High Order Nonlinear Elliptic Equations in the Subcoercive Case

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In the early 1960th, the works of G. Minty and F. Browder revealed the fundamental role of the monotonicity condition for the existence of a solution to nonlinear elliptic equations and systems of arbitrary order, which opened the way for the study of qualitative properties of energetic solutions. But mostly equations of strictly divergent form $\mathrm{div}^m A(x, D^m u) = f(x)$ were investigated. We consider equations and systems of nonstrictly divergent form $\mathrm{div}^t A(x, D^s u) = f(x)$, $s \neq t$, interpolating in a sense divergent equations with standard structure conditions and equations of Cordes type. We establish that equations of a nonstrictly divergent form admit some estimates of the solution even under degenerate structure conditions. We present the subsequent results, e.g. existence and uniqueness of the solution under degeneration of coerciveness. These results are completely different from the results possible for equations of a strictly divergent form.

All theorems and a definition below are ours, so we do not mark them by 'K'.

We consider equation (or system):

$$\mathcal{L}u := \operatorname{div}^t A(x, D^s u) = f(x),$$

$x \in \mathbb{R}^n$, $s + t$ is even, $s, t \in \mathbb{N}_0$. Structure condition on A:

$$|B(x, \xi) - B(x, \eta)| \leq |\xi - \eta|,$$

B defined by

$$\Delta^{(s+t)/2} u + \operatorname{div}^t B(x, D^s u) = \kappa \operatorname{div}^t A(x, D^s u),$$

$\kappa > 0$ – normalizing multiplier; and $A(x, 0) = 0$.
The condition allows degeneration of ellipticity, examples:

$$\Delta^m u + |D^{2m} u|, \qquad \Delta^{(s+t)/2} u - \partial_{x_1}^{s+t} u.$$

The stronger condition

$$|B(x,\xi) - B(x,\eta)| \leq K|\xi - \eta|, \quad K < 1,$$

provides coerciveness and monotonicity of $\mathcal{L}u$ in pair with $\Delta^{(s-t)/2}u$ in the space $H^s = \{u : D^s u \in L_2\}$.
For $s = t$ the condition with $K < 1$ coincides with the standard structure conditions for divergent equations and systems; for one nondivergent equation ($t = 0$) – with the Cordes condition.

If $K = 1$, the coerciveness fails. Nevertheless, in the case $s \neq t$ we establish the 'subcoercive' estimate:

$$\|D^{s-1}u\|_{a-2} \leq c\,\|I_{t-1}\,f\|_{a+2}$$

for some range $a \in (a_*, a^*)$; here $\|\cdot\|_a$ is a norm in the weighted space $L_2(\mathbb{R}^n; (1+|x|)^a)$, I_t is a Riesz potential of order t.
Let $H_a^s := \{u : \|D^s u\|_a < \infty\}$, $H_a^{-t} := \{f : \|I_t\,f\|_a < \infty\}$.

The estimate is valid for $u \in H_a^s$ — not a self-contained result.

Description of $a_*,\ a^*$.

Let $T^{s,t} := D^s \Delta^{-(s+t)/2} \mathrm{div}^t$ – vector Riesz transform;
$L_{2,a} = L_2(\mathbb{R}^n;\ |x|^a)$ (here the weight be a strict power of $|x|$);
$\mathcal{A} := \{a \in (-n, n) :\ \|T^{s,t}\|_{L_{2,a} \to L_{2,a}} = 1\}$,
$a_* := \inf \mathcal{A}, \quad a^* := \sup \mathcal{A}$.
The definition allows the possibility $a_* = a^* = 0$ (and it is so for $s = t$).

Theorem

Let $s \neq t$, $n > 3$. Then (a_*, a^*) is a nonempty half-neighbourhood of 0, negative for $s > t$ and positive for $s < t$. For $f \in L_{2,a}$, $a \in (a_*, a^*)$,

$$\| T^{s,t} f \|_a^2 \leq \| f \|_a^2 - c_a \| I_1 f \|_{a-2}^2 ,$$

$c_a = c_{s,t,n} (a - a_*)(a^* - a) > 0$.

Weak solutions

Let us consider weak solutions, in sense of integral identity.

Theorem (uniqueness)

Let $s \neq t$, $n > 3$. Then a solution is unique in the class $H^s_{loc} \cap H^{s-1}_{a_*-2}$.

For existence, let A does not depend on x, i.e., $\mathcal{L}u = \operatorname{div}^t A(D^s u)$.

Theorem (existence & uniqueness)

Let $s \neq t$, $n > 3$, $a, b \in (a_*, a^*)$. Then for $f \in H^{-t+2}_{b+2} \cap H^{-t+1}_{a+2}$ there exists a unique solution $u \in H^s_{b-2} \cap H^{s-1}_{a-2}$.

Gaps: for existence, no dependence of A on x is allowed; additional regularity $f \in H^{-t+2}_{b+2}$ is required.

For uniqueness, we do not need any estimates of solution in the space H^s; the requirement that $u \in H^s$ locally is needed to give the meaning for u as a solution in the sense of integral identity.

That's a reason to consider something weaker than a weak solution. We introduce the notion of a *subweak solution*, which can have at most $s-1$ derivatives. Our definition is quite similar to the notion of generalized pseudomonotonicity of Browder–Hess.

Subweak solutions

Definition

$u \in H^{s-1}_{a-2}$ be a solution for $\mathcal{L}u = f$ if there exists a sequence $u_j \in H^s_a$ such that $u_j \to u$ weakly in H^{s-1}_{a-2}, $\mathcal{L}u_j \to f$ weakly in H^{-t}_a,
$\lim_{j\to\infty}(f - \mathcal{L}u_j,\, lu_j) \geq 0$
$\forall\, l : H^{s-1}_{a-2} \to H^{\,t-1}_{-a-2}$ weakly continuous, $l(H^s_a) \subset H^{\,t}_{-a}$, and
$\lim_{j\to\infty}(\Delta^{(s+t)/2}v_j,\, lv_j) \geq 0$ for $\|v_j\|_{H^s_a} \leq c$, $\|v_j\|_{H^{s-1}_{a-2}} \to 0$.

Unlike the classical situation, spaces of convergence are not dual — the spaces H^{-t}_a and $H^{\,t-1}_{-a-2}$, the second – as an image $l(H^{s-1}_{a-2})$.

Theorem (existence)

Let $s \neq t$, $n > 3$, $a \in (a_*, a^*)$. Then for $f \in H^{-t+1}_{a+2}$ there exists a subweak solution $u \in H^{s-1}_{a-2}$.

For uniqueness, let A doesn't depend on x, i.e., $\mathcal{L}u = \operatorname{div}^t A(D^s u)$.

Theorem (existence & uniqueness)

Let $s \neq t$, $n > 3$, $a \in (a_*, a^*)$, and A does not depend on x. Then for $f \in H^{-t+1}_{a+2}$ there exists a unique subweak solution $u \in H^{s-1}_{a-2}$.

Still a gap, in a sense the opposite: for uniqueness, no dependence of A on x is allowed. But with subweak solutions (or 'generalized pseudo-monotone solutions') we no longer need additional regularity of data.

Qualitative theory

Qualitative properties of weak solutions, established in a known way on the basis of the integral identity, can be extended to subweak solutions.

Theorem

Let $F : H^{s-1}_{a-2} \to \mathbb{R}_+$ be weakly lower semicontinuous, I be some operator from the definition of subweak solutions, and

$$F(u) \le (\mathcal{L}u,\, Iu) \qquad \text{for } u \in H^s_a\,.$$

Then a subweak solution $u \in H^{s-1}_{a-2}$ of $\mathcal{L}u = f$ satisfies

$$F(u) \le (f,\, Iu).$$

References and Acknowledgments

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Estimates for the vector Riesz transforms are an old result, see Math. Notes 72, no. 6, 799-810 (2002); or Dokl. Math. 65, no. 2, 154-156 (2002).

Uniqueness for weak solutions had been presented at Nikolsky's seminar at the Steklov Math. Inst., Moscow, 2018.

Results for subweak solutions (or 'generalized pseudomonotone solutions') are recent.

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