Morphology of the Phase Space of the Mathematical Model of Autocatalytic Reaction with Diffusion

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Introduction

The mathematical model of an autocatalytic reaction with diffusion, based on a degenerate system of distributed brusselator equations, belongs to a wide class of Sobolev-type semilinear models. In this mathematical model, the rate of change of one of the components of the system can significantly exceed the other, which leads to a degenerate system of equations. Due to the degeneracy of the system of equations, the solution of the Cauchy problem does not exist for an arbitrary initial value. Consideration of the initial condition of Showalter – Sidorov allows one to avoid the difficulties of studying the Cauchy problem, however, the non-unique solution of the problem under consideration is possible if the phase space of the equation contains a Whitney fold or fold.

Problem

In the cylinder $\Omega \times R_+$ consider the degenerate system of equations

$$\begin{cases} \varepsilon_1 v_t = \alpha_1 v_{ss} + \beta_1 - (\beta_2 + 1)v + v^2 w, \\ \varepsilon_2 w_t = \alpha_2 w_{ss} + \beta_2 v - v^2 w, \end{cases} \qquad (1)$$

$$\varepsilon_1 = 0 \text{ or (and) } \varepsilon_2 = 0 \qquad (*)$$

with the Dirichlet condition

$$v(s,t) = 0, w(s,t) = 0, (s,t) \in \partial\Omega \times R_+. \qquad (2)$$

Here $v = v(s,t)$ and $w = w(s,t)$ are vector functions characterizing the concentrations of reagents, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R_+$. Let $U_M = (\overset{\circ}{W}{}_2^1(\Omega))^2$, $U = (L_2(\Omega))^2$, $U_N = (L_4(\Omega))^2$. Construct an auxiliary interpolation space X_α such that $\mathfrak{U}_M \hookrightarrow X_\alpha \hookrightarrow U_N \hookrightarrow U$.

Results

Case $\boldsymbol{\varepsilon_1 = 0}$. The phase space of problem (1), (2) takes the form

$$\mathfrak{P}_{\varepsilon_1} = \{x \in X_\alpha : \langle \textstyle\sum_{i=1}^n \alpha_1 v_{s_i}, \xi_{s_i} \rangle = \langle \beta_1 - (\beta_2 + 1)v - v^2 w, \xi \rangle\}.$$

Theorem 1. Let $\alpha_1, \alpha_2 \in R \backslash \{0\}$, $n \le 4$, $\beta_1 \in R$, $\beta_2 \in R \backslash \{\alpha_1 \nu_k - 1\}$, where ν_k are the eigenvalues of the homogeneous Dirichlet problem for the operator $(-\Delta)$, then the set $\mathfrak{P}_{\varepsilon_1}$ at the point x_0 is a simple Banach C^∞-manifold.

Theorem 2. Let $\alpha_1, \alpha_2 \in R \backslash \{0\}$, $n \le 4$, $\beta_1 \in R$, $\beta_2 = \alpha_1 \nu_k - 1$, then the phase space $\mathfrak{P}_{\varepsilon_1}$ contains the Whitney 1-fold.

Case $\boldsymbol{\varepsilon_2 = 0}$. The phase space of problem (1), (2) takes the form

$$\mathfrak{P}_{\varepsilon_2} = \{x = (v, w) \in X_\alpha : \textstyle\sum_{i=1}^n \langle -\alpha_2 w_{s_i}, \eta_{s_i} \rangle + \langle v^2 w, \eta \rangle = \langle \beta_2 v, \eta \rangle\}.$$

Theorem 3. Let $\alpha_1 \in R \backslash \{0\}$, $\alpha_2 \in R_-$, $\beta_1, \beta_2 \in R$, $n \le 4$, then the set $\mathfrak{P}_{\varepsilon_2}$ at the point x_0 is a simple Banach C^∞-manifold.

Case $\boldsymbol{\varepsilon_1 = \varepsilon_2 = 0}$. The phase space of problem (1), (2) takes the form

$$\mathfrak{P}_{\varepsilon_1 = \varepsilon_2 = 0} = \left\{ x \in X_\alpha : \begin{cases} \sum_{i=1}^n \langle \alpha_1 v_{s_i}, \xi_{s_i} \rangle = \langle \beta_1 - (\beta_2 + 1)v - v^2 w, \xi \rangle, \\ \sum_{i=1}^n \langle -\alpha_2 w_{s_i}, \eta_{s_i} \rangle + \langle v^2 w, \eta \rangle = \langle \beta_2 v, \eta \rangle \end{cases} \right\}.$$

Conclusion

The phase space of a model of an autocatalytic reaction with diffusion has been studied. Conditions are found under which the phase space of the system contains a Whitney 1-assembly or is a simple Banach C^∞-manifold.