differential operator (DO) $A_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$. Here $g(\mathbf{x})$ is a bounded and positive definite periodic $(m \times m)$-matrix-valued function; $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$ is an $(m \times n)$-matrix first-order DO. It is assumed that $m \geqslant n$ and the symbol $b(\boldsymbol{\xi})$ has maximal rank. We are interested in the behavior of the operators $\cos(\tau A_\varepsilon^{1/2})$ and $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$.

As $\varepsilon \to 0$, these operators converge in the norm of operators acting from the Sobolev space $H^s(\mathbb{R}^d;\mathbb{C}^n)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$ (with suitable s) to the similar operator-valued functions of the effective operator A^0, the error being of order $O(\varepsilon)$ for a fixed $\tau \in \mathbb{R}$. We will also discuss the possibility to obtain more accurate approximations in the $(H^s \to L_2)$-norm with error of order $O(\varepsilon^2)$, as well as approximations in the $(H^s \to H^1)$-norm with error of order $O(\varepsilon)$. It turns out that such approximations (with appropriate corrector terms) can be obtained for the operators $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$ and $\cos(\tau A_\varepsilon^{1/2})\,(I + \varepsilon K(\varepsilon))$, where $K(\varepsilon) = \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon$ involves a rapidly oscillating coefficient $\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\mathbf{x}/\varepsilon)$ and an auxiliary smoothing operator Π_ε. A special attention is paid to the sharpness of the results. The results are applied to the Cauchy problem for a hyperbolic equation $\partial_\tau^2 \mathbf{u}_\varepsilon = -A_\varepsilon \mathbf{u}_\varepsilon + \mathbf{F}$.

References

[1] M.Sh. Birman and T.A. Suslina, Operator error estimates in the homogenization problem for nonstationary periodic equations, *St. Petersburg Math. J.* **20** (2009), no. 6, 873–928.

[2] Yu.M. Meshkova, On operator error estimates for homogenization of hyperbolic systems with periodic coeffcients, *J. Spectr. Theory* **11** (2021), no. 2, 587–660.

[3] M.A. Dorodnyi and T.A. Suslina, Spectral approach to homogenization of hyperbolic equations with periodic coefficients, *J. Diff. Equ.* **264** (2018), no. 12, 7463–7522.

[4] M.A. Dorodnyi and T.A. Suslina, Homogenization of hyperbolic equations with periodic coefficients in $\mathbb{R}^d$: Sharpness of the results, *St. Petersburg Math. J.* **32** (2021), no. 4, 605–703.

[5] M.A. Dorodnyi and T.A. Suslina, Homogenization of hyperbolic equations with periodic coefficients: Results with correctors, *in preparation.*

Higher order boundary Harnack principle on nodal domains via degenerate equations

Susanna Terracini[1], Giorgio Tortone[2], Stefano Vita[1]

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[2] Università di Pisa, Italy

The ratio v/u of two solutions to a second order elliptic equation in divergence form solves a degenerate elliptic equation if u and v share the zero set; that is, $Z(u) \subseteq Z(v)$. The coefficients of the degenerate equation vanish on the nodal set as u^2. Developing a Schauder theory for such equations, we prove $C^{k,\alpha}$-regularity of the ratio from one side of the regular part of the nodal set in the spirit of the higher order boundary Harnack principle established by De Silva and Savin in [4]. Then, by a gluing lemma, the estimates extend across the regular part of the nodal set. Eventually, using conformal mapping in dimension $n = 2$, we provide local gradient estimates for the ratio which hold also across the singular part of the nodal set and depends on the highest value attained by the Almgren frequency function.

References

[1] S. Terracini, G. Tortone and S. Vita, Higher order boundary Harnack principle on nodal domains via degenerate equations. Preprint, 2020.

[2] Y. Sire, S. Terracini and S. Vita, Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions, *Comm. Partial Differential Equations* **46** (2021), no.2, 310–361.

[3] Y. Sire, S. Terracini and S. Vita, Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions, *Mathematics in Engineering* **3** (2021), no.1, 1–50.

[4] D. De Silva and O. Savin, A note on higher regularity boundary Harnack inequality, *DCDS-A* **35** (2015), no.12, 6155–6163.

High multiplicities interfaces for Allen-Cahn equation

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The relations between Allen-Cahn equation

$$\epsilon^2 \Delta u + u - u^3 = 0 \text{ in } (M, g)$$

and minimal surfaces have been explored intensively in recent years. However few is known in the case of interfaces with high multiplicities which are typically governed by Jacobi-Toda system ([1])

$$\epsilon^2 (\Delta_\Gamma + |A|^2 - Ric) u = e^{-u}$$

In this talk, I will describe surprising new solutions to Jacobi-Toda system with higher multiplicities and **finite Morse index**. Key idea is to use finite ended solutions to desingularize the singular solutions to the Jacobi field:

$$J[h] = \sum_{j=1}^{m} \delta_{p_j}$$

where p_j must satisfy some balancing conditions. Central to this construction is the nondegeneracy of finite-ended solutions of Allen-Cahn (proved in [2]), and the existence of Billiard type solutions to Jacobi field equation with Dirac singularities. We also desingularize the geodesic networks with crossings and compute the Morse index in terms of the Morse index of the each geodesics plus the number of crossings, which is conjectured by Chodosh and Mantoulidis (2021).

References

[1] M. del Pino, M. Kowalczyk, J. Wei and J.Yang, Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature, *Geom. Funct. Anal.* **20** (2010), no.4, 918–957.

[2] Y. Liu and J. Wei, A complete classification of finite Morse index solutions to elliptic sine-Gordon equation in the plane, Revista Matemtica Iberoamericana, to appear.

Invited speakers

Double periodic viscous flows in infinite space-periodic pipes

Hugo Beirão da Veiga[1], Jiaqi Yang[2]

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[2]Northwestern Polytechnical University, China

We study the motion of a viscous incompressible fluid in an $n+1$-dimensional infinite pipe Λ with an L-periodic shape in the $z = x_{n+1}$ direction. Denote by Σ_z the cross section of the pipe at the level z, and by v_z the $(n+1)$-th component of the velocity. We look for fully developed solutions $\mathbf{v}(x, z, t)$ with a given T-time periodic total flux $g(t) = \int_{\Sigma_z} v_z(x, z, t)\, dx$ which should be simultaneously T-periodic with respect to time and L-space-periodic with respect to z. We prove existence and uniqueness to the above problem. The results extend those proved by the first author in [1].

References

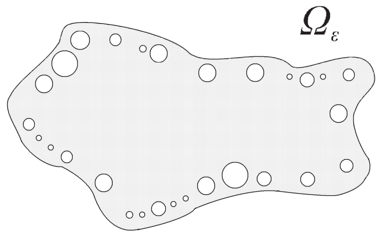
[1] H. Beirão da Veiga, Time-periodic solutions of the Navier–Stokes equations in unbounded cylindrical domains. Leray's problem for periodic flows, *Arch. Ration. Mech. Anal.* **178** (2005), no.3, 301–325.

On Meyers estimates

Gregory A. Chechkin

Lomonosov Moscow State University, Russia

We consider a domain $\Omega_\varepsilon \in \mathbb{R}^n$, $n > 1$, perforated along the boundary (see Figure).



We consider open balls B_j with radius $\frac{1}{2}\varepsilon$, centered in points located on the distance ε from the boundary, ε is a small positive parameter. Denote $B_\varepsilon = \bigcup\limits_{j\in J} B_j$, $J := \{1, 2, \ldots, M\}$. We consider in $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon}$ the following problem:

$$\Delta u_\varepsilon = l \text{ in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \text{ on } \partial B_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\frac{\partial u}{\partial n}$ is an outer normal derivative and l is a linear functional on $W_2^1(\Omega_\varepsilon, \partial B_\varepsilon)$ (the set of functions from $W_2^1(\Omega_\varepsilon)$ with zero trace on ∂B_ε. By the Hahn–Banach theorem $l(\varphi) = -\sum\limits_{i=1}^{n} \int\limits_{\Omega_\varepsilon} f_i \varphi_{x_i}\, dx$, where $f_i \in L_2(\Omega)$. The function u is a solution to problem (1), if $\int\limits_D \nabla u \cdot \nabla\varphi\, dx = \int\limits_{\Omega_\varepsilon} f \cdot \nabla\varphi\, dx$ for any $\varphi \in W_2^1(\Omega_\varepsilon, \partial B_\varepsilon)$.

Let $C_p(K)$, $1 < p < n$, be a p-capacity of a compact $K \subset \mathbb{R}^n$. Denote by $B_r^{x_0}$ an open circle of the radius r, centered in x_0. Assume that $p = 2n/(n+2)$ as $n > 2$ and $p = 3/2$ as $n = 2$. For x_0 from the ε-layer near the boundary containing B_ε, and $r \le r_0$ the inequality

$$C_p(\partial B_\varepsilon \cap \overline{B}_r^{x_0}) \ge c_0 r^{n-p} \tag{2}$$

holds true. The following Theorem is valid.

Theorem. *If $f \in L_{2+\delta_0}(\Omega)$, where $\delta_0 > 0$, then there exist constants $\delta(n, \delta_0) < \delta_0$ and C, such that for solutions to problem* (1)*, the estimate*

$$\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{2+\delta} dx \le C \int\limits_{\Omega_\varepsilon} |f|^{2+\delta}\, dx$$

holds, where C depends only on δ_0, n, and c_0, r_0 from (2).

The work is supported by RSF (project 20-11-20272).

Second-order estimates for solutions to nonlinear elliptic problems

Andrea Cianchi

University of Florence, Italy

Second-order regularity results are established for solutions to elliptic equations and systems with the principal part having Uhlenbeck structure and square integrable right-hand sides. Both local and global estimates are obtained. The latter applies to solutions to homogeneous Dirichlet problems under minimal regularity assumptions on the boundary of the domain. In particular, if the domain is convex, no regularity of its boundary is needed. A key step in the approach is a sharp pointwise inequality for the involved elliptic operator. This talk is based on joint investigations with A. Balci, L. Diening and V. Maz'ya.

Incomplete Cauchy type problem for quasilinear equations with Riemann — Liouville derivatives

Vladimir E. Fedorov[1], Tatyana A. Zakharova[1]

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Consider a fractional differential equation

$$D^{\alpha}z(t) + Az(t) = B\left(D^{\alpha_1}z(t), \dots, D^{\alpha_n}z(t), D^{\alpha-m-r}z(t), \dots, D^{\alpha-1}z(t)\right) \quad (1)$$

with Riemann — Liouville derivatives $D^{\beta}z$ at $\beta > 0$ and Riemann — Liouville integrals $D^{\beta}z$ at $\beta < 0$. Here $m-1 < \alpha \leq m \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha - 1$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{Z}$, $\alpha_l - m_l \neq \alpha - m$, $l = 1, 2, \dots, n$. A linear closed operator $-A$ in a Banach space $\mathcal{Z}$ belongs to the class of operators $\mathcal{A}_{\alpha}(\theta_0, a_0)$, which generate analytic resolving families of operators of linear equation (1) (with $B = 0$) [1]. Nonlinear equations of form (1) with an operator $-A \in \mathcal{A}_{\alpha}(\theta_0, a_0)$ in the linear part and with a nonlinear operator B depending on derivatives such that fractional parts of their orders are equal to the fractional part of α, were investigated in work [1] and others. Obtained abstract results were applied to study of initial boundary value problems for partial differential equations and systems of equations. But the used in [1] condition of the operator B contunuity with respect to the norm in the Banach space $\mathcal{Z}$, as rule, do not allow to use general results for partial differential equations with derivatives in spatial variables in the nonlinear part.

Here we define fractional powers A^{γ} of a continuously invertible operator A, such that $-A \in \mathcal{A}_{\alpha}(\theta_0, a_0)$, and prove the unique solvability of the Cauchy type problem to (1) with a locally Lipschitzian with respect to the graph norm of A^{γ} operator B, $\gamma \in (0, 1)$. This result we apply to the study of an initial boundary value problem for equation with partial derivatives in spatial variables.

Note also that for the Cauchy type problem to equations with several Riemann — Liouville derivatives there are difficulties with matching the derivatives in the vicinity of the initial point. The difficulties are solved in [2] by introducing a notion of the defect of the Cauchy type problem, it is the number of low-order initial conditions, which contains necessarily zero initial data. Here the notion of the defect is used also and the incomplete Cauchy type problem is considered.

The work is supported by the grant of President of the Russian Federation for state support of leading scientific schools, project number NSh-2708.2022.1.1.

References

[1] V.E. Fedorov and A.S. Avilovich, Semilinear fractional-order evolution equations of Sobolev type in the sectorial case, *Complex Variables and Elliptic Equations* **66** (2021), no.6–7, 1108–1121.

[2] V.E. Fedorov and M.M. Turov, The defect of a Cauchy type problem for linear equations with several Riemann — Liouville derivatives, *Siberian Mathematical Journal* **62** (2021), no.5, 925–942.

Error control for problems in Cosserat (micropolar) elasticity theory

Maksim E. Frolov

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A posteriori error estimates for Cosserat (micropolar) elasticity for problems in 2D ([2, 3]) and 3D ([4]) are considered. Majorants are based on the functional approach that guarantees the reliability property regardless of some additional assumptions, for instance, the Galerkin orthogonality (see [5, 6, 7] and the literature cited therein). Error estimates with such type of properties are as important for justification of mathematical methods in computational mechanics as well-known classical results on existence and uniqueness of solutions following from the theory of partial differential equations (see, for example, [1]).

References

[1] O. A. Ladyzhenskaya, *Boundary Value Prolems of Mathematical Physics*, Nauka, Moscow, 1973.

[2] S.I. Repin and M.E. Frolov, Estimates for deviations from exact solutions to plane problems in the Cosserat theory of elasticity, *Probl. Math. Anal.* **62** (2011), 153–161 [transl. *Journal of Mathematical Sciences* **181** (2012), no.2, 281–291].

[3] M.E. Frolov, Functional a posteriori estimates of the error in the solutions of plane problems in Cosserat elasticity theory, *Journal of Applied Mathematics and Mechanics* **78** (2014), no.4, 425–431.

[4] M.E. Frolov, Reliable a Posteriori Error Estimation for Cosserat Elasticity in 3D, *Lobachevskii J. Math.* **42** (2021), 96–103.

[5] P. Neittaanmäki and S. Repin, *Reliable methods for computer simulation. Error control and a posteriori estimates*, Studies in Mathematics and its Applications, Vol. 33, Elsevier, Amsterdam, 2004.

[6] S. Repin, *A posteriori estimates for partial differential equations*, Radon series on computational and applied mathematics, Vol. 4, de Gruyter, Berlin, 2008.

[7] O. Mali, P. Neittaanmäki and S. Repin, *Accuracy verification methods. Theory and algorithms*, Springer, Dordrecht, 2014.

Invariant foliations in Gene network models

Vladimir P. Golubyatnikov

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We consider the following dynamical system as a simple gene network model:

$$\frac{dx_1}{dt} = L_1(x_3) - k_1x_1; \quad \frac{dx_2}{dt} = L_2(x_1) - k_2x_2; \quad \frac{dx_3}{dt} = L_3(x_2) - k_3x_3. \qquad (1)$$

The constant coefficients k_j are positive, the step functions L_j defined by

$$L_j(w) = k_ja_j > 0 \quad \text{for } 0 \le w < 1; \;\; L_j(w) = 0 \quad \text{for } 1 \le w; \;\; a_j > 1,$$

describe negative feedbacks in the gene network. Here and below, $j = 1, 2, 3$, and all the variables are non-negative, they denote concentrations of components of the gene network. See [1] for some interpretations of similar dynamical systems. It is easy to show that the parallelepiped $Q = [0, a_1] \times [0, a_2] \times [0, a_3]$ is positively invariant with respect to trajectories of (1). Let $E = (1, 1, 1)$ be the point of discontinuities of the functions L_j. The planes $x_j = 1$ decompose Q to 8 blocks. Let $\{000\} \subset Q$, respectively $\{111\} \subset Q$, be defined by inequalities $0 < x_j < 1$, respectively $a_j \ge x_j \ge 1$, and let $W := Q \setminus (\{000\} \cup \{111\})$. Then W is positively invariant domain of the system (1) as well.

It was established in [2] that if $a_j > 1$ then W contains a cycle $\mathcal{C}$ of this system. This cycle is unique and stable.

Following the ideas of proof of the Grobman-Hartman theorem, see [3], we show now that the domain W is foliated to (non-smooth) surfaces which are invariant with respect to shifts along the trajectories of the system (1) in both directions, and that each of these invariant surfaces contains the cycle $\mathcal{C}$. One of these surfaces contains the point E.

Similar constructions and results can be reproduced for higher-dimensional dynamical systems of gene networks modeling; for example, this can be done for the 4D system considered in [4], which generalizes one of the systems in [1].

References

[1] L. Glass and J.C. Pasternack, Stable oscillations in mathematcal models of biological control systems, *J. of Mathematical Biology* **6** (1978), 207–223.

[2] V.P. Golubyatnikov and V.V. Ivanov, Uniqueness and stability of a cycle in three-dimensional block-linear circular gene network models, *Siberian Journal of Pure and Applied Mathematics* **18** (2018), no.4, 19–28.

[3] P. Hartman, *Ordinary Differential Equations*, J.Wiley & Sons Inc. New York-London, 1964.

[4] V.P. Golubyatnikov and L.S. Minushkina, On uniqueness and stability of a cycle in one gene network, *Siberian Electronic Mathematical Reports* **18** (2021), no.1, 464–473.

The nonlinear generalized Rayleigh quotient method and its applications

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A concept of the nonlinear generalized Rayleigh quotient [1, 2] is presented which is a new tool in nonlinear analysis allowing us to find branches of solutions to stationary equations, as well as to investigate their stability for the corresponding nonstationary problems.

The method is exhibited in the examples of the finding solutions with prescribed energy, the fundamental frequency solutions [3] and proving the orbital stability of ground states for the non-linear Schrödinger equation.

References

[1] Y.Sh. Ilyasov, On extreme values of Nehari manifold method via nonlinear Rayleigh's quotient, *Topological Methods in Nonlinear Analysis* **49** (2017), no.2, 683–714.

[2] Y. Il'yasov, Rayleigh quotients of the level set manifolds related to the nonlinear, *Minimax Theory and its Applications* **07** (2022), no.2, 277–302.

[3] Y. Il'yasov, Fundamental Frequency Solutions with Prescribed Action Value to Nonlinear Schrödinger Equations, *JMS* **259** (2021), no. 2, 187–204.

Blow up of solutions of nonliinear parabolic and wave equations with positive initial energy

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The talk will be devoted to the problem of blow up in a finite time of solutions with positive initial energy of initial boundary value problems for nonlinear parabolic amd wave equations. We are going to discuss also the problems of boundedness of non blowing up solutions and preventing blow up of solutions of nonlinear parabolic and wave equations by convective and damping terms.

The subharmonic bifurcation of Stokes waves with vorticity

Vladimir A. Kozlov

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My talk concerns 2D theory of steady water waves. There are two types of waves (Stokes and solitary) studied sufficiently good. The first one is a periodic wave with exactly one crest and one through on the period. The solitary wave is described by an even function having the same limit at $\pm\infty$ and monotone to both sides from its maximum. The proof of existence of small and large water waves together with long standing Stokes conjecture about waves of highest amplitude was a result of deep study of water waves during last century. In 1980th new type of waves were discovered numerically. They are periodic and have several crests of different height on the period. They appear as a result of bifurcation when Stokes waves approach a wave of highest amplitude. Their existence was proved in the irrotational case for the channel of infinite depth by Buffoni, Dancer and Toland in 2000.

In this talk I will present an existence result of subharmonic bifurcations of Stokes waves with vorticity in a channel of finite depth.

Mathematical modeling of plasma confinement in the helically corrugated magnetic field

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Novel concept of the longitudinal plasma flow suppression by active plasma pumping in helicoidal magnetic field in mirror traps was proposed recently. Concept exploration device SMOLA is now being constructed to prove the possibility of the suppression and determine basic scalings of ist effectiveness [1]. New mathematical model of plasma confinement in the helically corrugated magnetic field are described in the article. The calculated density distribution in the longitudinal cross-section and particle flow showing plasma pinching are obtained. Suppression effectiveness drops significantly to the magnetic axis compared to the confinement in the peripheral region and discharge always contracts. Radial and axial transport was described in [2]. The suppression length was calculated in [3] in assumption of the negligible diffusion.

References

[1] A. V. Sudnikov, et al., Plasma flow suppression by the linear helical mirror system, *J. Plasma Phys.* **88** (2022), no.1.

[2] A. D. Beklemishev, Radial and Axial Transport in Trap Sections with Helical Corrugation, *AIP Conference Proceedings* **1771** (2016), no.040006.

[3] A. V. Sudnikov, et al., SMOLA device for helical mirror concept exploration, *Fusion Eng. Design.* **122** (2017), no.85.

Higher order Hardy-type inequalities and higher order fractional Laplacians

Roberta Musina

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We discuss some Hardy and trace-Hardy type inequalities involving weighted polyharmonic operators, and their relations with higher order fractional Laplacians.

This is a joint work with Gabriele Cora, Università di Torino.

References

[1] The s-polyharmonic extension problem and higher-order fractional Laplacians, J. Functional Analysis, (2021), to appear. Preprint ArXiv:2106.11669

Threshold resonances in spectra of waveguides of different physical nature

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The threshold resonances at the lower bound λ_{tr} of the continuous spectrum of a quantum waveguide Ω occurs in the case when the corresponding Dirichlet problem for the Helmholtz equation in Ω with the spectral parameter $\lambda = \lambda_{tr}$ has a bounded solution, see [1]; notice that such problem always has a solution with a linear growth at infinity while this solution becomes bounded only for particular occasianal shapes. Such resonances lead to miscellaneous near-threshold anomalies, cf. [2], namely Wood's and Weinstein's ones, eigenvalues detaching from the cutoff point λ^* etc. This definition [1] does not work for thresholds inside the continuous spectrum as well as for vector problems and for higher-order differential equations, in particular, for periodic waveguides. An improved definition [3] will be given which serves for general elliptic boundary-value problems and new phenomena due to threshold resonances will be demonstrated in water-wave problems, elasticity and for Kirchhoff plates. The complete description of the threshold structure at $\lambda_{tr} = 0$ will be presented in the Neumann problem for a formally self-adjoint elliptic systems.

An unexpected effect of uplifting an eigenvalue from the null threshold of the elastic waveguides (a system of two or three differential equations) and semi-infinite two-dimensional Kirchhoff plate (bi-harmonic operator).

References

[1] S. Molchanov and B. Vainberg, Scattering solutions in networks of thin fibers: small diameter asymptotics, *Comm. Math. Phys.* **273** (2007), 1—31.

[2] S. A. Nazarov, Anomalies of scattering of acoustic waves near cutoff points of the continuous spectrum (a review), *Acoustic journal* **66** (2020), 477—494.

[3] S. A. Nazarov, Threshold resonances and virtual levels in the spectrum of cylindrical and periodic waveguides, *Math. Izvestiya* **84** (2020), 1105–1180.

[4] S. A. Nazarov, Constructing a trapped mode at low frequencies in an elastic waveguide, *Funct. Anal. Appl.* **54** (2020), 31–44.

Modeling and numerical analysis of surface fluids

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In this talk we focus on numerical analysis for systems of PDEs governing the motion of material viscous surfaces, the topic motivated by continuum-based modeling of lateral organization in plasma membranes. We shall consider several systems of fluid and phase-field equations defined on evolving surfaces and discuss some recent results about well-posedness of such problems. We further introduce a computational approach and numerical analysis for the resulting systems of PDEs. This dilivers a computationally tractable and thermodynamically consistent model describing the dynamics of a multi-component thin viscous layer.

Quasilinear elliptic equations with Morrey data

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We will discuss regularity issues regarding general quasilinear divergence-form coercive equations

$$\operatorname{div}\big(\mathbf{a}(x,u,Du)\big) = b(x,u,Du),$$

whose prototype is the m-Laplacian equation. The nonlinear terms are given by Carathéodory functions and satisfy controlled growth structure conditions in u and Du, while their behaviour with respect to x is modeled in Morrey spaces. The fairly non-smooth boundary of the underlying domain is supposed to support a capacity density condition that allows domains with exterior corkscrew property.

Global boundedness and Hölder continuity up to the boundary will be shown for the weak solutions of such equations, generalizing this way the classical L^p-results of Ladyzhenskaya and Ural'tseva ([1, 2, 3, 1]) to the settings of the Morrey spaces.

Some applications to nonlinear Calderón–Zygmund theory in Morrey spaces, and generalizations to componentwise coercive systems, will be discussed as well.

The results presented ([5]) are obtained in collaboration with S.-S. Byun (Seoul) and P. Shin (Suwon).

References

[1] O.A. Ladyzhenskaya and N.N. Ural'tseva, Quasilinear elliptic equations and variational problems in several independent variables, *Russian Math. Surveys* **16** (1961), no. 1, 17–91.

[2] O.A. Ladyzhenskaya and N.N. Ural'tseva, On the Hölder continuity of the solutions and the derivatives of linear and quasi-linear equations of elliptic and parabolic types, *Trudy Mat. Inst. Steklov* **73** (1964), 172–220.

[3] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Equations of Elliptic Type,* 2nd Edition revised, Nauka, Moscow, 1973.

[4] O.A. Ladyzhenskaya and N.N. Ural'tseva, A survey of results on the solvability of boundary value problems for uniformly elliptic and parabolic second-order quasilinear equations having unbounded singularities, *Russian Math. Surveys* **41** (1986), no.5, 1–31.

[5] S.-S. Byun, D.K. Palagachev and P. Shin, Global Hölder continuity of solutions to quasilinear equations with Morrey data, *Commun. Contemp. Math.* (2022), 2150062.

Non-Newtonian flows in thin tube structures

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Thin tube structures are finite unions of thin cylinders depending on the small parameter, ratio of the diameter of the cross section to the length of the cylinder. Flows in such domains model blood flow in a network of vessels. The asymptotic expansion of the solution of the steady Stokes and Navier-Stokes equations in these domains with no slip boundary condition was constructed in the papers [1], [2], and the book [3]. However, the blood exhibits a non-Newtonian rheology, when the viscosity depends on the strain rate. In the present talk we consider such rheology. Applying the Banach fixed point theorem we prove the existence and uniqueness of a solution and its regularity. An asymptotic approximation is constructed and justified by an error estimate. The results are published in the set of three papers [4], [5], [6]. It is a joint work with K.Pileckas and B.Vernescu.

References

[1] G. Panasenko, Asymptotic expansion of the solution of Navier-Stokes equation in a tube structure, *C.R. Acad. Sci. Paris, Série IIb* **326** (1998), 867–872.

[2] F. Blanc, O. Gipouloux, G. Panasenko and A.M. Zine, Asymptotic analysis and partial asymptotic decomposition of the domain for Stokes Equation in tube structure, *Mathematical Models and Methods in Applied Sciences* **9** (1999), no.9, 1351–1378.

[3] G. Panasenko, *Multi-Scale Modelling for Structures and Composites,* Springer, Dordrecht, 2005.

[4] G. Panasenko, K. Pileckas and B. Vernescu, Steady state non-Newtonian flow in thin tube structure: equation on the graph, *Algebra i Analiz* **33** (2021), no.2, 197–214.

[5] G. Panasenko, K. Pileckas and B. Vernescu, Steady state non-Newtonian flow with strain dependent viscosity in domains with cylindrical outlets at infinity, *Nonlinear Analysis: Modelling and control* **26** (2021), no.6, 1166–1199.

[6] G. Panasenko, K. Pileckas and B. Vernescu, Steady state non-Newtonian flow with strain rate dependent viscosity in thin tube structure with no slip boundary condition, *Mathematical Modeling of Natural Phenomena,* to appear, 2022.

Variational approach in homogenization of convolution type operators

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The talk will focus on homogenization of functionals of the form

$$F^{\varepsilon}(u) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)^2 dxdy$$

with a small positive parameter ε and periodic or random statistically homogeneous coefficients μ.

We assume that the function $a(z)$ is non-negative, even and satisfies the following moment conditions:

$$\int_{\mathbb{R}^d} a(z)|z|^k dz < \infty \quad \text{for } k = 0,\, 1,\, 2.$$

It is also assumed that $\mu(x,y) = \mu(y,x)$ for all x and y, and $0 < \mu_- \le \mu(x,y) \le \mu^+$ for some constants μ_- and μ^+.

We consider both the case of periodic coefficient $\mu(x,y)$ and the case of random stationary ergodic function $\mu(x,y)$. In both cases the family $\{F^{\varepsilon}\}$ Γ-converges, as $\varepsilon \to 0$, to a local functional that reads

$$F^{\hom}(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} a^{\hom}\nabla u(x)\cdot\nabla u(x)dx, & \text{if } u \in H^1(\mathbb{R}^d), \\ +\infty & \text{otherwise;} \end{cases}$$

here $a^{\hom}$ is a constant positive definite matrix.

We also consider the functionals defined in periodically and randomly perforated domains as well as functionals of the form

$$F^{\varepsilon}(u) = \frac{1}{\varepsilon^{d+p}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\big|u(y)-u(x)\big|^p dxdy$$

with $p > 1$.

As a consequence of these results we obtain the homogenization results for linear and nonlinear non-local convolution type operators.

References

[1] A. Braides and A. Piatnitski, Homogenization of random convolution energies, *J. London Math. Soc.*, **104** (2021), 295–319.

[2] A. Braides and A. Piatnitski, Homogenization of quadratic convolution energies in periodically perforated domains, *Advances in Calc. of Variations*, https://doi.org/10.1515/acv-2019-0083.

Error identities for a class of parabolic initial boundary value problems

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We consider a class of initial boundary value problems with linear and nonlinear spatial part. In particular, this class includes problems, which spatial part is presented by the α–Laplacian operator. Correctness of the problem and solution properties are well studied in [2] and many subsequent publications. Our goal is to deduce special (a posteriori) identities, which left hand sides is a measure of the distance between the exact solution u and any function v in the corresponding admissible (energy) class and the right hand side is a functional that depends on v and problem data. Identities of such a type serve as a source of fully computable bounds of computational and modeling errors [2, 3]. Identities derived for the class of parabolic problems possess and important consistency property and tend to zero under the conditions typical for approximations constructed by commonly used numerical methods. Therefore, the identities and corresponding error estimates can be applied for quantitative analysis of direct and inverse problems associated with parabolic equations.

References

[1] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York-London, 1968.

[2] S. Repin, A Posteriori Estimates for Partial Differential Equations, *Walter deGruyter, Berlin* 2008.

[3] S. Repin, S. Sauter, *Accuracy of Mathematical Models. Dimension Reduction, Simplification, and Homogenization*, EMS Tracts in Mathematics, 33, 2020.

Basic elliptic estimates with optimized constants and applications to the qualitative theory of elliptic PDE

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We develop a new, unified approach to the following two classical questions on elliptic PDE: (i) the strong maximum principle for equations with non-Lipschitz nonlinearities, and (ii) the at most exponential decay of solutions in the whole space or exterior domains (Landis conjecture). Our results apply to divergence and nondivergence operators with locally unbounded lower-order coefficients, in a number of situations where all previous results required bounded ingredients. Our approach, which allows for relatively simple and short proofs, is based on a (weak) Harnack inequality with optimal dependence of the constants in the lower-order terms of the equation and the size of the domain, which we establish. If time permits, we will report on some recent C1 estimates with optimized constants and refined Landis-type results. These are based on a new boundary weak Harnack inequality which also has applications in the boundary regularity theory of equations in divergence form.

References

[1] B. Sirakov, P. Souplet, The Vázquez maximum principle and the Landis conjecture for elliptic PDE with unbounded coefficients, *Adv. Math.* **387** (2021), 107838.

Lavrentiev phenomenon in partial Sobolev spaces of differential forms

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We extend previous examples (see [1]–[4]) for non-density of smooth functions in Sobolev-Orlicz spaces. Let Ω be a ball in $\mathbb{R}^n$, $n \geq 2$, and $\Phi : \Omega \times [0, \infty) \to [0, \infty)$ be a function defined by $\Phi(x, t) = \varphi(t) + a(x)\psi(t)$ where Orlicz functions $\varphi \lesssim \psi$ have growth of the order $\varphi(t) \sim t^p \ln^{-\beta} t$ and $\psi(t) \sim t^p \ln^{\alpha} t$ as $t \to \infty$. Here $p > 1$, $\alpha, \beta \in \mathbb{R}$, and $0 \leq a \in L^\infty(\Omega)$. This corresponds to "borderline" cases in double phase problems [5].

By $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k(\mathbb{R}^n))$, $k = 0, \ldots, n-1$, we denote the partial Sobolev-Orlicz space of differential k-forms $u \in L^1(\Omega, \Lambda^k(\mathbb{R}^n))$ with (weak) differential du in the Lebesgue-Orlicz space $L^{\Phi(\cdot)}(\Omega, \Lambda^{k+1}(\mathbb{R}^n))$.

Using the standard Friedrichs mollification one can approximate any form from $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k(\mathbb{R}^n))$ by smooth forms provided that the weight a satisfies $|a(x) - a(y)| \leq C \ln^{-\alpha-\beta}(e + |x-y|^{-1})$.

We show that if $\alpha + \beta > p$ then smooth k-forms, $k = 0, \ldots, n-2$, are not generally dense in $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k(\mathbb{R}^n))$. If $p = n/k$ we additionally require that $\alpha > p - 1$ or $\beta > 1$. We define a weight $a = a(x)$ and a nontrivial linear functional on $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k(\mathbb{R}^n))$ vanishing on smooth functions. The construction is based on a Cantor type "singular contact set".

Using this, we demonstrate the Lavrentiev phenomenon for corresponding integral functionals. That is, cohomological minimizers over the natural energy space and over its subspace obtained by closure of the set of smooth forms can be different together with corresponding energies.

The results are new even for the scalar case $k = 0$.

References

[1] V.V. Zhikov, On Lavrentiev's phenomenon, *Russ. J. Math. Phys.* **3** (1995), no.2, 249–269.

[2] I. Fonseca, J. Malý, G. Mingione, Scalar minimizers with fractal singular sets, *Arch. Ration. Mech. Anal.* **172** (2004), no.2, 295–307.

[3] A.K. Balci, L. Diening, M. Surnachev, New examples on lavrentiev gap using fractals, *Calc. Var. Partial Differential Equations* **59** (2020), no.5, Paper No. 180, 34 pp.

[4] A.K. Balci, M. Surnachev, Lavrentiev gap for some classes of generalized Orlicz functions, *Nonlinear Analysis* **207** (2021), Paper No. 112329, 22pp.

[5] P. Baroni, M. Colombo, G. Mingione, Nonautonomous functionals, borderline cases and related function classes, *Algebra Anal.* **27** (2015), no.3, 6–50.

Some methods for solving equations with an operator function

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Several applied problems are characterized by the need to numerically solve equations with an operator function (matrix function) [1]. In particular, in the last decade, mathematical models with a fractional power of an elliptic operator and numerical methods for their study have been actively discussed. Computational algorithms for such non-standard problems are based on approximations by the operator function. In this case, an approximate solution is determined by solving auxiliary standard problems. The paper [2] discusses the main directions for constructing acceptable approximations of operator functions when solving equations.

The most widespread are the approaches using various options for rational approximation. Also, we note the methods that relate to approximation by exponential sums. We propose to use a new approach, which is based on the application of approximation by exponential products. The solution of an equation with an operator function is based on the transition to standard stationary or evolutionary problems. Estimates of the accuracy of the approximate solution of the operator equation at known absolute or relative errors of approximation functions are obtained. The influence of the error in the solution of auxiliary problems is investigated separately. General approaches are illustrated by a problem with a fractional power of the operator. The first class of methods is based on the integral representation of the operator function under rational approximation, approximation by exponential sums, and approximation by exponential products. The second class of methods is associated with solving an auxiliary Cauchy problem for some evolutionary equation. In this case, we can distinguish a method based on approximation by exponential products.

References

[1] N. J. Higham, *Functions of Matrices: Theory and Computation*, Society for Industrial and Applied Mathematics, Philadelphia, 2008.

[2] P. N. Vabishchevich, Some methods for solving equations with an operator function and applications for problems with a fractional power of an operator, *Journal of Computational and Applied Mathematics* **407** (2022), 114096, 1–13.

Stable numerical schemes for modelling hemodynamic flows in time-dependent domains

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We present a unified numerical approach to finite-element modelling of incompressible flows in time-dependent domains. The approach features relatively large (independent of mesh size) time steps, solution of one linear system per time step, and relatively coarse computational meshes in space. The approach is monolithic and allows standard $P_2 - P_1$ (Taylor–Hood) finite element spaces. It is applicable to the Navier-Stokes equations in time-dependent domains, the fluid-structure interaction (FSI) problems, and the fluid-porous structure interaction (FPSI) problems. The properties of the schemes are shown on several benchmarks and hemodynamic applications.

References

[1] Lozovskiy A., Olshanskii M., Salamatova V. and Vassilevski Yu. An unconditionally stable semi-implicit FSI finite element method. *Comput.Methods Appl.Mech.Engrg.*, **297** (2015), 437–454.

[2] Lozovskiy A., Olshanskii M. and Vassilevski Yu. Analysis and assessment of a monolithic FSI finite element method. *Comp. & Fluids*, **179** (2019), 277–288.

[3] Danilov A., Lozovskiy A., Olshanskii M. and Vassilevski Yu. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling*, **32** (2017), no.4, 225–236.

[4] Lozovskiy A., Olshanskii M. and Vassilevski Yu. A quasi-Lagrangian finite element method for the Navier-Stokes equations in a time-dependent domain. *Comput.Methods Appl.Mech.Engrg.*, **333** (2018), 55–73.

[5] Vassilevski Yu. et al. A stable method for 4D CT-based CFD simulation in the right ventricle of a TGA patient. *Russian J. Numer. Anal. Math. Modelling*, **35** (2020), no.5, 315–324.

[6] Lozovskiy A., Olshanskii M. and Vassilevski Yu. A finite element scheme for the numerical solution of the Navier–Stokes/Biot coupled problem. *Russian J. Numer. Anal. Math. Modelling*, **37** (2022), no.3, 159–174.

Contributed talks

Inverse problem with a nonlinear gluing condition for a mixed type equation involving a nonlinear load

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Inverse problems for a mixed type equations fractional order was investigated by B. I. Islomov, U. Sh. Ubaydullayev [1], T.Yuladashev., E.Karimov [2] (and some references of these works). It should be noted, that all investigated problems the above works was considered only in rectangular domains.

For the equation:

$$g(x,y)+f(x)=\begin{cases} u_{xx} - {}_C D^{\alpha}_{oy}u + p_1\left(x,u(x,0)\right), & \text{at } y>0 \\ u_{xx}-u_{yy}+p_2\left(x,u(x,0)\right), & \text{at } y<0 \end{cases}$$

in the domain $\Omega = \{(x,t)\, 0<x<1;\, 0<t<h\}$, bounded with segments: $A_1A_2 = \{(x,t):\ x=1,\ 0<t<h\}$, $B_1B_2 = \{(x,t):\ x=0,\ 0<t<h\}$, $B_2A_2 = \{(x,t):\ t=h,\ 0<x<1\}$ and $B_1A_1 = \{(x,t):\ t=0,\ 0<x<1\}$, we investigate the following problem:

Problem NL. To find a pair of function $\{f(x),u(x,y)\}$ for equation (1) from the class of functions:

$$W=\Big\{u(x,y):\ u(x,y)\in C(\bar\Omega)\cap C^2(\Omega^-);\ u_{xx}, {}_C D^{\alpha}_{oy}u\in C(\Omega^+); \\ u(x,y)\in C^1(\bar\Omega^-\setminus A_1B_1)\Big\}$$

satisfies boundary conditions

$$u(x,y)\big|_{A_1A_2}=\varphi_1(y),\quad u(x,y)\big|_{B_1B_2}=\varphi_2(y),\quad 0\le y\le h;$$

$$\frac{d}{dx}u(\theta(x))=a_1(x)u_y(x,0)+a_2(x)u_x(x,0)+a_3(x)u(x,0)+a_4(x),\ 0<x<l;$$

$$u_n(x,y)\big|_{B_1C}=\psi_1(x),\ 0\le x\le\frac{l}{2},\ u_n(x,y)\big|_{A_1C}=\psi_2(x),\ \ \frac{l}{2}\le x\le l;$$

and integral gluing condition:

$$\lim_{y\to+0}y^{1-\alpha}u_y(x,y)=\lambda_1(x)u_y(x,-0)+\lambda_2(x)\int\limits_0^x r\left(t,u(t,0)\right)dt+\lambda_3(x),\ 0<x<l$$

where ${}_C D^{\alpha}_{oy}$ is the Caputo fractional derivative of order α, $(0<\alpha<1)$ and $\theta(x)=\theta\left(\frac{x}{2},\frac{-x}{2}\right)$, $g(x,t)$, $r(x,z(x))$, $p_i(x,z(x))$, $\varphi_i(y)$, $\psi_i(x)\,(i=1,2)$, $a_j(x)$, $\lambda_j(x)\,(j=\overline{1,3})$ are given functions, besides $\psi_1\left(\frac12\right)=\psi_2\left(\frac12\right)$, $\sum\limits_{k=1}^{2}\lambda_k^2(x)\neq0$.

References

[1] B. I. Islomov, U. Sh. Ubaydullayev, The inverse problem for a mixed type equation with a fractional order operator in a rectangular domain, *Russian Mathematics* **65** (2021), no.3, 25–42.

[2] T.Yuladashev., E.Karimov, Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters, *Axioms* **9** (2020), no.4.

Dirichlet problem for noncoercive nonlinear elliptic equations in unbounded domains

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We consider noncoercive nonlinear Dirichlet problems with discontinuous coefficients in unbounded domains. An existence result is proved exploiting suitable coercive nonlinear approximate Dirichlet problems.

References

[1] E. A. Alfano and S. Monsurrò, *Noncoercive nonlinear Dirichlet problems in unbounded domains*, Nonlinear Anal., **192** (2020).

[2] L. Boccardo, *Finite energy solutions of nonlinear Dirichlet problems with discontinuous coefficients*, Boll. Unione Mat. Ital., **5** (2012), 357–368.

[3] S. Monsurrò and M. Transirico, Noncoercive elliptic equations with discontinuous coefficients in unbounded domains, *Nonlinear Anal.*, **163** (2017), 86–103.

Fractional periodic problems with critical growth

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In this talk, I will discuss the existence of 2π-periodic solutions to the following fractional critical problem:

$$\begin{cases} [(-\Delta+m^2)^s - m^{2s}]u = W(x)|u|^{2^*_s-2}u + f(x,u) \ \text{in } (-\pi,\pi)^N, \\ u(x+2\pi e_i) = u(x) \quad \text{for all } x \in \mathbb{R}^N, \, i=1,\dots,N, \end{cases}$$

where $s \in (0,1)$, $N \geq 4s$, $m \geq 0$, $2^*_s = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, $W(x)$ is a 2π-periodic positive continuous function, and $f(x,u)$ is a superlinear 2π-periodic (in x) continuous function with subcritical growth and (e_i) is the canonical basis in $\mathbb{R}^N$. When $m > 0$, the existence of a nonconstant periodic solution will be established by combining the Linking theorem and a suitable variant of the extension method in periodic setting. The case $m = 0$ will be studied through a limit procedure.

Apriori estimates of minimizer in a control parabolic problem

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We consider the following mixed problem for an equation with a convective term and a depletion potential:

$$u_t = (a(x,t)u_x)_x + b(x,t)u_x + h(x,t)u,\ (x,t) \in Q_T = (0,1) \times (0,T), \qquad (1)$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \quad t \in (0,T),\ u(x,0) = \xi(x), \quad x \in (0,1), \quad (2)$$

where a, b and h are sufficiently smooth functions in $\overline{Q}_T$, and $0 < a_1 \leq a(x,t) \leq a_2 < \infty$, $\varphi, \psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. We consider the weak solution of problem (1), (2) which belongs to the space $V_2^{1,0}(Q_T)$ (see [1], Ch. 1, Par. 1) and study a control problem with a pointwise observation ([2], [3]). Namely, the task is to make the temperature $u(x_0,t)$ at some point $x_0 \in (0,1)$ close to a given function $z \in L_2(0,T)$ during the whole time interval $(0,T)$ by controlling the temperature φ at the left endpoint of the interval (the functions ψ and ξ are assumed to be fixed). Let nonempty, closed, and convex $\Phi \subset W_2^1(0,T)$ denote the set of control functions φ, and $Z \subset L_2(0,T)$ — the set of target functions z. Quality of the control is estimated by the cost functional $J[z,\rho,\varphi] = \int_0^T (u_\varphi(x_0,t) - z(t))^2 \rho(t)dt$, $\varphi \in \Phi$, $z \in Z$, where u_φ is the solution of problem (1), (2) with a given control function φ and a weight function $\rho \in L_\infty(0,T)$ such that $\operatorname{ess} \inf\limits_{t\in(0,T)} \rho(t) = \rho_1 > 0$.

Theorem 1. *Let u be a solution of problem (1), (2) with nonnegative boundary and initial functions such that $\inf\limits_{t\in(0,T)} \varphi \geq 0$, $\inf\limits_{t\in(0,T)} \psi \geq 0$, $\operatorname{ess} \inf\limits_{x\in(0,1)} \xi \geq 0$. Then the solution u is also nonnegative: $\operatorname{ess} \inf\limits_{(x,t)\in Q_T} u \geq 0$.*

Theorem 2. *Suppose that $a_t \geq 0$ and $b_x - h \geq 0$ for $(x,t) \in Q_T$, $b \geq 0$ for $(x,t) \in [0,x_0] \times [0,T]$ with $x_0 \in (0,1]$, and $b(1,t) \leq 0$ for $t \in [0,T]$. Then the solution of problem (1), (2) satisfies the inequality*

$$\|u(x_0,\cdot)\|_{L_1(0,T)} \leq \|\varphi\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right),$$

and for the control function the following inequality holds:

$$\|\varphi\|_{L_1(0,T)} \geq \|z\|_{L_1(0,T)} - \left(\frac{TJ[\varphi,\rho,z]}{\rho_1}\right)^{1/2} - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right).$$

Acknowledgement. The research was partially supported by Russian Science Foundation (scientific project 20-11-20272).

References

[1] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Academic Press, New York-London, 1968.

[2] I.V. Astashova, D.A. Lashin and A.V. Filinovskii, *Control with point observation for a parabolic problem with convection*, *Trans. Moscow Math. Soc.* **80** (2019), no.2, 221–234.

[3] I. Astashova, A. Filinovskiy and D. Lashin, *On the estimates in various spaces to the control function of the extremum problem for parabolic equation*, *WSEAS Trans. Appl. Theor. Mech.* **16** (2021), 187–192.

Maximal regularity of local and nonlocal equations with degenerate weights

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We study the regularity of elliptic equations with degenerate elliptic weights in the linear case as well as in the non-linear case We establish a novel condition on the weight M. Instead of a BMO (bounded mean oscillation) smallness condition for M, we use a BMO smallness condition on its logarithm , which is new even for the linear case. Under this condition we show that local higher integrability of the right hand side transfers to the gradient of the solution. The sharpness of our estimates is proved by examples. We obtain the results in the local case and up to the boundary. For the nonlocal model with the weight we provide the Meyers-type counterexample to the higher integrability. The talk is based on several joint works with Diening, Buyn, Giova, Kassmann, Lee and Passarelli di Napoli.

References

[1] A. Kh. Balci, L. Diening, R. Giova and A. P. di Napoli, *Elliptic Equations with Degenerate Weights*, SIAM Journal on Mathematical Analysis Vol. 54, Iss. 2, 2022.

[2] A. Kh. Balci, S.-S. Byun, L. Diening and H.-S. Lee, Global Maximal Regularity for Equations with Degenerate Weights. Preprint arXiv:2201.03524

Estimates on the spectral interval of validity of the anti-maximum principle

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The anti-maximum principle for the problem

$$-\Delta_p u = \lambda |u|^{p-2}u + f(x) \;\; \text{in } \Omega, \quad u = 0 \;\; \text{on } \partial\Omega,$$

with positive $f \in L^\infty(\Omega)$ states the existence of a critical value $\lambda_f > \lambda_1$ such that for every $\lambda \in (\lambda_1, \lambda_f)$, any solution of this problem is strictly negative. This result was obtained in [2] for $p = 2$ and in [3] for $p > 1$ by different techniques. Not much is known about the properties of λ_f, except that λ_f cannot be bounded away from λ_1 uniformly with respect to f. Our aim is to discuss some estimates for λ_f and their properties. In particular, we show that the critical value

$$\lambda_f^* := \inf \left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} : \int_\Omega f u \, dx = 0, \; u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}$$

is a variational upper bound for λ_f whenever $\lambda_f^* < \lambda_2$ or $p = 2$ or $N = 1$.

As an important supplementary result, we investigate the branch of ground state solutions of the considered boundary value problem in (λ_1, λ_2) and explain the role of λ_f^* in the corresponding bifurcation diagram.

The talk is based on [1].

References

[1] V. Bobkov, P. Drábek and Y. Ilyasov, Estimates on the spectral interval of validity of the anti-maximum principle, *Journal of Differential Equations* **269** (2020), no.4, 2956–2976.

[2] P. Clément and L.A. Peletier, An anti-maximum principle for second-order elliptic operators, *Journal of Differential Equations* **34** (1979), no.2, 218–229.

[3] J. Fleckinger, J.-P. Gossez, P. Takáč, and F. de Thélin, Existence, nonexistence et principe de l'antimaximum pour le p-laplacien, *Comptes rendus de l'Académie des sciences. Série 1, Mathématique* **321** (1995), no.6, 731–734.

Group analysis of kinetic equations and the problem of the moment system closing

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One of the main mathematical problems associated with the models of the kinetic theory is the problem of the moment system closure, which goes back to the works of J.C. Maxwell.We propose to solve this problem using the invariants of the symmetry group of the corresponding kinetic equation. The feasibility of this idea is established for the one-dimensional kinetic equation $f_t + cf_x + (Ff)_c = 0$ (t – time, x – spatial coordinate, c – velocity, $F = F(t, x, c)$ – external force field, unknown function $f(t, x, c)$ is the phase density of particle distribution.It turned out that the problem of group analysis of this equation should be accompanied by additional conditions imposed on the group of transformations. These are the conditions of the invariance of: a) relations$dx = c\,dt$ and $dc = F\,dt$; b) a family of lines $dx = dt = 0$ and c) the form$(1 + c\theta_x + F\theta_c)f(t, x, c)dxdc$.We establish that the group of point transformations of the space of variables (t, x, c, f, F) satisfying all these conditions coincides with the group of diffeomorphisms of the space of variables (t, x) (generating transformations of other variables).A group classification of one-dimensional equations is carried out in the specified class of transformations, the maximum symmetry group turned out to be eight-dimensional (for $F = 0$ and equivalent ones) and coinciding with the projective group in $\mathbb{R}^2$.For the symmetry groups obtained we explicitly describe the action of these groups on moment quantities and find invariants. In the case of $F = 0$, the found differential invariant led to the well known system $\rho_t + (\rho u)_x = 0$, $u_t + uu_x = 0$. The exact formulations of the obtained results will be presented in the report; one can find them in [1-3].

References

[1] K.S. Platonova and A.V. Borovskikh, Group analysis of the one-dimensional Boltsmann equation. III. Condition for the moment quantities to be physically meaningful, *Theor. and Math. Phys.* **195** (2018), no.3, 886–915.

[2] K.S. Platonova and A.V. Borovskikh, Group analysis of the one-dimensional Boltsmann equation. Invariants and the problem of moment system closure, *Theor. and Math. Phys.* **208** (2021), no.3, 1165–1181.

[3] K.S. Platonova and A.V. Borovskikh, Group analysis of the Boltsmann and Vlasov equations, *Theor. and Math. Phys.* **203** (2020), no.3, 794–823.

Subordination principle, stochastic solutions and Feynman-Kac formulae for generalized time fractional evolution equations

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We consider generalized time-fractional evolution equations of the form

$$u(t) = u_0 + \int_0^t k(t,s)Lu(s)ds$$

with a fairly general memory kernel k and an operator L being the generator of a strongly continuous semigroup. In particular, L may be the generator L_0 of a Markov process ξ on some state space Q, or $L := L_0 + b\nabla + V$ for a suitable potential V and drift b, or L generating subordinate semigroups or Schrödinger type groups. This class of evolution equations includes in particular time- and space- fractional heat and Schrödinger type equations.

We show that the subordination principle holds for such evolution equations and obtain Feynman-Kac formulae for solutions of these equations with the use of different stochastic processes, such as subordinate Markov processes and randomely scaled Gaussian processes. In particular, we obtain some Feynman-Kac formulae with generalized grey Brownian motion and other related self-similar processes with stationary increments.

The talk is based on the joint work with Ch. Bender and M. Bormann.

References

[1] Ch. Bender, M. Bormann and Ya.A. Butko, Subordination principle and Feynman-Kac formulae for generalized time-fractional evolution equations. Preprint ArXiv:2202.01655

[2] Ch. Bender and Ya.A. Butko, Stochastic solutions of generalized time-fractional evolution equations, *Frac. Calc. Appl. Anal.* **25** (2022), no.2, to appear.

Study of one class semilinear Sobolev type equations by the Galerkin method

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Of concern is an initial-boundary value problem for the modified Boussinesq equation (IMBq equation) is considered. The equation is often used to describe the propagation of waves in shallow water under the condition of mass conservation in the layer and taking into account capillary effects. In addition, it is used in the study of shock waves. The modified Boussinesq equation belongs to the Sobolev type equations. Earlier, using the theory of relatively p-bounded operators, the theorem of existence and uniqueness of the solution to the initial-boundary value problem was proved. We proved that the solution constructed by the Galerkin method using the system orthornormal eigenfunctions of the homogeneous Dirichlet problem for the Laplace operator converges *-weakly to an precise solution. Based on the compactness method and Gronwall's inequality, the existence and uniqueness of solutions to the Cauchy–Dirichlet and the Showalter–Sidorov–Dirichlet problems for the modified Boussinesq equation are proved.

References

[1] O.A. Ladyzhenskaya, *Matematicheskie voprosy dinamiki vyazkouprugoy neszhimaemoy zhidkosti [Mathematical Problems in the Dynamics of a Viscoelastic Incompressible Fluid]*, Fizmatgiz, Moscow, 1961.

[2] A.A. Zamyshlyaeva and E.V. Bychkov, The Cauchy problem for the Sobolev type equation of higher order, *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software* **11** (2018), no. 1, 5–14.

[3] E.V. Bychkov Analytical Study of the Mathematical Model of Wave Propagation in Shallow Water by the Galerkin Method, *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software* **14** (2021), no. 1, 26–38.

Boundedness of the solutions to a kind of parabolic systems

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We deal with nonlinear systems of parabolic type satisfying componentwise structural conditions. The nonlinear terms are Carathéodory maps having controlled growth with respect to the solution and the gradient and the data are in anisotropic Lebesgue spaces. Under these assumptions we obtain essential boundedness of the weak solutions.

References

[1] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Math. Mon., Vol. 23, Amer. Math. Soc.

[2] F. Leonetti, P.V. Petricca, Regularity for solutions to some nonlinear elliptic systems, *Complex Var. Elliptic Eq.* **12** (2011), 1099-1113.

[3] S.-S. Byun, D.K. Palagachev, P. Shin, Boundedness of solutions to quasilinear parabolic equations, *J. Differential Equations* **261** (2016), 6790-6805.

[4] D.K. Palagachev, L.G. Softova, Boundedness of solutions to a class of coercive systems with Morrey data, *Nonlinear Anal.* (2020), 16 p.

[5] L.G. Softova, Boundedness of the solutions to nonlinear systems with Morrey data, *Complex Var. Elliptic Equ.*, **63** (2018).

The non-uniqueness of the Showalter-Sidorov problem for the Barenblatt – Zheltov – Kochina equation with Wentzell boundary conditions in a bounded domain

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Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}\backslash\{1\}$, be a bounded connected domain with the boundary $\partial\Omega$ of the class C^∞. In cylinder $Q_T = \Omega \times (0, T)$, $T \in \mathbb{R}_+$, let us consider the Barenblatt – Zheltov – Kochina equation

$$(\lambda - \Delta)u_t(x, t) = \alpha_0 \Delta u(x, t) + f(x, t),\ (x, t) \in Q_T, \tag{1}$$

which describes dynamics of pressure of a filtered fluid in a fractured-porous medium [1], whose solutions must satisfy the Wentzel boundary condition

$$\Delta_{LB} u(x, t) + \alpha_1 \frac{\partial u}{\partial \nu}(x, t) + \beta_1 u(x, t) = 0,\ (x, t) \in \partial\Omega \times (0, T), \tag{2}$$

and the condition

$$\varphi(x) = \frac{\partial u}{\partial \nu}(x, t),\ x \in \partial\Omega. \tag{3}$$

Here α and λ are the material parameters characterizing the environment, the parameter $\alpha \in \mathbb{R}_+$, the function $f = f(x, t)$ plays the role of external loading. The paper considers the initial Showalter – Sidorov condition for problem (1) – (3).

$$\lim_{t \to 0+} (\lambda - \Delta)(u(x, t) - u_0(x)) = 0,\ x \in \Omega. \tag{4}$$

In particular, the paper intends to discuss the non-uniqueness of the Showalter-Sidorov problem (1)–(4).

Theorem For any $f \in C^1((0, \tau); \mathfrak{F}^0) \cap C^0([0, \tau]; \mathfrak{F}^1)$, $u_0 \in \mathfrak{U}$, and $\varphi \in C^1((0, \tau); \mathfrak{F}^0) \cap C^0([0, \tau]; \mathfrak{F}^1)$ there is a unique solution $u \in C^1((0, \tau); \mathfrak{U}) \cap C([0, \tau]; \mathfrak{U})$ for Showalter-Sidorov problems (1)–(4).

References

[1] G.I. Barenblatt, Yu.P. Zheltov and I.N. Kochina, Fundamental equations for the flow of homogeneous fluids through fissured rocks, *Dokl. Akad. Nauk SSSR* **24** (1960) no.5, 58–73.

Existence and Multiplicity of solutions to N-Kirchhoff equations with critical exponential growth and a perturbation term

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The aim of this article is twofold; firstly, we deal with the existence and multiplicity of weak solutions to the Kirchhoff problem:

$$\begin{cases} -a\Big(\int_\Omega |\nabla u|^N\Big)\Delta_N u = \dfrac{f(x,u)}{|x|^b} + \lambda h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and $0 \leq b < N$. Secondly, we deal with the existence and multiplicity of weak solutions to the Kirchhoff problem

$$-a\left(\int_{\mathbb{R}^N} |\nabla u|^N + V(x)|u|^N dx\right)(\Delta_N u + V(x)|u|^{N-2}u) = \frac{g(x,u)}{|x|^b} + \lambda h(x) \text{ in } \mathbb{R}^N$$

where $N \geq 2$, $0 \leq b < N$, λ is a suitably small real parameter and the perturbation term $h > 0$ belongs to the dual of some suitable Sobolev space. The function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and depends on the norm of u, that makes the equations non-pointwise. We assume that f and g have critical exponential growth at infinity. To establish our existence results, we use the mountain pass theorem, Ekeland variational principle and Moser-Trudinger inequality.

References

[1] W. Chen and F. Yu, On a nonhomogeneous Kirchhoff-type elliptic problem with critical exponential in dimension two, *Appl. Anal.*, (2020), 1–16.

[2] FJSA Corrêa and G. M. Figueiredo, On an elliptic equation of p-Kirchhoff type via variational methods, *Bull. Austral. Math. Soc.* **74** (2006), 263–277.

[3] M. de Souza, J.M.B. do Ó and T. da Silva, On a class quasilinear Schrödinger equations in $\mathbb{R}^N$, *Appl. Anal.* **95** (2016), 323–340.

[4] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.* **17** (1977), 473–483.

The fundamental gap conjecture

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At the end of his study [J. Statist. Phys., 83] on thermodynamic functions of a free boson gas, van den Berg conjectured that the difference between the two smallest eigenvalues

$$\Gamma^V(\Omega) := \lambda_2^V(\Omega) - \lambda_1^V(\Omega);$$

of the Schrödinger operator $-\Delta + V$ on a convex domain Ω in $\mathbb{R}^d$, $d \geq 1$, equipped with homogeneous Dirichlet boundary conditions satisfies

$$\Gamma^V(\Omega) \geq \Gamma(I_D) = \frac{3\pi^2}{D^2}, \tag{1}$$

where I_D is the interval $(-D/2, D/2)$ of length $D = \text{diameter}(\Omega)$. The term $\Gamma^V(\Omega)$ is called the *fundamental gap* and describes an important physical quantity: for example, in statistical mechanics, $\Gamma^V(\Omega)$ measures the energy needed to jump from the ground state to the next excited eigenstate, or computationally, it can control the rate of convergence of numerical methods to compute large eigenvalue problems [SIAM, 2011]. Thus, one is interested in (optimal) lower bounds on $\Gamma^V(\Omega)$. Since the late 80s, the *fundamental conjecture* (1) attracts consistently the attention of many researcher including M. S. Ashbaugh & R. Benguria [Proc. Amer. Math. Soc., 89], R. Schoen and S.-T. Yau [Camb. Press, 94.] (see also [Geneva, 86]), B. Andrews and J. Clutterbuck [J. Amer. Math. Soc., 11].

In this talk, I present new results on the fundamental gap conjecture (1) for the Schrödinger operator $-\Delta + V$ on a convex domain Ω equipped with *Robin boundary conditions*. In particular, we present a proof of this conjecture in dimension one, and mention results for the p-Laplacian.

The talk is based on the joint works [1, 2] with B. Andrews and J. Clutterbuck.

References

[1] B. Andrews, J. Clutterbuck and D. Hauer, Non-concavity of the Robin ground state, *Camb. J. Math.* **8** (2020), no.2, 243–310.

[2] B. Andrews, J. Clutterbuck and D. Hauer, The fundamental gap for a one-dimensional Schrödinger operator with Robin boundary conditions, *Proc. Amer. Math. Soc.* **149** (2021), no.4, 1481–1493.

Partial regularity for elliptic and parabolic systems with Orlicz growth

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In this talk I will speak about partial regularity results for weak solutions to elliptic and parabolic systems satisfying ellipticity and growth conditions in terms of Orlicz functions. The main result is obtained by using a new $\mathcal{A}$-caloric approximation lemma compatible with an Orlicz setting.

This talk is based on joint works with M. Foss, C. Leone and A.Verde.

Elliptic differential-difference equations with incommensurable sifts of arguments

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Consider the boundary value problem:

$$A_R u = -\sum_{i,j=1}^{n} (R_{ij} u_{x_j})_{x_i} = f(x) \quad (x \in Q), \tag{1}$$

$$u(x) = 0 \quad (x \notin Q), \tag{2}$$

Q is a bounded domain in $\mathbb{R}^n$ with a smooth boundary ∂Q, $f \in L_2(Q)$, difference operators $R_{ij} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ are as follows:

$$R_{ij} u(x) = \sum_{h \in M_{ij}} a_{ijh}(u(x+h) + u(x-h)) \quad (a_{ijh} \in \mathbb{R}),$$

where $M_{ij} \subseteq M$ is a finite set of vectors with incommenshurable coordinates. The solution u of problem (1), (2) belongs to the Sobolev space $\mathring{H}^1(Q)$.

For elliptic differential-difference equations with commensurable shifts of variables the theory of boundary-value problems is created in the works of A.Skubachevskii (see [1]). Unlike the problems studied in [1], eq. (1) contains incommensurable shifts of arguments, which greatly complicates the study. However, in the case where the orbit of the boundary ∂Q under the influence of shifts of the difference operator is finite, methods developed for problems with integer shifts are applicable. In particular, problem (1)–(2) can be reduced to a boundary problem for a differential equation with nonlocal boundary conditions.

For the case when the orbit of the boundary under the influence of shifts is infinite, the nature of the problem changes fundamentally. In particular, its solutions can have a dense set of derivative break points almost everywhere. A method for obtaining the conditions of strong ellipticity (the fulfillment of the Gårding-type inequality) based on the construction of a system of interrelated matrix polynomials is proposed [2]. These conditions are stable relative to small perturbations of the shifts of the difference operator.

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References

[1] A.L. Skubachevskii, *Elliptic Functional Differential Equations and Applications*, Basel–Boston–Berlin: Birkhäuser, 1997.

[2] E.P. Ivanova, On coercivity of differential -difference equations with incommensurable translations of arguments, *Journal of Mathematical Sciences* **239** (2019), no.6, 802–816.

Generalized analytical solution of the problem of determining the universal profile of the turbulent flow of an incompressible fluid

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The possibility of describing both laminar and turbulent modes of fluid flow based on the same equations has been investigated. It is proposed to consider the Navier-Stokes equations in a phase space expanded by the introduction of an additional stochastic variable and to supplement the full time derivative with a term characterized by entropy production due to the excitation of stochastic disturbances. In this approach, the problems of Hagen-Poiseuille, Poiseuille and the plane Couette flow are solved. It is shown that the occurrence and maintenance of stochastic processes in a liquid is possible in those systems where there are incompatible boundary conditions. In this case, the existence of one smooth solution becomes impossible, and we can only talk about the presence of two or more non-intersecting or non-smoothly intersecting asymptotes of the solution. The region located between these asymptotes (or in the vicinity of the point of "discontinuity" of derivatives) is a domain of uncertainty that generates a stochastic process. In this approach, the problems of Hagen-Poiseuille, Poiseuille and the plane Couette flow are solved. In all these problems, both "laminar" and generalized "turbulent" solutions are found. It is shown that, despite the difference between the solved equations for the Hagen-Poiseuille problems, the plane Poiseuille flow, and also for the flow along a flat plate (as the limiting case of the Poiseuille problem), the velocity profiles for the turbulent flow regime are the same, that may indicate the existence of a universal wall velocity profile for these problems.

References

[1] O.N. Khatuntseva, On finding a generalized analytical solution to the Hagen–Poiseuille problem for a turbulent flow mode, *"Trudy' MAI"* **118** (2021), paper ID 158211.

Klein-Gordon equation with mean field interaction. Orbital and asymptotic stability of solitary waves

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We consider $\mathbf{U}(1)$-invariant nonlinear Klein–Gordon equation

$$\ddot{\psi}(x,t)=\psi''(x,t)-m^2\psi(x,t)+\rho(x)f(|\langle\psi(\cdot,t),\rho\rangle|^2)\langle\psi(\cdot,t),\rho\rangle,\ \psi(x,t)\in\mathbb{C},\ x\in\mathbb{R}.$$

Here $m>0$, ρ is real-valued coupling function, $\langle\psi,\rho\rangle=\int\psi(x)\rho(x)dx$. The equation admits finite energy solutions of the form $\psi_\omega(x,t)=e^{-i\omega t}\varphi_\omega(x)$ called *solitary waves*. The solitary waves form a two-dimensional *solitary manifold* in the Hilbert space of finite energy states of the system. We prove that for this equation the standard criterion for orbital stability of solitary waves (see [1] and references therein) holds:

$$\partial_\omega\mathcal{Q}(\varphi_\omega,-i\omega\varphi_\omega)=\partial_\omega(\omega\|\varphi_\omega\|_{L^2(\mathbb{R})}<0,$$

where the charge $\mathcal{Q}(\psi,\pi)=\operatorname{Im}\int\psi(x)\overline{\pi}(x)\,dx$ is conserved for any solution $(\psi(x,t),\dot{\psi}(x,t))$. We show that in the case when $f(z)=|z|^\kappa$ for $|z|\geq 1$, and ρ is close to δ-function, the criterion holds for any $\kappa<0$ and $\omega\in(-m,m)$; if $\kappa>0$, it holds only for $m\sqrt{\kappa}<|\omega|<m$.

Our second result is asymptotic stability of solitary waves. Namely, we prove scattering asymptotics of type

$$\begin{bmatrix}\psi(t)\\ \dot{\psi}(t)\end{bmatrix}\sim\begin{bmatrix}\psi_{\omega_\pm}(t)\\ \dot{\psi}_{\omega_\pm}(t)\end{bmatrix}+W(t)\begin{bmatrix}\chi_\pm\\ \pi_\pm\end{bmatrix}\qquad t\to\pm\infty,$$

where $W(t)$ is the dynamical group of the free Klein–Gordon equation, $(\chi_\pm,\pi_\pm)\in E:=H^1(\mathbb{R})\oplus L^2(\mathbb{R})$ are the corresponding asymptotic scattering states, and the remainder decays to zero as $\mathcal{O}(|t|^{-1/2})$ in global norm of E. The asymptotics holds for solutions with initial states close to the *stable part* of the solitary manifold, extending the results of [2].

References

[1] M. Grillakis, J. Shatah, and W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, *J. Funct. Anal.* **7**4 (1987), 160–197.

[2] V. Buslaev, A. Komech, E. Kopylova, and D. Stuart, On asymptotic stability of solitary waves in Schrödinger equation coupled to nonlinear oscillator, *Comm. Part. Differ. Equat* **33** (2008), 669–705.

[3] E. Kopylova, Klein–Gordon equation with mean field interaction. Orbital and asymptotic stability of solitary waves, accepted in *Nonlinearity*, (2022)

Electric impedance tomography of surfaces with boundary

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Let (M, g) be a smooth compact two-dimensional Riemannian manifold (*surface*) with a smooth metric tensor g and smooth boundary Γ. Its *DN-map* $\Lambda : C^\infty(\Gamma) \to C^\infty(\Gamma)$ is associated with the (forward) elliptic problem $\Delta_g u = 0$ in $M \setminus \Gamma$, $u = f$ on Γ, and acts by $\Lambda f := \partial_\nu u^f$ on Γ, where Δ_g is the Beltrami-Laplace operator, $u = u^f(x)$ the solution, ν the outward normal to Γ. The corresponding *inverse problem* (EIT-problem) is to determine the surface (M, g) from its DN-map Λ.

An algebraic version of the Boundary Control method (Belishev' 2003) is developed:

1) the version is extended to the case of *nonorientable* surfaces and a criterion of orientability (in terms of Λ) is provided [2];

2) a procedure that determines surfaces with (unknown) internal holes, is proposed [1];

3) a characteristic description of Λ that provides the necessary and sufficient conditions for solvability of the inverse problem for orientable surfaces, is given [3];

4) for the surfaces M and M' with the mutual boundary Γ and DN-maps Λ and Λ', we show that $\|\Lambda' - \Lambda\| \to 0$ leads to $M' \to M$ in a relevant sense [4].

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References

[1] A.V. Badanin, M.I. Belishev and D.V. Korikov, Electric impedance tomography problem for surfaces with internal holes, *Inverse Problems* **37** (2021), no. 10.

[2] M.I. Belishev and D.V. Korikov, On Determination of Nonorientable Surface via its Diriclet-to-Neumann Operator, *SIAM Journal on Mathematical Analysis* **53** (2021), no.5, 5278–5287.

[3] M.I. Belishev and D.V. Korikov, On Characterization of Hilbert Transform of Riemannian Surface with Boundary, *Complex Analysis and Operator Theory* **16** (2022), no.10.

[4] M.I. Belishev and D.V.Korikov, On stability of determination of Riemann surface from its DN-map. Preprint arXiv:2112.14816

Fine properties of steady water waves

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In this talk we will discuss some recent results on two dimensional steady water waves; see [1]. We will explain how the Benjamin and Lighthill conjecture can be significantly refined and will prove a new bound for the amplitude of an arbitrary Stokes wave in terms of the non-dimensional Bernoulli constant. Our result, in particular, implies the inequality $a \leq Cc^2/g$, where a is the amplitude, c is the speed of the wave, and g is the gravitational constant. This fact is valid for arbitrary Stokes waves irrespectively of the amplitude with an absolute constant C. Another observation is that any extreme Stokes wave over a sufficiently deep stream has necessarily a small amplitude, provided the non-dimensional mass flux is much smaller than the depth.

References

[1] E. Lokharu, On the amplitude and the flow force constant of steady water waves, *J. Fluid Mech.* **921** (2021).

Coefficient inverse problems for the filtration equations

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In this work we study a new class of the inverse problems on identification of the coefficients in the filtration equations of sobolev type and the associated stationary inverse problems. The unknown coefficients are recovered by additional integral boundary data. In particular, we consider the following inverse problems.

Problem 1. For a given constant η and functions $f(t,x)$, $g(t,x)$, $\mu_1(t,x)$, $\mu_2(t,x)$, $u_0(x)$, $\sigma(x)$, $\omega(t,x)$, $\varphi_1(t)$, $\varphi_2(t)$ find the pair of functions $(u(t,x), k(t))$ satisfying the equation

$$u_t + \eta M u_t + k(t)\, M u = f(t,x), \quad (t,x) \in Q_T,$$

and the conditions

$$u\big|_{t=0} = u_0(x), \quad x \in \Omega,$$

$$\Big\{\eta \frac{\partial u_t}{\partial \overline{N}} + k(t)\frac{\partial u}{\partial \overline{N}} + \sigma(x)(\eta u_t + k(t)u)\Big\}\Big|_{S_T} + k(t)\mu_1(t,x) = \mu_2(t,x),$$

$$\int_{\partial\Omega} (\eta u_t + k(t)u)\omega(t,x)\, dS + \varphi_1(t)k(t) = \varphi_2(t), \quad t \in (0,T).$$

Here Ω is a bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega$, $Q_T = (0,T)\times\Omega$ is a cylinder with the lateral surface $S_T(0,T)\times\partial\Omega$; $M = -\mathrm{div}(\mathcal{M}(x)\nabla) + m(x)I$, $\mathcal{M}(x)$ is a matrix of functions, $m(x)$ is a scalar function, I is the identity operator; $\frac{\partial}{\partial \overline{N}} = (\mathbf{n}, \mathcal{M}(x)\nabla)$ and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

Problem 2. For given functions $f(x), \sigma(x), \beta(x), h(x)$ and a constant μ find the pair of function $u(x)$ and constant g satisfying the equation

$$Mu + gu = f,$$

the boundary condition

$$\Big(\frac{\partial u}{\partial \overline{N}} + \sigma(x)u\Big)\Big|_{\partial\Omega} = \beta(x),$$

and the condition of overdetermination

$$\int_{\partial\Omega} u\omega(x) ds = \mu.$$

The sufficient conditions for existence and uniqueness of the strong solutions to Problems 1 and 2 are found. The continuous dependence of solutions on the input data is established. The results concern with the identification of the hydraulic properties of a fissured medium $k(t)$ (Problem 1) and the absorption coefficient g (Problem 2).

On resolvent approximations of elliptic differential operators with periodic coefficients under low regularity condition

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We consider elliptic non-selfadjoint differential operators $A_\varepsilon = -div\, a(x/\varepsilon)\nabla$ in $\mathbb{R}^d$ with ε-periodic measurable matrix $a(x/\varepsilon)$ and study the asymptotic behaviour of resolvents $(A_\varepsilon + 1)^{-1}$, as the period ε goes to zero, extending the results obtained in [1] and [2]. We provide the "operator asymptotics" of $(A_\varepsilon+1)^{-1}$ in the sense of L^2-operator-norm convergence with order ε^2 remainder term. Namely, $(A_\varepsilon+1)^{-1}=(A_0+1)^{-1}+\varepsilon C_\varepsilon+O(\varepsilon^2)$, where A_0 is the well known homogenized operator and the corrector C_ε consists of several terms constructed with help of solutions to so-called cell problems (see [3]). The class of operators covered by our analysis includes uniformly elliptic families, first, with bounded coefficients; second, with unbounded coefficients in the skew-symmetric part of the operator which are from the John–Nirenberg space BMO.

Decomposing the matrix a into the symmetric and skew-symmetric parts $a = a^s + a^c$, we can write the operator A_ε in the form $A_\varepsilon = -div\, a^s(x/\varepsilon)\nabla - \varepsilon^{-1}b(x/\varepsilon)\cdot\nabla$, where $b(x)=div\, a^c(x)$, thereby, $div\, b=0$, $b\in BMO^{-1}(\mathbb{R})$. Then A_ε corresponds to the convection-diffusion operator in a stationary incompressible flow, b is called the drift vector. Elliptic and parabolic equations with the divergence-free drift having low regularity is of particular interest for applications to incompressible flows (see [4] and references there).

We apply the modified method of the first approximation with the usage of Steklov's smoothing proposed in [5] (see also [1]).

References

[1] V. V. Zhikov and S. E. Pastukhova, Operator estimates in homogenization theory, *Russian Math. Surveys* **71** (2016), 417–511.

[2] N. N. Senik, Homogenization for non-self-adjoint periodic elliptic operators on an infinite cylinder, *SIAM J. Math. Anal.* **49** (2017), 874–898.

[3] S. E. Pastukhova, On resolvent approximations of elliptic differential operators with periodic coefficients, *Applicable Analysis*, published online 17 Dec 2020, https://doi.org/10.1080/00036811.2020.1859493.

[4] G. Seregin, L. Silvestre, V. Sverak and A. Zlatos, On divergence-free drifts, *J. Differential Equations* **252** (2012), 505–540.

[5] V. V. Zhikov and S. E. Pastukhova, On operator estimates for some problems in homogenization theory, *Russian Journal of Math.Physics* **12** (2005), 515–524.

Optimal control problems for fractional-order diffusion and diffusion-wave equations

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We consider an optimal control problems for a model system, which described by a one-dimensional non-homogeneous diffusion-wave equation with a time derivative of fractional-order:

$$r(x)\ {}_0^C D_t^\alpha Q(x,t) = \frac{\partial}{\partial x}\left[w(x)\frac{\partial Q(x,t)}{\partial x}\right] - q(x)Q(x,t) + u(x,t), \tag{1}$$

where $Q(x,t)$ — system state, ${}_0^C D_t^\alpha$ — Caputo left-side fractional time derivative operator, $\alpha \in (0,2]$, $u(x,t)$ — distributed control, $t \geq 0$, $x \in [0,L]$, $(x,t) \in \Omega = [0,L] \times [0,\infty)$. Functions $r(x) > 0$, $w(x) > 0$ and $q(x)$ allowed to be continuous at segment $[0,L]$.

Initial conditions for Eq. (1) we define as follows:

$$\frac{\partial^k Q(x,0+)}{\partial x^k} = \varphi^k(x), \quad x \in [0,L], k = 0, [\alpha]\,.$$

Boundary conditions for Eq.(1) we will choose in the following form:

$$\left[b_i\frac{\partial Q(x,t)}{\partial x} + a_i Q(x,t)\right]_{x=x^i} = h_i(t) + u^i(t), \quad t \geq 0, i = 1,2,$$

where a_i and b_i — constant coefficients, $b_1 \leq 0$, $b_2 \geq 0$; $u^i(t)$ — boundary controls, $h_i(t)$ — some known completely regular functions, $x^1 = 0$, $x^2 = L$.

In general case we consider both of boundary and distributed controls which are p-integrable functions (including $p = \infty$). In this case two types of optimal control problem is posed and analyzed: the problem of control norm minimization at given control time and the problem of time-optimal control at given restriction on control norm. In general case we use an explicit solution for equation (1) and reduce the optimal control problem to an infinite-dimensional l-problem of moments. We also consider the finite-dimensional l-problem of moments derived using an approximate solution of the equation. For this problem the correctness and solvability is analyzed. Also we analyze the conditions, which cause an existence of control with minimal norm but non-existence of time-optimal control. Finally, we study an example of boundary control calculation using finite-dimensional l-problem of moments.

Steady states of the Vlasov equation with a Lennard-Jones type potential

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We consider a system of mutually gravitating particles with possible collisions. As it is known, when pair collisions of an infinitely large number of gravitating particles are taken into account, the probability density function evolves in accordance with the Vlasov-Boltzmann-Poisson system of equations. The collisions can be described using the theory of inelastic interaction of solids with Newton's recovery coefficient for the relative velocity of colliding particles. In numerical implementation, the main difficulty of this approach is to track and refine a huge number of time moments of particle collisions.

As another approach, we suggest to add to the gravitational potential the potential of repulsive forces, similarly to the intermolecular Lennard-Jones forces. Numerical experiments show that when the Jacobi stability condition is satisfied, both models lead to a qualitatively identical character of evolution with the possible formation of stable configurations. [1] The probability density function is determined by the Vlasov kinetic equation with a modified gravitational potential called the Lennard-Jones type potential. The existence and nonlinear stability of steady states of the proposed dynamical model with a modified gravitational potential is under consideration.

Unlike many works on this topic for the gravitational and electromagnetic interactions of particles, there is no Poisson equation in our system. Using the energy-Casimir method ([2]), the existence of a large class of nonlinearly stable equilibrium solutions of this equation is proved. For this mathematical model it is shown that the total energy of the system has a minimizer under prescribed mass-Casimir constraint. This minimizer is a steady state, and its nonlinear stability is derived from its minimizing property.[3] This analytical study justifies the use of such mathematical model with the potential of Lennard-Jones type to describe the evolution of the relevant physical systems.

References

[1] T. V. Salnikova, E. I. Kugushev and S. Ya. Stepanov, Jacobi stability of a many-body system with modified potential, *Doklady Mathematics*, **101** (2020), no.2, 154–157.

[2] G.Rein, *Collisionless Kinetic Equations from Astrophysics: The Vlasov-Poisson System*, Evolutionary Equations, Amsterdam: Elsevier, 2007.

[3] T. V. Salnikova, Existence and stability of equilibrium solutions of the Vlasov equation with a modified gravitational potential, *Doklady Mathematics*, **104** (2021), no.5, 277–281.

On probability solutions to the Kolmogorov equations with coefficients of low regularity

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We study the stationary Kolmogorov equation

$$\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho) = 0 \tag{1}$$

in the case where the positive definite diffusion matrix A satisfies the Dini mean oscillation condition and the drift coefficient b is locally integrable to a power greater than the dimension. Note that even in the one-dimensional case the function ϱ can fail to have the Sobolev derivative. For example, this is the case if $b = 0$ and $A = 1/\varrho$, where $\varrho > 0$ is Hölder continuous and non-differentiable. Moreover there is an example of a positive definite and continuous diffusion matrix A for which the equation $\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) = 0$ has a locally unbounded solution.

Theorem *Suppose that* $\varrho \in L^1_{loc}(\Omega)$ *is a solution to the equation* (1). *Then*
(i) ϱ *has a continuous version,*
(ii) *if* $\varrho \geq 0$, *then the continuous version of* ϱ *satisfies the Harnack inequality,*
(iii) *if* $\varrho \geq 0$ *and* $\sigma \geq 0$ *are continuous solutions, then* $\frac{\sigma}{\varrho} \in W^{2,1}_{loc}(\mathbb{R}^d)$.

A solution ϱ is a probability solution if

$$\varrho \geq 0, \qquad \int \varrho(x)\,dx = 1.$$

In the general case the stationary Kolmogorov equation can have several different probability solutions.

Theorem *Let* ϱ *be a probability solution to equation* (1) *and let one of the following conditions be fulfilled:*
(i) $(1+|x|)^{-2}|a^{ij}(x)|, (1+|x|)^{-1}|b^i(x)| \in L^1(\varrho\,dx)$,
(ii) *there exists a function* $V \in C^2(\mathbb{R}^d)$ *with* $\lim\limits_{|x|\to\infty} V(x) = +\infty$ *and*

$$LV \leq C_1 + C_2 V.$$

Then ϱ *is a unique probability solution.*

References

[1] V.I. Bogachev and S.V. Shaposhnikov, Uniqueness of a probability solution to the Kolmogorov equation with a diffusion matrix satisfying Dini's condition, *Doklady Math.* **501** (2021), 11-15.

[2] V.I. Bogachev and S.V. Shaposhnikov, Integrability and continuity of solutions to double divergence form equations, *Annali di Matematica* **196** (2017), 1609–1635.

A strange non-local monotone operator arising in the homogenization of a diffusion equation with dynamic nonlinear boundary conditions on particles of critical size and arbitrary shape

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We characterize the homogenization limit of the solution of a Poisson equation in a bounded domain, either periodically perforated or containing a set of asymmetric periodical small particles and on the boundaries of this particles a nonlinear dynamic boundary conditions holds involving a Höder nonlinear function $\sigma(u)$. We consider the case in which the diameter of the perforations (or the diameter of particles) is critical in terms of the period of the structure.For this case of asymmetric critical particles we prove that the effective equation is a semilinear elliptic equation in which the time arises as a parameter and the nonlinear expression is given in terms of a nonlocal operator H which is monotone and Lipschitz continuous on $L^2(0,T)$, independently of the regularity of σ.

Some new periodic solutions to semilinear equations in $\mathbb{R}^n$ with fractional Laplacian

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Let $n \geq 2$, and let $s \in (0,1)$. Denote by $2_s^* = \dfrac{2n}{n-2s}$ the critical embedding exponent for the Sobolev–Slobodetskii space $H^s(\mathbb{R}^n)$. For $q \in (2, 2_s^*)$, we consider the equation

$$(-\Delta)^s u + u = |u|^{q-2}u \qquad \text{in } \mathbb{R}^n, \tag{1}$$

where $(-\Delta)^s$ is the conventional fractional Laplacian in $\mathbb{R}^n$ defined for any $s > 0$ by the Fourier transform $(-\Delta)^s u := F^{-1}\big(|\xi|^{2s} Fu(\xi)\big)$.

Semilinear equations driven by fractional Laplacian have been studied in a number of papers. We construct some new classes of solutions to the equation (1), which, apparently, were not considered earlier.

In [1], for the model equation

$$-\Delta u + u = u^3 \qquad \text{in } \mathbb{R}^n,$$

a variational approach was suggested. It is based on the concentration-compactness principle by P.-L. Lions and on the reflections. This method, also applicable to the equations driven by p-Laplacian, allows to construct in a unified way the solutions with various symmetries which can also decay in some directions.

Let $\Omega \subset \mathbb{R}^n$ be a convex polyhedron. For a positive sequence $R \to +\infty$, we define a family of expanding domains $\Omega_R = \{x \in \mathbb{R}^n : x/R \in \Omega\}$ and consider the problem

$$(-\Delta)^s_{\Omega_R} u + u = |u|^{q-2}u \quad \text{in} \quad \Omega_R, \tag{2}$$

where $(-\Delta)^s_{\Omega_R}$ stands for some fractional Laplacian in Ω_R, such as spectral fractional Dirichlet or Neumann Laplacian, etc.

Lemma 1. *There exists a least energy solution u of* (2), *positive and smooth in Ω_R.*

Now we assume that the polyhedron Ω has the following property: the space $\mathbb{R}^n$ can be filled with its reflections, colored checkerwise. Then we can extend the function u to the function $\mathbf{u}$ in the whole space by reflections consistent with the boundary conditions of $(-\Delta)^s_{\Omega_R}$.

Theorem 1. *The function $\mathbf{u}$ is a solution of the equation* (1) *in $\mathbb{R}^n$.*

In this way, we construct solutions of the equation (1) with various symmetries. Among them, there are: positive and sign-changing periodic solutions with various periodic lattices, quasi-periodic complex-valued solutions, breather-type solutions. These classes of solutions, apparently, were not studied earlier. In the local case, similar solutions are considered in [1]. However, some of our solutions are new even for $s = 1$.

This talk is based on joint work with Alexander Nazarov, see [2]. A part of our results was announced in the short communication [3].

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References

[1] L.M. Lerman, P.E. Naryshkin and A.I. Nazarov, Abundance of entire solutions to nonlinear elliptic equations by the variational method, *Nonlin. Analysis* **190** (2020), paper No. 111590.

[2] A.I. Nazarov and A.P. Shcheglova, Solutions with various structures for semilinear equations in $\mathbb{R}^n$ driven by fractional Laplacian. Preprint ArXiv:2111.07301

[3] A.I. Nazarov and A.P. Shcheglova, New classes of solutions to semilinear equations in $\mathbb{R}^n$ with fractional Laplacian, *ZNS POMI* **508** (2021), 124–133 (Russian).

Homogenization of non-local operators of convolution type

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The talk is based on joint works with A.L. Piatnitskii, T.A. Suslina and E.A. Zhizhina. I will give some results [1] on homogenization of non-local operators of convolution type.

In $L_2(\mathbb{R}^d)$, we consider a self-adjoint bounded operator given by

$$(\mathbb{A}_\varepsilon u)(x) := \varepsilon^{-d-2} \int_{\mathbb{R}^d} a((x-y)/\varepsilon)\mu(x/\varepsilon, y/\varepsilon)(u(x) - u(y))\, dy,$$
$$x \in \mathbb{R}^d, \quad u \in L_2(\mathbb{R}^d), \quad \varepsilon > 0.$$

Such operators are used to describe the behavior of random systems of large (infinite) number of particles. It is assumed that $a(x)$ is an even non-negative function, $a \in L_1(\mathbb{R}^d)$, $\|a\|_{L_1} = 1$; $\mu(x, y)$ is a bounded positive definite function; $\mu(x+m, y+n) = \mu(x, y)$, $m, n \in \mathbb{Z}^d$, $\mu(x, y) = \mu(y, x)$. Besides, it is assumed that $M_k = \int_{\mathbb{R}^d} |x|^k a(x)dx < +\infty$, $k = 1, 2, 3$. Under these assumptions the operator $\mathbb{A}_\varepsilon$ is bounded, self-adjoint and non-negative; moreover, $\min \sigma(\mathbb{A}_\varepsilon) = 0$.

We study the resolvent $(\mathbb{A}_\varepsilon + I)^{-1}$ for small ε. We prove that $(\mathbb{A}_\varepsilon + I)^{-1}$ converges to the effective resolvent $(\mathbb{A}^0 + I)^{-1}$, $\varepsilon \to 0$, in the operator norm on $L_2(\mathbb{R}^d)$. The effective operator $\mathbb{A}^0 = -\operatorname{div} g^0 \nabla$ is elliptic second-order differential operator; the matrix g^0 is defined in terms of the solution of some auxiliary problem on the cell of periodicity $\Omega := [0, 1)^d$. The following estimate for the norm of the difference of the resolvents

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(a, \mu)\varepsilon, \quad \varepsilon > 0.$$

holds true.

The method is based on the operator-theoretic approach, which was developed by M.Sh. Birman and T.A. Suslina. We will discuss a number of features of the non-local convolution-type operator, which requires some very interesting modifications of the operator-theoretic approach.

References

[1] E.A. Zhizhina, A.L. Piatnitski, V.A. Sloushch and T.A. Suslina, On operator estimates in homogenization of nonlocal convolution-type operators, *Petersburg Department of Steklov Institute of Mathematics, PREPRINTS 2021*, Preprint 08/2021

Qualitative properties of the weak solutions of a kind of nonlinear elliptic systems

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We study the boundedness and Morrey regularity of the solutions of a kind of nonlinear systems satisfying controlled growth conditions. The ellipticity of the operator is espressed through componentwise coercivity condition. This result permits us to study the higher integrability and Morrey regularity of the solutions of quasilinear elliptic systems.

References

[1] L. Fattoruso and L. Softova, Precise Morrey regularity of the weak solutions to a kind of qua-silinear systems with discontinuous data, *Electr. J. Qual. Theory Differ. Equ.* **2020** (2020), Paper no. 36, 13 p.

[2] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York-London, 1968.

[3] F. Leonetti and P.V. Petricca, Regularity for solutions to some nonlinear elliptic systems, *Complex Variabl. Elliptic Equ.*, **56** (2011), no.3, 1099–1113.

[4] L. Softova, L^p-Integrability of the gradient of solutions to quasilinear systems with discon-tinuous coefficients, *Differ. Integr. Equ.*, **26** (2013), no.9-10, 1091–1104.

[5] L. Softova, Boundedness of the solutions to non-linear systems with Morrey data, *Complex Variabl. Elliptic Equ.*, **63** (2018), no.11, 1581–1594.

Solution to stochastic partial Loewner equation with several complex variables using Nevanlinna theory

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Becker (1973) studied solutions to the Loewner differential equation in one complex variable using Carath´eodory class of holomorphic functions. However, in several complex variables point singularities are removable and other approaches necessary. Pfaltzgraff generalized to higher dimensions the Loewner differential equation and developed existence and uniqueness theorems for its solutions. The existence and regularity theory has been considered by several authors, and applications given to the characterization of subclasses of biholomorphic mappings, univalence criteria, growth theorems and coefficient bounds for restricted classes of biholomorphic mappings. Duren et al. (2010) studied general form of solutions to the Loewner differential equation under common assumptions of holomorphicity and uniquely determined univalent subordination chains. To our knowledge, to date stochastic Loewner equation has not been studied in a several complex variable setting. We solve the equation in its partial and ordinary differential form by firstly appropriately defining Brownian motion in high dimensions, following Pitman and Yor (2018). We translate the problem in meromorphic form using generalizations of Nevanlinna theory for several complex variables (Noguchi and Winkelman, 2013). Finally, the equation is solved using techniques from rough paths theory (see e.g. Hairer, 2013). Solution allows to study stochastic phenomena such as Schramm-Loewner evolution in high dimensions and we shortly study convergence to a scaling limit for multidimensional lattice models as a power series problem on Hartogs domain.

References

[1] J. Becker, Löwnersche Differentialgleichung und Schlichtheitskriterien, *Math. Annalen* **202** (1973), 321–335.

[2] P. Duren, I. Graham, H. Hamada and G. Kohr, Solutions for the generalized Loewner differential equation in several complex variables, *Math. Ann.* **347** (2010), 411–435.

[3] M. Hairer, Solving the KPZ Equation, *Annals of Mathematics* **178** (2013), 559–664.

[4] J. Noguchi and J. Winkelmann, *Nevanlinna Theory in Several Complex Variables and Diophantine Approximation*, Springer Japan, Tokyo, 2014.

[5] J. Pitman and M. Yor, A guide to Brownian motion and related stochastic processes. Preprint ArXiv:1802.09679

Optimal regularity for supercritical parabolic obstacle problems

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The nonlocal parabolic obstacle problem for an elliptic operator L with zero obstacle is

$$\begin{cases} \min\{u_t - Lu, u - \varphi\} &= 0 \text{ in } \mathbb{R}^n \times (0,T) \\ u &= \varphi \text{ at } t = 0. \end{cases}$$

When L is the Laplacian, this problem is closely related to the Stefan problem, that models phase transitions. When L is a nonlocal operator such as the fractional laplacian $(-\Delta)^s$, the equation serves as a model for stock pricing and other random processes with jumps.

The elliptic (time-stationary) version of this problem has been thoroughly studied since the pioneer works of Caffarelli, Salsa and Silvestre around 2007. However, much less is known about the parabolic problem.

When L is the fractional Laplacian, Caffarelli and Figalli proved in 2013 that the solutions are $C^{1,s}$ in space and $C^{1,\alpha}$ in time. Still in the case of $(-\Delta)^s$, for $s > 1/2$, Barrios, Figalli and Ros-Oton proved that the free boundary is $C^{1,\alpha}$ at regular points.

In this talk, we present our recent results with X. Ros-Oton, where we proved the optimal $C^{1,1}$ regularity of the solutions and a global $C^{1,\alpha}$ free boundary regularity for the case $s < 1/2$.

References

[1] X. Ros-Oton, D. Torres-Latorre, Optimal regularity for supercritical parabolic obstacle problems. Preprint arXiv:2108.12339

Posters

On p-Laplacian with rapidly changing boundary conditions

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Denote by $D \subset \mathbb{R}^2$ the domain $\{(x,y) \ : \ 0<x<1, \ 0<y<1\}$ and let $F \subset \partial D$. Let $W_p^1(D,F)$ be a completion of the set of smooth functions vanishing in a neighbourhood of F by the norm $\| \, u \, \|_{W_p^1(D,F)} = \left(\int\limits_D |v|^p \, dx + \int\limits_D |\nabla v|^p \, dx \right)^{1/p}$. Denoting $G = \partial D \setminus F$, we consider the Zaremba problem

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = l \quad \text{in} \quad D, \quad u = 0 \text{ on } F, \quad \frac{\partial u}{\partial n} = 0 \text{ on } G, \qquad (1)$$

where $\frac{\partial u}{\partial n}$ is an outer normal derivative and l is a linear functional on $W_p^1(D,F)$. By the Hahn–Banach theorem $l(\varphi) = \sum\limits_{i=1}^{n} \int\limits_D f_i \varphi_{x_i} \, dx$, where $f_i \in L_{p'}(D)$, $p' = p/(p-1)$. The function u is a solution to problem (1), if

$$\int\limits_D |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = - \int\limits_D f \cdot \nabla \varphi \, dx$$

for any $\varphi \in W_p^1(D,F)$.

Define the capacity $C_q(K)$, $1<q<2$ of a compact $K \subset \mathbb{R}^2$ by

$$C_q(K) = \inf \left\{ \int\limits_{\mathbb{R}^n} |\nabla \varphi|^q \, dx : \ \varphi \in C_0^\infty(\mathbb{R}^n), \ \varphi \geq 1 \text{ on } K \right\}.$$

Denote by $B_r^{x_0}$ an open circle of the radius r, centered in x_0, and by $mes_1(E)$ the linear measure of $E \subset \partial D$.

I. If $1 < p \leq 2$, then for $q = (p+1)/2$ and for an arbitrary point $x_0 \in F$ and $r \leq 1$ we have either

$$C_q(F_1 \cap \overline{B}_r^{x_0}) \geq c_0 r^{2-q}, \qquad (2)$$

dequation or

$$mes_1(F_1 \cap \overline{B}_r^{x_0}) \geq c_0 r. \qquad (3)$$

II. If $p > 2$, then $F \neq \emptyset$.

The following Theorem is valid.

Theorem 1. *If $f \in L_{p'+\delta_0}(D)$, where $\delta_0 > 0$, then there exists a positive constant $\delta < \delta_0$, such that for a solution to* (1) *the following estimate*

$$\int\limits_D |\nabla u|^{p+\delta} \, dx \leq C \int\limits_D |f|^{p'(1+\delta/p)} \, dx$$

holds, where C depends only on p, δ_0 and c_0 from (2) *and* (3) *as* $1 < p \leq 2$. *And if $p > 2$ the constant C depends only on p and δ_0.*

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A posteriory estimates on solutions for a forth order elliptic problem with obstacle

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We study variational inequalities generated by a fourth order elliptic problem with obstacle. It can be formulated as follows

$$J(v) = \int\limits_{\Omega} \left(\frac{1}{2} |\Delta v|^2 - f v \right) dx \to \min_{\mathbb{K}_1} . \tag{1}$$

Here $\Omega \in \mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $f \in L_2(\Omega)$, and $\mathbb{K}_1 = \left\{ v \in H^2(\Omega) : v\big|_{\partial\Omega} = 0,\ v \geqslant \varphi \text{ a.e. in } \Omega \right\}$. Function $\varphi \in C^2(\overline{\Omega})$ such that $\varphi \leqslant 0$ on $\partial\Omega$ is called an obstacle. This problem is useful in studying behaviour of plates or beams when they are free-supported. Other applications of the fourth order elliptic problems with obstacle can be related to the elasticity theory, hydrodynamics, etc., see [1] and bibliography therein.

Adjusting to (1) the duality theory and using natural energy norms we obtain estimates for the distance between the exact solution and its approximation, i.e., any function from energy class that satisfies the boundary condition and conditions generated by the obstacle. A posteriory estimates of solutions and verifications for them in the case of rigidly supported plates and beams can be found in [2].

Results of this work were obtained in collaboration with D.E. Apushkinskaya.

References

[1] P. Neittaanmäki and S.I. Repin, A posteriori error estimates for boundary-value problems related to the biharmonic operator, *East-West Journal of Numerical Mathematics* **9** (2001), no.2, 157–178.

[2] D.E. Apushkinskaya and S.I. Repin, Biharmonic obstacle problem: guaranteed and computable error bounds for approximate solutions, *Computational Mathematics and Mathematical Physics* **60** (2020), no.11, 1823–1838.

Morphology of the phase space of the mathematical model of autocatalytic reaction with diffusion

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In the cylinder $\Omega \times \mathbb{R}_+$ (by $\Omega \subset \mathbb{R}^n$ we mean a bounded domain with boundary $\partial\Omega$ of class C^∞) consider the degenerate system of equations [1]

$$\begin{cases} \varepsilon_1 v_t = \alpha_1 v_{ss} + \beta_1 - (\beta_2 + 1)v + v^2 w, \\ \varepsilon_2 w_t = \alpha_2 w_{ss} + \beta_2 v - v^2 w, \end{cases} \tag{1}$$

$$\varepsilon_1 = 0 \text{ or (and) } \varepsilon_2 = 0 \tag{$*$}$$

with the Dirichlet condition

$$v(s,t) = 0,\ w(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbb{R}_+. \tag{2}$$

Here $v = v(s,t)$ and $w = w(s,t)$ are vector functions characterizing the concentrations of reagents, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}_+$. Let $\mathfrak{U}_M = (\overset{\circ}{W}{}^1_2(\Omega))^2$, $\mathfrak{U} = (L_2(\Omega))^2$, $\mathfrak{U}_N = (L_4(\Omega))^2$. Construct an auxiliary interpolation space $\mathfrak{X}_\alpha$ such that $\mathfrak{U}_M \hookrightarrow \mathfrak{X}_\alpha \hookrightarrow \mathfrak{U}_N \hookrightarrow \mathfrak{U}$. Let's consider three cases.

Case $\varepsilon_1 = 0$. The phase space of problem (1), (2) takes the form

$$\mathfrak{P}_{\varepsilon_1} = \{x \in \mathfrak{X}_\alpha : \langle \textstyle\sum_{i=1}^n \alpha_1 v_{s_i}, \xi_{s_i}\rangle = \langle \beta_1 - (\beta_2+1)v - v^2 w, \xi\rangle\}.$$

Theorem 1. [2] *Let $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$, $n \le 4$, $\beta_1 \in \mathbb{R}$, $\beta_2 \in \mathbb{R}\backslash\{\alpha_1\nu_k - 1\}$, where ν_k are the eigenvalues of the homogeneous Dirichlet problem for the operator $(-\Delta)$, then the set $\mathfrak{P}_{\varepsilon_1}$ at the point x_0 is a simple Banach C^∞-manifold.*

Theorem 2. [2] *Let $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$, $n \le 4$, $\beta_1 \in \mathbb{R}$, $\beta_2 = \alpha_1\nu_k - 1$, then the phase space $\mathfrak{P}_{\varepsilon_1}$ contains the Whitney 1-fold.*

Case $\varepsilon_2 = 0$. The phase space of problem (1), (2) takes the form

$$\mathfrak{P}_{\varepsilon_2} = \{x = (v,w) \in \mathfrak{X}_\alpha : \textstyle\sum_{i=1}^n \langle -\alpha_2 w_{s_i}, \eta_{s_i}\rangle + \langle v^2 w, \eta\rangle = \langle \beta_2 v, \eta\rangle\}.$$

Theorem 3. *Let $\alpha_1 \in \mathbb{R}\backslash\{0\}$, $\alpha_2 \in \mathbb{R}_-$, $\beta_1, \beta_2 \in \mathbb{R}$, $n \le 4$, then the set $\mathfrak{P}_{\varepsilon_2}$ at the point x_0 is a simple Banach C^∞-manifold.*

Case $\varepsilon_1 = \varepsilon_2 = 0$. The phase space of problem (1), (2) takes the form

$$\mathfrak{P}_{\varepsilon_1=\varepsilon_2=0} = \left\{ x \in \mathfrak{X}_\alpha : \left\{ \begin{array}{l} \sum_{i=1}^n \langle \alpha_1 v_{s_i}, \xi_{s_i}\rangle = \langle \beta_1 - (\beta_2+1)v - v^2 w, \xi\rangle, \\ \sum_{i=1}^n \langle -\alpha_2 w_{s_i}, \eta_{s_i}\rangle + \langle v^2 w, \eta\rangle = \langle \beta_2 v, \eta\rangle \end{array} \right. \right\}.$$

References

[1] R. Lefevr and I. Prigogin, Symmetry-breaking instabilities in dissipative system, *Journal of American Physics* (1968), no.48, 1965–1700.

[2] O.V. Gavrilova, Numerical study on the non-uniqueness of solutions to the Showalter–Sidorov problem for one degenerate mathematical model of an autocatalytic reaction with diffusion, *Journal of Computational and Engineering Mathematics* **6** (2019), no.4, 3–17.

Solvability of the nonlinear equation with the Dzhrbashyan – Nersesyan fractional derivatives

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Let $\mathcal{Z}$ be a Banach space, $\mathcal{L}(\mathcal{Z})$ be the Banach space of all linear bounded operators on $\mathcal{Z}$, $A \in \mathcal{L}(\mathcal{Z})$, D^{σ_n} be the Dzhrbashyan — Nersesyan fractional derivative, which is defined by a set of numbers $\{\alpha_k\}_0^n = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, $0 < \alpha_k \leq 1$, $k = 0, 1, \ldots, n \in \mathbb{N}$. Denote by Z an open set in $\mathbb{R} \times \mathcal{Z}^n$, the operator $B : Z \to \mathcal{Z}$ is nonlinear, generally speaking.

Consider the initial value problem for nonlinear equation

$$D^{\sigma_n} z(t) = Az(t) + B(t, D^{\sigma_0} z(t), D^{\sigma_1} z(t), \ldots, D^{\sigma_{n-1}} z(t)), \tag{1}$$

$$D^{\sigma_k} z(t_0) = z_k, \quad k = 0, 1, \ldots, n-1. \tag{2}$$

A function $z \in C((t_0, t_1]; \mathcal{Z})$ is called a solution of problem (1), (2) on $(t_0, t_1]$, if $D_t^{\sigma_k} z \in C([t_0, t_1]; \mathcal{Z})$, $k = 0, 1, \ldots, n-1$, $D_t^{\sigma_n} z \in C((t_0, t_1]; \mathcal{Z})$, the elements $(t, D^{\sigma_0} z(t), D^{\sigma_1} z(t), \ldots, D^{\sigma_{n-1}} z(t))$ belong to the set Z for all $t \in (t_0, t_1]$, equality (1) is satisfied and conditions (2) are valid.

Lemma 1. *Let $A \in \mathcal{L}(\mathcal{Z})$, $z_k \in \mathcal{Z}$, $0 < \alpha_k \leq 1$, $k = 0, 1, \ldots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $B \in C(Z; \mathcal{Z})$, $(t_0, z_0, \ldots, z_{n-1}) \in Z$. Then function $z \in C((t_0, t_1]; \mathcal{Z})$, such that $D^{\sigma_k} z \in C([t_0, t_1]; \mathcal{Z})$, $k = 0, 1, \ldots, n-1$, is a solution of problem* (1), (2) *on $(t_0, t_1]$, if and only if for $t \in (t_0, t_1]$*

$$z(t) = \sum_{k=0}^{n-1} (t - t_0)^{\sigma_k} E_{\sigma_n, \sigma_k + 1}((t - t_0)^{\sigma_n} A) z_k +$$

$$+ \int_{t_0}^{t} (t - s)^{\sigma_n - 1} E_{\sigma_n, \sigma_n}((t - s)^{\sigma_n} A) B(s, D^{\sigma_0} z(s), D^{\sigma_1} z(s), \ldots, D^{\sigma_{n-1}} z(s)) ds.$$

Theorem 1. *Let $A \in \mathcal{L}(\mathcal{Z})$, $z_k \in \mathcal{Z}$, $0 < \alpha_k \leq 1$ for $k = 0, 1 \ldots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, Z be open set in $\mathbb{R} \times \mathcal{Z}$, $B \in C^2(Z; \mathcal{Z})$. Then for each $(t_0, z_0, \ldots, z_{n-1}) \in Z$ there exists an unique solution of problem* (1), (2) *on $(t_0, t_1]$ at some $t_1 > t_0$.*

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References

[1] M.V. Plekhanova, E.M. Izhberdeeva, Local unique solvability of a quasilinear equation with the Dzhrbashyan – Nersesyan derivatives, *Lobachevskii Journal of Mathematics*, 2022, to appear.

Nonlinear Elliptic Equations of Nonstrictly Divergent Form

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We consider nonlinear elliptic equations and systems of the form

$$\operatorname{div}^t A(x, D^s u) = f(x)$$

in $\mathbb{R}^n$, $s+t$ is even. We assume that A satisfies the structure condition

$$|B(x,\xi) - B(x,\eta)| \le |\xi - \eta|,$$

B defined by $\Delta^{(s+t)/2}u + \operatorname{div}^t B(x, D^s u) = \kappa \operatorname{div}^t A(x, D^s u)$, $\kappa > 0$ be appropriate normalizing multiplier (in plain words, 'perturbation of poly-Laplacian should be Lipschitz-continuous with constant 1'), and $A(x,0) = 0$. Under the stronger condition $|B(x,\xi) - B(x,\eta)| \le K|\xi - \eta|$, $K < 1$, the operator be coercive in pair with $\Delta^{(s-t)/2}u$ in H^s. In case $s = t$, the condition with $K < 1$ coincides with standard structure conditions for divergent equations and systems; in case of a single nondivergent ($t = 0$) equation – with Cordes condition. Whereas the condition with $K = 1$ allows degeneration of ellipticity, e.g. the operator $\Delta^m u + |D^{2m}u|$ as well as the linear operator $\Delta^{(s+t)/2}u - \partial_{x_1}^{s+t}u$ satisfies it.

The nonstrictly divergent case $s \ne t$ (in contrast to the strictly divergent case $s = t$) allows to establish some estimates of solutions even under degenerate structure condition:

$$\|D^{s-1}u\|_{a-2} \le c_a \, \|I_{t-1}f\|_{a+2}$$

for some range of $a \in (a_*, 0)$ for $s > t$ and $a \in (0, a^*)$ for $s < t$; here $\|\cdot\|_a$ be a norm in $L_2(\mathbb{R}^n)$ with power weight $(1+|x|)^a$, I_t be a Riesz potential of order t.

We will discuss existence and uniqueness results in various settings under more or less strong restrictions: for solutions in sense of integral identity; in sense of maximal monotone extension; in sense of generalized pseudomonotonicity of Browder-Hess type.

On the stability of the stochastic Hoff equation

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The Hoff equation

$$(\lambda + \Delta)\dot{u} = \alpha u + \beta u^3 \tag{1}$$

is a model of buckling of an I-beam from the equilibrium position. Here the parameter $\lambda \in \mathbb{R}_+$ is the parameter responsible for the load applied to the beam and parameters $\alpha,\ \beta \in \mathbb{R}$ are parameters responsible for the material from which the beam is made.

The stability of solutions of the equation (1) in a neighborhood of the point zero is devoted in [1], which shows the existence of stable and unstable invariant manifolds. Consider the stochastic analogue of the equation (1). Let $\mathfrak{U} = \overset{\circ}{W}{}_2^1$, $\mathfrak{F} = W_2^{-1}$ and $\{\nu_k\}$ be a sequence of eigenvalues of the Laplace operator numbered by non-increment taking into account multiplicity. We construct the spaces of random $\mathbf{K}$-values $\mathbf{U_K L}_2$, $\mathbf{F_K L_2}$ and spaces of differentiable "noise" $\mathbf{C}^l\mathbf{U_K L}_2$, $l \in \{0\} \bigcup \mathbb{N}$. Let $\mathbf{K} = \{\lambda_k\}$ be a sequence such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. Operators L, M and N are defined by formulas

$$L : \eta \to (\lambda + \Delta)\eta,\ M : \eta \to \alpha\Delta\eta,\ N : \eta \to \beta\eta^3,\ \eta \in \mathbf{U_K L}_2.$$

Then the stochastic analogue of the Hoff equation (1) is represented as an equation

$$L \overset{o}{\eta} = M\eta + N(\eta). \tag{2}$$

Here, $\overset{o}{\eta}$ denotes the Nelson – Gliklikh derivative of the stochastic process $\eta = \eta(t)$.

Theorem. [2] *Let $\alpha,\ \beta,\ \lambda \in \mathbb{R}_+$.*

(i) If $\lambda \leq -\nu_1$ then the equation (2) has only a stable invariant manifold that coincides with $\mathbf{M_K L}_2$;

(ii) If $-\nu_1 < \lambda$ then there are a finite-dimensional unstable invariant the manifold $\mathbf{M}^+_{\mathbf{K}}\mathbf{L}_2$ and an infinite-dimensional stable invariant manifold $\mathbf{M}^-_{\mathbf{K}}\mathbf{L}_2$ of the equation (2) in the neighborhood of point zero.

References

[1] G.A. Sviridyuk and O.G. Kitaeva, Invariant Manifolds of the Hoff Equation, *Mathematical Notes* **79** (2006), no.3, 408–412.

[2] O.G. Kitaeva, Invariant Manifolds of the Hoff Model in “Noise”, *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software* **14** (2021), no.4, 24–35.

Improved Poincaré inequality for a sublinear embedding

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Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with $C^{1,\alpha}$-boundary $\partial\Omega$. Poincaré inequality states that the best constant for embedding of $W_0^{1,2}(\Omega)$ into $L_2(\Omega)$ is $\frac{1}{\lambda_1}$, where λ_1 is the first eigenvalue. The equality is achieved only on multiples of the first eigenfunction. A well-known refinement of Poincaré inequality is that the gap in Poincaré inequality can be estimated rom below:

$$\int_\Omega |\nabla u|^2 dx - \lambda_1 \int_\Omega u^2 dx \geqslant (\lambda_2 - \lambda_1) \int_\Omega |\nabla u^\perp|^2 dx,$$

where $u^\perp = u - u^\parallel \varphi_1$, φ_1 is the normalized first eigenfunction, and $u^\parallel = \int_\Omega \varphi_1 u \; dx$. Similar statement for embedding of $W_0^{1,p}(\Omega)$ into $L_p(\Omega)$ for $p \in (2, +\infty)$ was proven in [1].

We extend these results to embedding of $W_0^{1,p}$ into L_q for $1 \leqslant q < p < \infty$.

Theorem 1. *Let $2 \leq q < p < \infty$, and let Ω be of class $C^{1,\alpha}$.*

Let

$$\lambda_1 = \min_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^q dx\right)^{\frac{p}{q}}}$$

and let φ_1 be a normalized minimizer of this Rayleigh quotient, i.e. $\int_\Omega |u|^q dx = 1$.

Let $u^\parallel = \int_\Omega |\varphi_1|^{q-2} \varphi_1 u \; dx$ and $u^\perp = u - u^\parallel \varphi_1$.

Then there exists $C > 0$ such that

$$\|\nabla u\|_p^p - \lambda_1 \|u\|_q^p \geq C \left(|u^\parallel|^{p-2} \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla u^\perp|^2 \, dx + \int_\Omega |\nabla u^\perp|^p \, dx \right)$$

for any $u \in W_0^{1,p}(\Omega)$.

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References

[1] J. Fleckinger-Pellé and P. Takáč, An improved Poincaré inequality and the p-Laplacian at resonance for $p > 2$, *Advances in Differential Equations* **7** (2002), no.8, 951–971.

Error control for approximate solutions of 4th-order elliptic equations with reaction cofficients having large jumps

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There are numereous papers on the a posteriori error bounds for the conform, mixed and other types FEM's (finite element methods) for the equation $\Delta^2 u + \kappa^2 u = f$. However, these bounds are still in the process of improvement and simplification whereas some special cases of the problem were attended only quite recently [1, 2]. We derive computable a posteriori error bounds, which are not improvable in the orders of accuracy, robust in the wide range of the reaction coefficients $\sigma = \kappa^2$, and applicable as to conform FEM's for the primal formulation of the problem so to the Ciarlet-Raviart mixed methods. At that, σ can have considerable jumps and, in particular, it can be an element wise constant function taking any bounded positive value on each finite element. The bounds under consideration contain free functions in their right parts, called sometimes testing functions, which at the evaluation of an a posteriori bound admit an efficient choice from relatively wide range. For their derivation, it was implemented the technique similar to the one introduced in [3], which, in turn, comes back to J.-P. Aubin's technique [4], upgraded in [1]–[3] with the purpose of improving coefficients before the norms in the right parts of the bounds. Some of the derived a posteriori error bounds are supplemented with the corresponding bounds of efficiency.

References

[1] V.G. Korneev, A note on a posteriori error bounds for numerical solutions of elliptic equations with piece wise constant reaction coefficient having large jumps, *Computational Mathematics and Mathematical Physics* **60** (2020), no.11, 1754–1760.

[2] V.G. Korneev, On Error Control at Numerical Solution of Forth Order Elliptic Equations with Strongly Discontinuous Reaction Coefficient, *Mesh Methods for Boundary-Value Problems and Applications*, Lecture Notes in Computational Science and Engineering **141** (2022).

[3] V.G. Korneev, On the accuracy of a posteriori functional error majorants for approximate solutions of elliptic equations, *Doklady Mathematics* **96** (2017), no.1, 380–383.

[4] J.-P. Aubin. *Approximation of elliptic boundary-value problems.* Wiley-Interscience, 1972.

The Aleksandrov-Bakelman type maximum principle for elliptic equations on a "book" type stratified set

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The Aleksandrov-Bakelman maximum principle is a well-known approach for achieving a priori estimates for solutions of nondivergence type second order elliptic equations. The first investigations on this approach were introduced by A.D. Aleksandrov in [1], [2] and I.Ya. Bakelman in [3]. Further researches of this topic have been made by many authors. See e.g. recent servey [4].

In the last decades studies of partial differential equations and boundary-value problems on complicated structures have been gaining their popularity. In particular we notice equations on so called *stratified set* which is a cell complex with specific properties, see for example [5].

We discuss a local Aleksandrov-Bakelman type estimate on a “book” type stratified set. The talk is based on a joint work with A.I.Nazarov.

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References

[1] A.D. Aleksandrov, Certain estimates for the Dirichlet problem, *Dokl. Akad. Nauk SSSR* **134** (1960), 1001–1004 [Russian]; English transl. in *Soviet Math. Dokl.* **1** (1961), 1151–1154.

[2] A.D. Aleksandrov, Uniqueness conditions and bounds for the solution of the Dirichlet problem, *Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom.* **18** (1963), no.3, 5–29 [Russian].

[3] I.Ya. Bakelman, On the theory of quasilinear elliptic equations, *Sibirsk. Mat. Z.* **2** (1961), 179–186 [Russian].

[4] A.I. Nazarov and D.E. Apushkinskaya, Around the normal derivative lemma, *Uspekhi Mat. Nauk* **77** (2022), no. 2(464), 3–68 [Russian].

[5] S.N. Oshchepkova and O.M.Penkin, The mean-value theorem for elliptic operators on stratified sets, *Mathematical Notes* **81** (2007), 365–372.

Guaranteed error estimates for approximate solutions of the problem with an obstacle for the p-Laplace operator

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We consider an elliptic variational inequality, that arises in a problem with an obstacle for a nonlinear p-Laplace operator ($p > 1$). This free boundary problem reduces to minimizing the energy functional

$$J[v] = \int\limits_{\Omega} (\frac{1}{p} |\nabla v|^p - fv) \, \mathrm{dx} \tag{1}$$

on a closed convex set $\mathbb{K} = \left\{ v \in W_p^1(\Omega) : v|_{\partial\Omega} = 0, v \geq \phi \text{ in } \Omega \right\}$. Here Ω is a bounded domain in the space $\mathbb{R}^n$, $f \in L^q(\Omega)$ ($\frac{1}{p} + \frac{1}{q} = 1$), and the function ϕ is a sufficiently smooth obstacle function from the $C^{\max\{2,p\}}$ space.

One of the line of research into problems with free boundaries is the derivation of estimates for the deviation of the exact solution from the approximate one (see [1]), and this issue is considered here.

Based on the general dual theory for convex variational problems, for the formulated problem (1) the main error identity was obtained. The left side of this identity contains non-negative quantities, which are a refinement for the full measure of the deviation of the approximate solutions from the exact solutions of the primal and dual problems. The right side of the resulting identity is a fully computable expression. This identity allows us to estimate the quality of the approximate solution without knowing the exact one, but it contains restrictions on dual variables. For the superquadratic case ($p \geq 2$), an extension of the admissible set for approximations of the dual problem was obtained. With such an extension, the required estimate of the deviation is expressed not in terms of an identity, but in terms of a fully computable majorant, which also depends on known functions and fully computable expressions.

Main results of this work were obtained under supervision of D.E. Apushkinskaya.

References

[1] D.E. Apushkinskaya and S.I. Repin, Biharmonic obstacle problem: guaranteed and computable error bounds for approximate solutions, *Computational Mathematics and Mathematical Physics* **60** (2020), no.11, 1823–1838.

Investigation of start control problem for one nonlinear model of filtration

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In the cylinder $Q = \Omega \times [0, T]$ consider the Showalter–Sidorov condition

$$(\lambda - \Delta)(x(s, 0) - u(s)) = 0, \ s \in \Omega, \tag{1}$$

and the Dirichlet condition

$$x(s, t) = 0, \ (s, t) \in \partial\Omega \times [0, T], \tag{2}$$

for equation

$$(\lambda - \Delta)x_t - \operatorname{div}(|\nabla x|^{p-2}\nabla x) = y, \quad \lambda \in \mathbb{R}. \tag{3}$$

The problem (1)–(3) describes the process of changing the concentration potential of a viscoelastic fluid filtering in a porous medium (the process of nonlinear diffusion of a substance) [1] and forms a mathematical model of nonlinear diffusion. As the control space, we take the space $\mathfrak{U} = \overset{0}{W}{}^1_p(\Omega)$ and fix in it a non-empty, closed, and convex subset $\mathfrak{U}_{ad} \subset \mathfrak{U}$. Consider the start control problem of solutions (1)–(3)

$$J(x, u) = \vartheta \|x - x_f\|^p_{L_p(0,T;\overset{0}{W}{}^1_p(\Omega))} + (1 - \vartheta)\|u\|^p_{\overset{0}{W}{}^1_p(\Omega)} \to \inf, \ \vartheta \in (0, 1), \tag{4}$$

$x_f = x_f(s, t)$ is fix state of the system to be achieved with minimal initial impact u. The pair $(\hat{x}, \ \hat{u}) \in L_p(0, T; \overset{0}{W}{}^1_p(\Omega)) \times \mathfrak{U}_{ad}$ called a solution (1)–(4), if $J(\hat{x}, \hat{u}) = \inf\limits_{(x,u)} J(x, u)$. Let $\{\lambda_k\}$ is the sequence of eigenvalue of the operator $(-\Delta)$ in the Sturm–Liouville problem with the homogeneous Dirichlet condition.

Theorem 1. *For all fixed values of the parameter $\lambda \geq -\lambda_1$ and for all $u \in \overset{0}{W}{}^1_p(\Omega)$, $T \in \mathbb{R}_+$, $y \in L_q(0, T; W_q^{-1}(\Omega))$ exists weak generalized solution there exists a unique weak generalized solution $x \in L_\infty(0, T; \operatorname{coim} L) \cap L_p(0, T; \overset{0}{W}{}^1_p(\Omega))$ problem* (1)–(3).

Theorem 2. *For all fix parameter values $\lambda \geq -\lambda_1$, $T \in \mathbb{R}_+$, $y \in L_q(0, T; W_q^{-1}(\Omega))$ exist solution $(\hat{x}, \hat{u})$ problem* (1)–(4).

References

[1] P.L. Davis, A Quasilinear Parabolic and a Related Third Order Problem, *Journal of Mathematical Analysis and Applications,* **40** (1971), no.2, 327–335.

Construction of approximate solutions to an elliptic variational problem with a thin obstacle

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We consider elliptic variational inequalities generated by obstacle type problems with thin obstacles.

This problem with a free boundary is reduced to minimizing the functional

$$J\left[v\right] = \frac{1}{2}\int\limits_{\Omega} \left|\nabla v\right|^2 \mathrm{dx}$$

on a closed convex set $\mathbb{K} = \left\{v \in H^1\left(\Omega\right) : \; v \geq \psi \text{ on } \mathcal{M} \cap \Omega, \; v = \varphi \text{ on } \partial\Omega\right\}$. Here Ω is an open, connected, and bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, $\mathcal{M}$ is smooth one – dimensional manifold in $\mathbb{R}^2$, which divides Ω into two Lipschitz subdomains Ω_- and Ω_+. The given functions $\psi : \mathcal{M} \to \mathbb{R}$ is a sufficiently smooth and $\varphi \in H^{1/2}(\partial\Omega)$. For this class of problems, we construct an approximate solution.

An algorithm was developed to find an approximate solution. The algorithm consists in iterative application of piecewise affine approximation and coordinate relaxation method. It was implemented in the Python programming language and an approximate solution was built using the matplotlib library. Comparing the exact and approximate solutions, we were convinced of a fairly good accuracy (Fig.1).

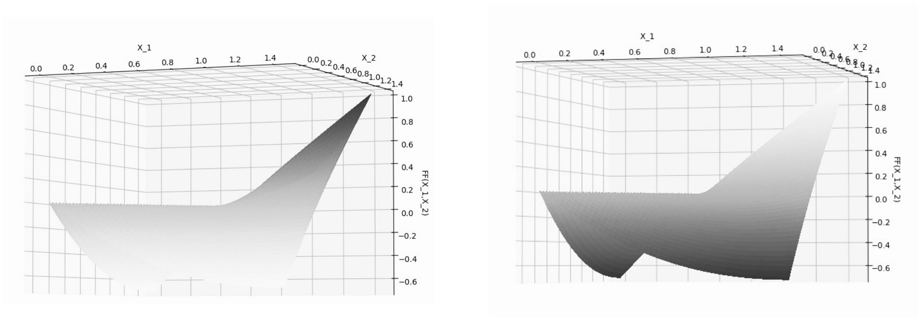

Figure 1: Approximate and exact solutions

Main results of this work were obtained under supervision of D.E. Apushkinskaya.

Dissipation ratio dependent solutions of the Riemann problem for the chemical flooding conservation laws system

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We study the solutions of the Riemann problem arising in chemical flooding models. Vanishing viscosity admissibility criterion is used to distinguish physically meaningful weak solutions. We demonstrate that when the flow function depends non-monotonically on the chemical agent concentration, non-classical undercompressive shocks appear. We prove the monotonic dependence of the shock velocity on the ratio of dissipative coefficients. For that purpose we classify the phase portraits for the travelling wave dynamical systems based on their nullcline configuration and study the saddle-saddle connections.

Joint work by F. Bakharev, A. Enin, Yu. Petrova.

References

[1] F. Bakharev, A. Enin, Yu. Petrova and N. Rastegaev, Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. Preprint arXiv:2111.15001

On a Sobolev-type stochastic equation in spaces of differential forms given on a Riemannian manifold without boundary

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We consider the Ginzburg – Landau equation from the phenomenological theory of superconductivity. This equation belongs to the Sobolev-type equations. To study the solvability of the Cauchy or Showalter–Sidorov problems for this type of equations, we use the Sviridyuk phase space method. The equation is considered in the space of smooth differential forms defined on a smooth compact oriented connected Riemannian manifold without boundary. As operators, generalizations of ordinary operators into the space of differential forms are used, such as Laplace – Beltrami instead of the standard Laplace. Moreover, the coefficients of these forms are Wiener stochastic processes, differentiable in the sense of Nelson – Glicklich.

References

[1] O.G. Kitaeva, D.E. Shafranov and G.A. Sviridyuk, Degenerate holomorphic semigroups of operators in spaces of K-'noises' on Riemannian manifolds, *Semigroups of Operators – II Theory and Applications SOTA 2018. Springer Proc. Math. Stat., Cham: Springer* **325** (2020), 279–292.

[2] D.E. Shafranov, O.G. Kitaeva and G.A. Sviridyuk, Stochastic equations of Sobolev-type with relatively p-radial operators in spaces of differential forms, *Differential equations* **57** (2021), no.4, 526–535.

Singularities of solutions to the equations of an isentropic gas flow and singularities of solutions to the linear wave equation

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It is shown that the catastrophe germs of smooth mappings that define all three typical features of solutions of the system of equations of a one-dimensional isentropic gas flow, which are typical in the sense of the mathematical theory of catastrophes, coincide with germs corresponding to similar singularities of solutions of the linear wave equation with constant coefficients. A hypothesis is put forward that a similar inheritance should take place for typical singularities of solutions of systems of equations of an isentropic gas in spatially non-one-dimensional cases as well.

References

[1] A.K. Rakhimov, Singularities of Riemannian invariants, *Functional Analysis and Its Applications* **27** (1993), no.1, 39–50.

[2] B.I. Suleimanov, Generic singularities in solutions of the shallow water equations , *Doklady Mathematics* **85** (2012), no.1, 125–128.

[3] B.I. Suleimanov and A.M. Shavlukov, A Typical dropping cusp singularity of solutions to equations of a one-dimensional isentropic gas flow , *Bulletin of the Russian Academy of Sciences: Physics* **84** (2020), no.5, 664–666.

Synchronization between delay chaotic systems via matrix projective approach

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In this article, matrix projective synchronization between delay chaotic systems has been studied. The disturbances are also introduced in chaotic systems to analyze the physical importance of systems. The sufficient conditions are discussed to achieve matrix projective synchronization. Further, numerical results agree with theoretical study.

Solvability of mixed control problems for the class of degone Nonlinear Equations with fractional derivatives

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In a reflexive Banach spaces $\mathcal{X}$, $\mathcal{Y}$, a continuous linear operator $L \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is given $\mathcal{U}$ is a Banach space, $M \in \mathcal{C}l(\mathcal{X}, \mathcal{Y})$ is a linear closed operator with domain D_M dense in $\mathcal{X}$, $B \in \mathcal{L}(\mathcal{U}; \mathcal{X})$, $N : \mathbb{R} \times \mathcal{X}^n \to \mathcal{Y}$ is a nonlinear operator, control space $\mathfrak{U} = L_q(t_0, T; \mathcal{U}) \times \mathcal{X}^m$ with the norm $\|(u,v)\|^2_{\mathfrak{U}} = \|u\|^2_{L_q(t_0,T;\mathcal{U})} + \|v\|^2_{\mathcal{X}^m}$. Consider the mixed control problem with controls $(u, v) \in \mathcal{U}_\partial$

$$LD_t^\alpha x(t) = Mx(t) + N(t, D_t^{\alpha_1} x(t), \dots, D_t^{\alpha_n} x(t)) + Bu(t), \quad t \in (t_0, T), \quad (1)$$

$$(Px)^{(k)}(t_0) = v_k, \ k = 0, 1, \dots, m-1, \quad (2)$$

$$(u, v) = (u, v_0, v_1, \dots, v_{m-1}) \in \mathcal{U}_\partial, \quad (3)$$

$$J(x, u, v) \to \inf, \quad (4)$$

where $0 \le \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m - 1 < \alpha \le m$, $m \in \mathbb{N}$, D_t^β is the Gerasimov — Caputo derivative, $k = 0, 1, \dots, m-1$, $\mathcal{U}_\partial$ is a set of admissible controls, $\mathcal{U}_\partial \subset \mathfrak{U}$, J is the cost functional.

An operator M is said to be $(L, 0)$-bounded, if L-spectrum $\sigma^L(M)$ of operator M is bounded, then $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$. The initial data are given only for the projector of unknown function on the subspace $\mathcal{X}^1 = \mathrm{im}P$ without degeneration.

A strong solution of (1), (2) is a function in the space

$$\mathcal{Z}_{\alpha,q}(t_0, T; \mathcal{X}) := \{x \in L_q(t_0, T; D_M) \cap C^{m-1}([t_0, T]; \mathcal{X}) : \\ J_t^{m-\alpha}\left(x - \sum_{k=0}^{m-1} x^{(k)}(t_0)\tilde{g}_{k+1}\right) \in W_q^m(t_0, T; \mathcal{X})\}.$$

The paper proves that problem (1)–(4) has a unique solution $(\hat{x}, \hat{u}, \hat{v}) \in \mathcal{Z}_{\alpha,q}(t_0, T; \mathcal{X}) \times \mathcal{U}_\partial$ under the condition $N(t, z_1, \dots, z_n) = N_1(t, Pz_1, \dots, Pz_n)$.

The work was supported by RFBR and VANT grant 21-51-54003.

References

[1] M.V. Plekhanova, Degenerate distributed control systems with fractional time derivative, *Ural Mathematical Journal* (2016), 58–71.

[2] A.V. Fursikov, *Optimal control of distributed systems. Theory and Applications*, Scientific book, Novosibirsk, 1999.

A posteriori estimates for solutions to a problem with a thin obstacle for a second-order elliptic operator

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We study a variational problem with a thin obstacle generated by a second-order elliptic operator. It can be formulated as follows: minimize the functional

$$J(v) = \frac{1}{2} \int\limits_{\Omega} A(x) \nabla v \cdot \nabla v dx - \int\limits_{\Omega} f v dx$$

over the closed convex set

$$\mathbb{K} := \left\{ v \in \ H^1(\Omega) : \ v \geq \Psi \text{ on } \mathcal{M} \cap \Omega, \ v = \varphi \text{ on } \partial\Omega \ \right\}.$$

Here Ω is an open, connected, and bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, and $\mathcal{M}$ is a smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$, which divides Ω into two Lipschitz subdomains Ω_+ and Ω_-. The given function $\varphi : \partial\Omega \to \mathbb{R}$ is assumed to satisfy $\varphi \in H^{1/2}(\partial\Omega)$, while the given obstacle sufficiently smooth function $\Psi : \mathcal{M} \to \mathbb{R}$ should satisfy $\varphi \geq \Psi$ on $\mathcal{M} \cap \partial\Omega$. It is assumed that A is a symmetric matrix subject to the condition

$$A(x)\xi \cdot \xi \geq \nu |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n, \quad \nu > 0,$$

almost everywhere in Ω. Under the assumptions made, the unique solution $u \in \mathbb{K}$ exists.

We deduce the error identity for this class of free boundary problems. The left-hand side of this identity gives true form of the deviation between the exact solution u and arbitrary approximate solution $v \in \mathbb{K}$, while the right-hand side is a fully computable expression. In real life computations, this identity can be used to control the accuracy of approximate solutions.

Main results of this work were obtained under supervision of D.E. Apushkinskaya. Our results generalized the ones from [1] where the case of unit matrix A was studied.

References

[1] D.E. Apushkinskaya and S.I. Repin, Thin obstacle problem: estimates of the distance to the exact solution, *Interfaces Free Boundaries* **20** (2018), no.4, 511–531.

Sectorial Tuples of Operators and Fractional Multi-Term Linear Equations

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Let $\mathcal{Z}$ be a Banach space, $\mathcal{C}l(\mathcal{Z})$ is a set of all linear closed operators in $\mathcal{Z}$, which are densely defined in $\mathcal{Z}$, $m-1<\alpha\leq m\in\mathbb{N}$, $0<\alpha_1<\alpha_2<\dots<\alpha_n<\alpha$, $\beta_1>\beta_2>\dots>\beta_r\geq 0$. Denote $\underline{\alpha}:=\max\{\alpha_l : l\in\{1,2,\dots,n\},\ \alpha_l-m_l<\alpha-m\}$, $\overline{\alpha}:=\max\{\alpha_l : l\in\{1,2,\dots,n\},\ \alpha_l-m_l>\alpha-m\}$, $\underline{m}=\lceil\underline{\alpha}\rceil$, $\overline{m}=\lceil\overline{\alpha}\rceil$, $m^*:=\max\{\underline{m}-1,\overline{m}\}$ is the defect of the Cauchy type problem for multi-term equation [1]. Consider the linear homogeneous multi-term fractional equation

$$D_t^\alpha z(t)=\sum_{j=1}^{m-1}A_jD_t^{\alpha-m+j}z(t)+\sum_{l=1}^{n}B_lD_t^{\alpha_l}z(t)+\sum_{s=1}^{r}C_sJ_t^{\beta_s}z(t) \tag{1}$$

with initial conditions $D_t^{\alpha-m+k}z(0)=0,\ k=m^*,m^*+1,\dots,m-1$.

A tuple of operators $(A_1,A_2,\dots,A_{m-1},B_1,B_2,\dots,B_n,C_1,C_2,\dots,C_r)$ belongs to the class $\mathcal{A}_\alpha^{n,r}(\theta_0,a_0)$ for some $\theta_0\in(\pi/2,\pi)$, $a_0\geq 0$, if $D:=\bigcap_{j=1}^{m-1}D_{A_j}\cap\bigcap_{l=1}^{n}D_{B_l}\cap\bigcap_{s=1}^{r}D_{C_s}$ is dense in $\mathcal{Z}$; for all $\lambda\in S_{\theta_0,a_0}:=\{\mu\in\mathbb{C}: |\arg(\mu-a_0)|<\theta_0\}$, $p=0,1,\dots,m-1$, there exist bounded operators $R_p(\lambda):=$

$$\left(\lambda^\alpha I-\sum_{j=1}^{m-1}\lambda^{\alpha-m+j}A_j-\sum_{l=1}^{n}\lambda^{\alpha_l}B_l-\sum_{s=1}^{r}\lambda^{-\beta_s}C_s\right)^{-1}\left(I-\sum_{j=p+1}^{m-1}\lambda^{j-m}A_j\right);$$

for any $\theta\in(\pi/2,\theta_0)$, $a>a_0$, there exists $K(\theta,a)$ such that for all $\lambda\in S_{\theta,a}$, $p=0,1,\dots,m-1$, $\|R_p(\lambda)\|_{\mathcal{L}(\mathcal{Z})}\leq K(\theta,a)|\lambda-a|^{-1}|\lambda|^{1-\alpha}$.

Theorem. Let $m-1<\alpha\leq m\in\mathbb{N}$, $A_j\in\mathcal{C}l(\mathcal{Z})$, $j=1,2,\dots,m-1$, $\alpha_1<\alpha_2<\dots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leq m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $B_l\in\mathcal{C}l(\mathcal{Z})$, $l=1,2,\dots,n$, $\beta_1>\beta_2>\dots>\beta_r\geq 0$, $C_s\in\mathcal{C}l(\mathcal{Z})$, $s=1,2,\dots,r$. Then there exist analytic p-resolving operators families for equation (1) of type $(\theta_0,a_0,0)$, $p=m^*,m^*+1,\dots,m-1$ (of type $(\theta_0,a_0,m-\alpha)$ for $p=m^*=0$), if and only if $(A_1,A_2,\dots,A_{m-1},B_1,B_2,\dots,B_n,C_1,C_2,\dots,C_r)\in\mathcal{A}_\alpha^{n,r}(\theta_0,a_0)$.

The notion of p-resolving operators is described in [2].

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References

[1] V. E. Fedorov and M. M. Turov, The Defect of a Cauchy Type Problem for Linear Equations with Several Riemann — Liouville Derivatives, *Siberian Mathematical Journal* **62** (2021), no.5, 925–942.

[2] V. E. Fedorov, W.-S Du and M. M. Turov, On the Unique Solvability of Incomplete Cauchy Type Problems for a Class of Multi-Term Equations with the Riemann — Liouville Derivatives, *Symmetry* **14** (2022), no.2.

On the constancy of the extremal function in the embedding theorem of fractional order

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Let $n \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Assume that $s \in (0,1)$, $2^*_s := 2n/(n-2s)$ and

$$q \in \begin{cases} [1, 2^*_s] & \text{if } n \geq 2 \text{ or } n = 1 \text{ and } s < 1/2; \\ [1, \infty) & \text{if } n = 1 \text{ and } s = 1/2; \\ [1, \infty] & \text{if } n = 1 \text{ and } s > 1/2. \end{cases}$$

We consider the fractional embedding theorem $\mathcal{H}^s(\Omega) \hookrightarrow L_q(\Omega)$

$$\inf_{u \in \mathcal{H}^s(\Omega)} \mathcal{I}^{\Omega}_{s,q}[u] := \inf_{u \in \mathcal{H}^s(\Omega)} \frac{\|u\|^2_{\mathcal{H}^s(\Omega)}}{\|u\|^2_{L_q(\Omega)}} > 0. \tag{1}$$

For $q \in [1, 2^*_s)$, this embedding is compact, and the extremal in (1) exists. Moreover, in [2] it was shown that, for $q = 2^*_s$, the extremal in (1) exists in any $\mathcal{C}^2$ domain Ω for $n \geq 3$ and $2s > 1$.

The properties of extremals in (1) depend on the shape of the domain Ω, on its size, and on the norm of $\mathcal{H}^s(\Omega)$. We define

$$\|u\|^2_{\mathcal{H}^s(\Omega)} := \langle (-\Delta)^s_{Sp} u, u \rangle + \|u\|^2_{L_2(\Omega)}, \tag{2}$$

where the quadratic form $\langle (-\Delta)^s_{Sp} u, u \rangle$ is defined by

$$\langle (-\Delta)^s_{Sp} u, u \rangle := \sum_{j=1}^{\infty} \lambda_j^s \cdot (u, \phi_j)^2_{L_2(\Omega)}$$

with λ_j as eigenvalues and ϕ_j as orthonormal eigenfunctions of the Neumann Laplacian in Ω. The operator $(-\Delta)^s_{Sp}$ generated by (3), is called the *spectral Neumann fractional Laplacian.*

In this talk we discuss the problem of the constancy of the minimizer in (1). The simple fact here is that for $q \in [1, 2]$ such minimizer is constant and unique. For the interesting case $q > 2$ the answer depends on the domain size: for the family of domains $\varepsilon\Omega$, we prove that, for small dilation coefficients ε, the unique minimizer is constant, whereas for large ε, a constant function is not even a local minimizer. For the local case $s = 1$ similar effects were established in [1].

The research was supported by RFBR grant 20-01-00630.

References

[1] A.I. Nazarov and A.P. Scheglova, On some properties of extremals in a variational problem generated by the Sobolev embedding theorem, *Probl. Mat. Anal.* **27** (2004), 109–136.

[2] N.S. Ustinov, On solvability of a critical semilinear problem with the spectral Neumann fractional Laplacian, *Algebra i Analiz* **33** (2021), no.1, 194–212.

* * *

Remembering Olga Ladyzenskaya

Informal Talk

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March 7, 2022, we celebrated the centenary of the great Russian mathematician Olga A. Ladyzhenskaya (1922-2004). She played a crucial role in development of Leningrad (St. Petersburg) school of PDE and Mathematical Physics. In the second half of the XX century, this scientific school and Olga Ladyzhenskaya herself have mainly influenced the development of the theory of PDEs.

The story of Olga's life and diverse activities will be accompanied by a display of famous and less-known photographs from private collections.

References

[1] *Remembering Olga Ladyzhenskaya*, Internet-project available at https://pdmi.ras.ru/pdmi/en/memoirs/ladyzhenskaya

The Queen of Rhymes and the Queen of Formulas

Darya E. Apushkinskaya, Alexander I. Nazarov

Excursion to Komarovo cemetery (in Russian)

How often do the poets dedicate poems to mathematicians? During the excursion we will talk about the friendship between the great Russian poet of the twentieth century Anna Akhmatova and the prominent mathematician Olga Ladyzhenskaya, and about the history of Akhmatova's poem "In Vyborg".

References

[1] D.E. Apushkinskaya and A.I. Nazarov, *The Queen of Rhymes and the Queen of Formulas (around the poem "In Vyborg")*, preprint available at `https://www.researchgate.net/publication/337012058_The_Queen_of_Rhymes_and_the_Queen_of_Formulas_around_the_poem_In_Vyborg`