On p-Laplacian with rapidly changing boundary conditions

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Denote by $D \subset \mathbb{R}^2$ the domain

$$\{(x,y)\ :\ 0<x<1,\ 0<y<1\}$$

and let $F \subset \partial D$. Let $W_p^1(D,F)$ be a completion of the set of smooth functions vanishing in a neighbourhood of F by the norm

$$\| u \|_{W_p^1(D,F)} = \left(\int\limits_D |v|^p\, dx + \int\limits_D |\nabla v|^p\, dx \right)^{1/p}.$$

Denoting $G = \partial D \backslash F$, we consider the Zaremba problem

$$\begin{gathered} \Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) = l \quad \text{in} \quad D, \\ u = 0 \text{ on } F, \quad \frac{\partial u}{\partial n} = 0 \text{ on } G, \end{gathered} \tag{1}$$

where $\frac{\partial u}{\partial n}$ is an outer normal derivative and l is a linear functional on $W_p^1(D,F)$.

By the Hahn–Banach theorem

$$l(\varphi) = \sum_{i=1}^{n} \int_D f_i \varphi_{x_i} \, dx,$$

where $f_i \in L_{p'}(D)$, $p' = p/(p-1)$.

The function u is a solution to problem (1), if

$$\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = -\int_D f \cdot \nabla \varphi \, dx$$

for any $\varphi \in W_p^1(D, F)$.

Define the capacity $C_q(K)$, $1 < q < 2$ of a compact $K \subset \mathbb{R}^2$ by

$$C_q = \inf \left\{ \int_{\mathbb{R}^2} |\nabla \varphi|^q \, dx : \ \varphi \in C_0^\infty(\mathbb{R}^2), \ \varphi \geq 1 \text{ on } K \right\}.$$

Denote by $B_r^{x_0}$ an open circle of the radius r, centered in x_0, and by $mes_1(E)$ the linear measure of $E \subset \partial D$.

I. If $1 < p \leq 2$, then for $q = (p+1)/2$ and for an arbitrary point $x_0 \in F$ and $r \leq 1$ we have either

$$C_q(F \cap \overline{B}_r^{x_0}) \geq c_0 r^{2-q}, \tag{2}$$

or

$$mes_1(F \cap \overline{B}_r^{x_0}) \geq c_0 r. \tag{3}$$

II. If $p > 2$, then $F \neq \emptyset$.

The following **Theorem** is valid:

If $f \in (L_{p'+\delta_0}(D))^2$, where $\delta_0 > 0$, then there exists a positive constant $\delta < \delta_0$, such that for a solution to (1) the following estimate

$$\int_D |\nabla u|^{p+\delta}\, dx \leq C \int_D |f|^{p'(1+\delta/p)}\, dx$$

holds, where C depends only on p, δ_0 and c_0 from (2) and (3) as $1 < p \leq 2$. And if $p > 2$ the constant C depends only on p and δ_0.