Improved Friedrichs inequality for a subhomogeneous embedding

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Jul. 21th, 2022

Main result

Let $p \geqslant q \geqslant 2$, and
let Ω be a bounded domain of class $C^{1,\alpha}$, $\alpha \in (0,1)$.
Then there exists $C = C(p,q,\Omega) > 0$ such that for any $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^p\,dx - \lambda_1 \left(\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}} \geqslant C\left(|u^\parallel|^{p-2}\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla u^\perp|^2\,dx + \int_\Omega |\nabla u^\perp|^p\,dx\right), \tag{1}$$

where:

- $u = u^\parallel \varphi_1 + u^\perp$, $\quad u^\parallel = \int_\Omega \varphi_1^{q-1}u\,dx \Big/ \int_\Omega \varphi_1^q\,dx \in \mathbb{R}$ and $\quad \int_\Omega \varphi_1^{q-1}u^\perp\,dx = 0$
- λ_1 is the "first eigenvalue", i.e.
$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\left(\int_\Omega |u|^q\right)^{\frac{p}{q}}} \tag{2}$$
- φ_1 is the "first eigenfunction", i.e. the minimizer of (1) (unique up to multiplication by real constant)

Steklov inequality

$$\int_\Omega |\nabla u|^2 \geqslant \lambda_1 \int_\Omega |u|^2,$$

provided that $u = 0$ on $\partial\Omega$.
λ_k is the k-th eigenvalue of $-\Delta$ with Dirichlet BC.

Refinement of Steklov inequality

Let φ_1 be the first eigenfunction of $-\Delta$ with Dirichlet BC. Then

$$\int_\Omega |\nabla u|^2 - \lambda_1 \int_\Omega |u|^2 \geqslant \frac{\lambda_2 - \lambda_1}{\lambda_2}\int_\Omega |\nabla u^\perp|^2,$$

where $u = u^\parallel\varphi_1 + u^\perp$ and $\int_\Omega u^\perp \varphi_1 = 0$.
Equivalently:

$$u^\parallel = \frac{\int_\Omega \varphi_1 u}{\int_\Omega \varphi_1^2}; \qquad u^\perp = u - u^\parallel \varphi_1.$$

Takáč (2002)

Let $p > 2$, and
let Ω be a bounded domain of class $C^{1,\alpha}$, $\alpha \in (0,1)$.
Then there exists $C = C(p,q,\Omega) > 0$ such that for any $u \in W_0^{1,p}(\Omega)$ (1) holds, where:

- $u = u^\parallel \varphi_1 + u^\perp$
- $\int_\Omega \varphi_1 u^\perp\,dx = 0$
- λ_1 is the "first eigenvalue", i.e.
$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$
- φ_1 is the "first eigenfunction"

Proving (1): general idea

Consider

$$\lambda_1 = \min_{u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$$

and

$$\lambda_2 = \min_{u\in W_0^{1,p}(\Omega);\quad u\perp\varphi_1} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}.$$

Steklov inequality can be refined if and only if

$$\lambda_2 > \lambda_1$$

.
The same idea applies to Friedrichs inequality ($p \geqslant q \geqslant 2$): refinement is possible if a certain quotient is strictly greater than λ_1.

Proving (1): rough sketch

We prove the inequality (1) in two cones:

$$\mathcal{C}_\gamma = \{u \in W_0^{1,p}(\Omega):\ \|\nabla u^\perp\|_p \leqslant \gamma |u^\parallel|\},$$

and

$$\mathcal{C}'_\gamma = \{u \in W_0^{1,p}(\Omega):\ \|\nabla u^\perp\|_p \geqslant \gamma |u^\parallel|\}.$$

Proof in $\mathcal{C}'_\gamma$

In this case u is "far" from $\mathbb{R}\varphi_1$,
so for any $\gamma > 0$ inequality (1) holds with some constant $C(\gamma)$.

However, $\lim_{\gamma\to 0} C(\gamma) = 0$.

Proof in $\mathcal{C}_\gamma$

In this case u is "close" to $\mathbb{R}\varphi_1$.

- Assume the contrary: quotient
$$\frac{\int_\Omega |\nabla u|^p\,dx - \lambda_1\left(\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}}}{|u^\parallel|^{p-2}\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla u^\perp|^2\,dx + \int_\Omega |\nabla u^\perp|^p\,dx}$$
can be arbitrarily close to 0 for some sequence u_n.
- W.l.o.g. assume that $u_n^\parallel = 1$ and thus $u_n = \varphi_1 + v_n$, where $\int_\Omega \varphi_1^{q-1}v_n dx = 0$.

Taylor expansion and "quadratic" form

Consider

$$J[u] = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\left(\int_\Omega |u|^q\right)^{\frac{p}{q}}.$$

Then

$$J[\varphi + v] = J[\varphi] + DJ[\varphi]v + \int_0^1 D^2J[\varphi + sv](v,v)(1-s)ds.$$

As $J[u] \geqslant 0$ and $J[\varphi_1] = DJ[\varphi_1] = 0$, we obtain that $Q_v(v,v) \geqslant 0$, where

$$\begin{aligned}
Q_u(v,v) &:= \int_0^1 D^2J[\varphi_1 + su](v,v)\,(1-s)\,ds \\
&= \int_0^1 \left(\int_\Omega \langle \mathrm{A}(\nabla\varphi_1 + s\nabla u)\nabla v, \nabla v\rangle\,dx\right)(1-s)\,ds \\
&- \lambda_1(q-1)\int_0^1 \left(\int_\Omega |\varphi_1 + su|^q\,dx\right)^{\frac{p-q}{q}}\left(\int_\Omega |\varphi_1 + su|^{q-2}v^2\,dx\right)(1-s)\,ds \\
&-\lambda_1(p-q)\int_0^1 \left(\int_\Omega |\varphi_1 + su|^q\,dx\right)^{\frac{p-2q}{q}}\left(\int_\Omega |\varphi_1 + su|^{q-2}(\varphi_1 + su)v\,dx\right)^2(1-s)\,ds = \\
&= \mathcal{P}_u^1(v,v) - \lambda_1\mathcal{P}_u^0(v,v).
\end{aligned}$$

Quotient of quadratic forms

Under our asumptions

$$\lambda_1 = \lim_{n\to\infty} \frac{\mathcal{P}_{v_n}^1(v_n,v_n)}{\mathcal{P}_{v_n}^0(v_n,v_n)} = \lim_{n\to\infty} \frac{\mathcal{P}_{v_n}^1(t_nv_n,t_nv_n)}{\mathcal{P}_{v_n}^0(t_nv_n,t_nv_n)}.$$

We can choose t_n to be such that $\mathcal{P}_{v_n}^0(t_nv_n,t_nv_n) = 1$ and denote $w_n = t_nv_n$.

Passing to the limit

Observe that (up to subsequence) $v_n \to 0$ strongly in $W_0^{1,p}(\Omega)$. We also prove that $w_n \to w \neq 0$. Then

$$\lambda_1 = \lim_{n\to\infty} \frac{\mathcal{P}_{v_n}^1(w_n,w_n)}{\mathcal{P}_{v_n}^0(w_n,w_n)} \geqslant \frac{\mathcal{P}_0^1(w,w)}{\mathcal{P}_0^0(w,w)}.$$

The last quotient is equal to λ_1 if and only if $w = t\varphi_1$. This contradicts the orthogonality condition.