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A posteriori estimares on solutions for a fourth order elliptic problem with obstacle

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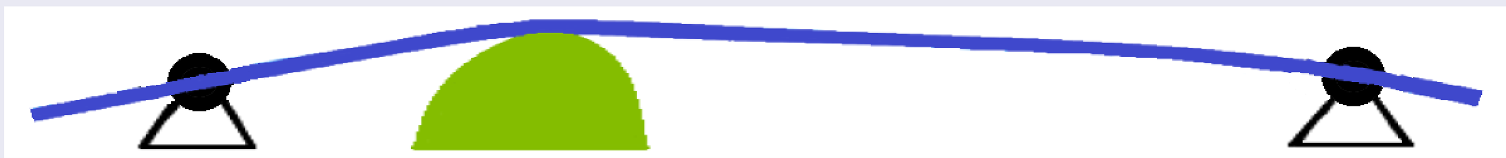
Based on joint work with Darya Apushkinskaya

Problem

$$J(v) = \int_\Omega \left(\frac{1}{2}|\Delta v|^2 - fv\right) dx \to \min_{\mathbb{K}_1} \qquad (1)$$

- $\Omega \in \mathbb{R}^d$ — bounded connected domain with $\partial\Omega \in$ Lip,
- $\mathbb{K}_1 = \left\{ v \in W_2^2(\Omega) : v\big|_{\partial\Omega} = 0,\ \ v \geqslant \varphi \text{ a.e. in } \Omega \right\}$,
- $f \in L_2(\Omega)$ — given function,
- $\varphi \in C^2(\overline{\Omega})$, $\varphi \leqslant 0$ on $\partial\Omega$ — obstacle (given).

Application: study of the frictionless equilibrium contact for free-supported elastic beams over a rigid obstacle.



Research

Theoretical part

Known: problem (1) has* a unique solution u (minimizer) such that $\Delta^2 u \geq f$, $u \geq \varphi$, $(\Delta^2 u - f)(u - \varphi) = 0$ a.e. in Ω.

Open: ¿optimal regularity of the solution?

Numerical part

Function v that satisfies boundary conditions and restrictions generated by the obstacle we call an approximation.

Open: ¿how good is the approximation?

References

1. D.E. Apushkinskaya and S.I. Repin, Biharmonic obstacle problem: guaranteed and computable error bounds for approximate solutions, *Comp. Math. and Math. Phys.*, **60** (2020), no.11, 1823–1838.

The very first result

We use spaces

- $V = \left\{ v \in W_2^2(\Omega), w\big|_{\partial\Omega} = 0 \right\}$,
- $Y = Y^* = L_2(\Omega, M_{Sym}^{d\times d})$ — space of symmetrical matrices $(d \times d)$ with elements from $L_2(\Omega)$ equipped with the norm $||y^*||_M^2 = (y^*, y^*)_M = \int_\Omega \sum_{i,j} y_{ij}^* \cdot y_{ij}^*\, dx$,
- $Y_f^* = \left\{ y^* \in Y^* : (y^*, \nabla\nabla w)_M = (f, w) \quad \forall w \in V \right\}$.

We denote

- $m(v)$ — distance between the minimizer u and approximation v,
- p^* and y^* — the exact solution of problem dual to (1) and its approximation, respectively,
- $m^*(y^*)$ — distance between p^* and y^*.

Theorem

For any $v \in \mathbb{K}_1$ and for any $y^* \in Y_f^*$ holds the equality

$$m(v) + m_\varphi(v) + m^*(y^*) = \frac{1}{2}||\nabla\nabla v - y^*||_M^2, \qquad (2)$$

where $m_\varphi(v) \geq 0$ is a nonlinear adjustment term.

The proof is based on the **error identity method**.

Sketch of the proof. 1. Preliminaries

Represent problem (1) as $J(v) = G(\Lambda v) + F(v) \to \min_{\mathbb{K}_1}$.

- $\Lambda : V \to Y$, $\ G : Y \to \mathbb{R}$, and $\ F : V \to \mathbb{R} \cup \{+\infty\}$,
- $\Lambda = \nabla\nabla$, $\ G(h) = \frac{1}{2}||h||_M^2$, $F = -(f, v) + \chi_{\mathbb{K}_1}(v)$, where $\chi_{\mathbb{K}_1}(v) = 0$ if $v \in \mathbb{K}_1$, $\chi_{\mathbb{K}_1}(v) = +\infty$ if $v \notin \mathbb{K}_1$.

The dual problem reduces to $I^*(y^*) = -G^*(y^*) \to \max_{Y_f^*}$,

with $G^* : Y_f^* \to \mathbb{R}$, $G^*(y^*) = \frac{1}{2}||y^*||_M^2$ due to the integral condition in Y_f^*.

Sketch of the proof. 2. Applying the method

Consider **compound functionals**

- $D_G(\Lambda v, p^*) = G(\Lambda v) + G^*(p^*) - (\Lambda v, p^*)$,
- $D_F(v, -\Lambda^* p^*) = F(v) + F^*(-\Lambda^* p^*) + < \Lambda^* p^*, v >$

$\Lambda^* : Y^* \to V^*$, $\ < \Lambda^* p^*, v > := (p^*, \Lambda v)$, and $F^* : V^* \to \mathbb{R}$.

Lemma (Error identity)

For any $v \in \mathbb{K}_1$ and for any $y^* \in Y_f^*$ holds the equality

$$\underbrace{D_G(\Lambda v, p^*) + D_F(v, -\Lambda^* p^*)}_{\text{error measure for approximation } v} + \underbrace{D_G(\Lambda u, y^*) + D_F(u, -\Lambda^* y^*)}_{\text{error measure for approximation } y^*}$$

$$= \underbrace{D_G(\Lambda v, y^*) + D_F(v, -\Lambda^* y^*)}_{\text{no } u,\text{ no } p^*,\text{ thus, computable}}. \quad (3)$$

Note: the r.h.s. of (3) equals the duality gap $J(v) - I^*(y^*)$.

It is directly shown that

- $D_G(\Lambda v, p^*) = \frac{1}{2}||\nabla\nabla v - p^*||_M^2 = \frac{1}{2}||\nabla\nabla(v - u)||_M^2 =: m(v)$,
- $D_G(\Lambda u, y^*) = \frac{1}{2}||\nabla\nabla u - y^*||_M^2 = \frac{1}{2}||p^* - y^*||_M^2 =: m^*(y^*)$,
- $D_G(\Lambda v, y^*) = \frac{1}{2}||\nabla\nabla v - y^*||_M^2$,
- $D_F(u, -\Lambda^* y^*) = D_F(v, -\Lambda^* y^*) = 0$.

Finally, $\quad D_F(v, -\Lambda^* p^*)$

$$= \underbrace{\Big((p^*, \nabla\nabla v)_M - (f, v) \Big) - \inf_{v \in \mathbb{K}_1} \Big\{ (p^*, \nabla\nabla v)_M - (f, v) \Big\}}_{\geq 0}$$

$$= \left(p^*, \nabla\nabla(v - \varphi)\right)_M - \left(f, (v - \varphi)\right) =: m_\varphi(v). \quad \square$$

Remark: a kin problem

Problem (1) over the set $\mathbb{K} = \left\{ v \in \mathbb{K}_1 : \frac{\partial v}{\partial \nu}\big|_{\partial\Omega} = 0 \right\}$ deals with the case of rigidly-supported beams and plates, see [1].

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