Solvability of the Nonlinear Equation with the Dzhrbashyan — Nersesyan Fractional Derivatives

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Dzhrbashyan — Nersesyan Derivative

Let $\mathcal{Z}$ be a Banach space, $z : \mathbb{R}_+ \to \mathcal{Z}$. The Riemann — Liouville fractional integral of an order $\alpha > 0$ for a function z has the form

$$J_t^\alpha z(t) := \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds, \quad t > 0.$$

The Riemann — Liouville fractional derivative of an order $\alpha > 0$ for a function z is defined as ${}^R D_t^\alpha z(t) := D_t^m J_t^{m-\alpha} z(t)$, where $m - 1 < \alpha \leq m \in \mathbb{N}$, $D_t^m := \frac{d^m}{dt^m}$ is the integer order derivative. Further, we use the notation $D_t^{-\alpha} := J_t^\alpha$ for $\alpha > 0$.
Let $\{\alpha_k\}_0^n = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ be the set of real numbers $\alpha_k \in (0, 1]$, $k = 0, 1, \dots, n \in \mathbb{N}$. Denote $\sigma_k := \sum\limits_{j=0}^k \alpha_j - 1, \quad k = 0, 1, \dots, n$, so, $-1 < \sigma_k \leq k - 1$. Further, it is assumed everywhere that the condition $\sigma_n > 0$ is met. The fractional Dzhrbashyan — Nersesyan derivative of an order σ_n, associated with a sequence $\{\alpha_k\}_0^n$, is determined by relations

$$D^{\sigma_0} z(t) := D^{\alpha_0 - 1} z(t),$$

$$D^{\sigma_k} z(t) := D^{\alpha_k - 1} D^{\alpha_{k-1}} D^{\alpha_{k-2}} \dots D^{\alpha_0} z(t), \quad k = 1, 2, \dots, n.$$

Special Banach spaces

The space of functions $v \in C((t_0, T]; \mathcal{Z})$, such that $(t - t_0)^\gamma v(t) \in C([t_0, T]; \mathcal{Z})$ at some $\gamma > 0$ denote by $C_\gamma([t_0, T]; \mathcal{Z})$ and endow with the norm

$$\|v\|_{C_\gamma([t_0,T];\mathcal{Z})} := \max_{t \in [t_0,T]} \|(t - t_0)v(t)\|_{\mathcal{Z}}. \tag{1}$$

On the space $C^1_\gamma([t_0, T]; \mathcal{Z})$ of functions $v \in C([t_0, T]; \mathcal{Z}) \cap C^1((t_0, T]; \mathcal{Z})$, such that $(t - t_0)^\gamma v'(t) \in C([t_0, T]; \mathcal{Z})$ at some $\gamma > 0$, define the norm

$$\|v\|_{C^1_\gamma([t_0,T];\mathcal{Z})} := \max_{t \in [t_0,T]} \|v(t)\|_{\mathcal{Z}} + \max_{t \in [t_0,T]} \|(t - t_0)^\gamma v'(t)\|_{\mathcal{Z}}. \tag{2}$$

In these notations $C_0([t_0, T]; \mathcal{Z}) := C([t_0, T]; \mathcal{Z})$, $C^1_0([t_0, T]; \mathcal{Z}) := C^1([t_0, T]; \mathcal{Z})$.

Lemma (1)

Let $\gamma \in (0, 1)$, then $C_\gamma([t_0, T]; \mathcal{Z})$ with norm (1) and $C^1_\gamma([t_0, T]; \mathcal{Z})$ with norm (2) are Banach spaces.

Special Banach spaces

Lemma (2)

Let $\gamma \in (0,1)$, $\beta \geq \gamma$, *then* $J_t^{\beta} \in \mathcal{L}(C_\gamma([t_0,T];\mathcal{Z}); C([t_0,T];\mathcal{Z}))$.

Let $\alpha_0 = 1$, $\alpha_k \in (0,1]$, $k = 1,2,\ldots,n$. Consider the set $C^{\{\alpha_k\}_0^{n-1}}([t_0,T];\mathcal{Z})$ of functions $v \in C^1_{1-\alpha_1}([t_0,T];\mathcal{Z})$, such that there exist the derivatives $D^{\sigma_k} v \in C^1_{1-\alpha_{k+1}}([t_0,T];\mathcal{Z})$, $k = 1,2,\ldots,n-1$. Endow this set with the norm

$$\|v\|_{C^{\{\alpha_k\}_0^{n-1}}([t_0,T];\mathcal{Z})} := \|v\|_{C^1_{1-\alpha_1}([t_0,T];\mathcal{Z})} + \sum_{k=1}^{n-1} \|D^{\sigma_k} v\|_{C^1_{1-\alpha_{k+1}}([t_0,T];\mathcal{Z})}. \tag{3}$$

Theorem (1)

Let $\alpha_0 = 1$, $\alpha_k \in (0,1]$, $k = 1,2,\ldots,n$. *Then the set* $C^{\{\alpha_k\}_0^{n-1}}([t_0,T];\mathcal{Z})$ *with norm* (3) *is a Banach space.*

Solvability of the Nonlinear Equation

Denote by Z an open set in $\mathbb{R} \times \mathcal{Z}^n$, the operator $B : Z \to \mathcal{Z}$ is nonlinear, generally speaking. Consider the initial value problem for nonlinear equation

$$D^{\sigma_n} z(t) = Az(t) + B(t, D^{\sigma_0} z(t), D^{\sigma_1} z(t), \dots, D^{\sigma_{n-1}} z(t)), \tag{4}$$

$$D^{\sigma_k} z(t_0) = z_k, \quad k = 0, 1, \dots, n-1. \tag{5}$$

A function $z \in C((t_0, t_1]; \mathcal{Z})$ is called a solution of problem (4), (5) on $(t_0, t_1]$, if $D^{\sigma_k} z \in C([t_0, t_1]; \mathcal{Z})$, $k = 0, 1, \dots, n-1$, $D^{\sigma_n} z \in C((t_0, t_1]; \mathcal{Z})$, for all $t \in (t_0, t_1]$ the elements $(t, D^{\sigma_0} z(t), D^{\sigma_1} z(t), \dots, D^{\sigma_{n-1}} z(t))$ belong to the set Z, equality (4) is satisfied and conditions (5) are valid.

Theorem (2)

Let $A \in \mathcal{L}(\mathcal{Z})$, $z_k \in \mathcal{Z}$, $0 < \alpha_k \leq 1$, $k = 0, 1 \dots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, Z be open set in $\mathbb{R} \times \mathcal{Z}$, $B \in C^2(Z; \mathcal{Z})$. Then for each $(t_0, z_0, \dots, z_{n-1}) \in Z$ there exists an unique solution of problem (4), (5) on $(t_0, t_1]$ at some $t_1 > t_0$.

The proof of Theorem 2 is based on the following lemma.

Lemma (3)

Let $A \in \mathcal{L}(\mathcal{Z})$, $z_k \in \mathcal{Z}$, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $B \in C(Z; \mathcal{Z})$, $(t_0, z_0, \dots, z_{n-1}) \in Z$. *Then function* $z \in C((t_0, t_1]; \mathcal{Z})$, *such that* $D^{\sigma_k} z \in C([t_0, t_1]; \mathcal{Z})$, $k = 0, 1, \dots, n-1$, *is a solution of problem* (4), (5) *on* $(t_0, t_1]$, *if and only if for* $t \in (t_0, t_1]$

$$z(t) = \sum_{k=0}^{n-1} (t - t_0)^{\sigma_k} E_{\sigma_n, \sigma_k + 1}((t - t_0)^{\sigma_n} A) z_k +$$

$$+ \int_{t_0}^{t} (t - s)^{\sigma_n - 1} E_{\sigma_n, \sigma_n}((t - s)^{\sigma_n} A) B(s, D^{\sigma_0} z(s), D^{\sigma_1} z(s), \dots, D^{\sigma_{n-1}} z(s)) ds. \tag{6}$$

Application to Initial Boundary Value Problems

Consider the problem

$$D^{\sigma_k} u(s, t_0) = u_k(s),\ k = 0, 1, \ldots, n-1,\ s \in \Omega, \tag{7}$$

$$u(s,t) = 0,\ \ (s,t) \in \partial\Omega \times (t_0, T), \tag{8}$$

$$D^{\sigma_n}(\lambda - \Delta)u(s,t)d\alpha = \Delta u(s,t) + F(t, D^{\sigma_0}u(s,t), D^{\sigma_1}u(s,t), D^{\sigma_{n-1}}u(s,t)),\ \ (s,t) \in \Omega \times (t_0, T), \tag{9}$$

where D^{σ_k} are the Dzhrbashyan — Nersesyan derivatives with respect to t, $k = 0, 1, \ldots, n$. Take

$$\mathcal{X} = \{v \in H^2(\Omega) : v(s) = 0,\ x \in \partial\Omega\}, \mathcal{Y} = L_2(\Omega).$$

Take $P_1(\lambda) = \kappa - \lambda$, $P_1(\lambda) \neq 0$, $\kappa > 0$, $Q_1(\lambda) = \lambda$.

Theorem 2 implies the next statement.

Theorem (3)

Let $0 < \alpha_k \leq 1$, $k = 0, 1, \ldots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, *the spectrum* $\sigma(\Lambda)$ *do not contain the origin and zeros of the polynomial* $P_1(\lambda)$, $d < 4$, $u_k \in \mathcal{X}$, $k = 0, 1 \ldots, n-1$, $F \in C^\infty(\Omega \times \mathbb{R}^n; \mathbb{R})$. *Then at some* $t_1 > t_0$ *there exists a unique solution of problem* (7)–(9).

Note only that $Z = \mathbb{R} \times \mathcal{X}^n$ and due to the condition $d < 4$ we have condition $F(\cdot, x_0(\cdot), x_1(\cdot), \ldots, x_{n-1}(\cdot))) \in C^\infty((H^2(\Omega))^n; H^2(\Omega))$ by theorem, hence, a mapping $B(x_0(\cdot), x_1(\cdot), \ldots, x_{n-1}(\cdot))) := L^{-1}F(\cdot, x_0(\cdot), x_1(\cdot), \ldots, x_{n-1}(\cdot))) \in C^\infty((H^2(\Omega))^n; L_2(\Omega))$.