On the stability of the stochastic Hoff equation

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The Hoff model

$$(\lambda + \Delta)\dot{u} = \alpha u + \beta u^3, \tag{1}$$

$$u(x,0) = u_0,\ u \in \Sigma,\ u(t,0) = 0, (x,t) \in \partial\Sigma \times \mathbb{R} \tag{2}$$

is a model of buckling of an I-beam from the equilibrium position. Here $\Sigma \subset \mathbb{R}^n$ is a bounded region with a smooth boundary $\partial\Sigma$, the parameter $\lambda \in \mathbb{R}_+$ is the parameter responsible for the load applied to the beam and parameters $\alpha,\ \beta \in \mathbb{R}$ are parameters responsible for the material from which the beam is made.

Paper [1] considers the set of valid initial data of the problem (1), (2) understood as a phase space. Here equation (1) was reduced to a semilinear Sobolev type equation. In [2], the proof of smoothness and simplicity of the phase space of the equation for positive values of parameters α and β is considered. The stability of solutions of the equation (1) in a neighborhood of the point zero is devoted in [3], which shows the existence of stable and unstable invariant manifolds.

[1] Sviridyuk G.A., Sukacheva T.G. [Phase Spaces of a Class of Operator Equations]. *Differential Equations*, 1990, vol. 26, no. 2, pp. 250–258. (in Russian)

[2] Sviridyuk G.A., Kazak V.O. The Phase Space of an Initial-Boundary Value Problem for the Hoff Equation. *Mathematical Notes*, 2002, vol. 71, no. 2, pp. 262–266.

[3] Sviridyuk G.A., Kitaeva O.G. Invariant Manifolds of the Hoff Equation. *Mathematical Notes*, 2006, vol. 79, no. 3, pp. 408–412.

The space of differential "noises"

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space with a probability measure $\mathbf{P}$ associated with a σ-algebra $\mathcal{A}$ of subsets of the set Ω. Let $\mathbf{R}$ be the set of real numbers endowed with the σ-algebra. Then the mapping $\xi : \mathcal{A} \to \mathbf{R}$ is called a *random variable*. Consider the set of random variables $\{\xi\}$ having zero expectation $(\mathbf{E}\xi = 0)$ and finite variance $(\mathbf{D}\xi < +\infty)$. The set forms Hilbert space $\mathbf{L_2}$ with scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$ and norm $\|\xi\|_{\mathbf{L_2}}$. Let $\mathcal{A}_0$ be a σ-subalgebra of the σ-algebra $\mathcal{A}$. Construct a subspace $\mathbf{L_2^0} \subset \mathbf{L_2}$ of random variables measurable with respect to $\mathcal{A}_0$. Denote the orthoprojector by $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$. Let $\xi \in \mathbf{L_2}$ be a random variable, then $\Pi\xi$ is called *conditional expectation* and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$.
Further, a measurable mapping $\eta : \mathcal{I} \times \mathcal{A} \to \mathbf{R}$, where $\mathcal{I} \subset \mathbf{R}$ is an interval, we call a *stochastic process*, a random variable $\eta(\cdot, \omega)$ we call a *cut* of the stochastic process, and a function $\eta(t, \cdot)$, $t \in \mathcal{I}$ we call a *trajectory* of the stochastic process. A stochastic process $\eta = \eta(t, \cdot)$ is called *continuous,* if almost sure (i.e. for almost all $\omega \in \mathcal{A}$) the trajectories $\eta(t, \omega)$ are continuous functions. The set $\{\eta = \eta(t, \omega)\}$ of continuous stochastic processes forms Banach space $\mathbf{CL}_2$ with norm

$$\|\eta\|_{\mathbf{CL_2}} = \sup_{t \in \mathcal{I}} (\mathbf{D}\eta(t, \omega))^{1/2}.$$

Consider a stochastic process $\eta \in \mathbf{CL_2}$. The *Nelson–Gliklikh derivative* of the *stochastic process* η *at a point* $t \in \mathcal{I}$ is a random variable

$$\overset{o}{\eta}(\cdot,\omega) = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot)-\eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot)-\eta(t-\Delta t,\cdot)}{\Delta t}\right)\right), \quad (1)$$

if the limit exists in the sense of a uniform metric on R. Here $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$, and $\mathcal{N}_t^\eta \subset \mathcal{A}$ is the σ-algebra generated by the random variable $\eta(t,\omega)$.

We say that the Nelson–Gliklikh derivative $\overset{o}{\eta}(\cdot,\omega)$ of the stochastic process $\eta(\cdot,\omega)$ exists (or almost sure exists) on the interval $\mathcal{I}$, if there exist the Nelson–Gliklikh derivatives $\overset{o}{\eta}(\cdot,\omega)$ at all (or almost all) points of $\mathcal{I}$. The set of continuous stochastic processes that have continuous Nelson–Gliklikh derivatives $\overset{o}{\eta} \in \mathbf{CL}_2(\mathcal{I})$ forms Banach space $\mathbf{C}^1\mathbf{L}_2(\mathcal{I})$ with norm

$$\|\eta\|_{\mathbf{C}^1\mathbf{L}_2} = \sup_{t\in\mathcal{I}}(\mathbf{D}\eta(t,\omega) + \mathbf{D}\overset{o}{\eta}(t,\omega))^{1/2}.$$

Hence, by induction, we define Banach spaces $\mathbf{C}^l\mathbf{L}_2(\mathcal{I})$, $l \in N$, of stochastic processes that have continuous Nelson–Gliklikh derivatives on $\mathcal{I}$ up to the order $l \in N$ inclusive.

Let us construct the space of *random* $\mathbf{K}$*-variables.* Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{\varphi_k\}$, $\mathbf{K}=\{\lambda_k\}\subset\mathbf{R}_+$ be a monotone sequence such that $\sum\limits_{k=1}^{\infty}\lambda_k^2<+\infty$, $\{\xi_k\}=\xi_k(\omega)\subset\mathbf{L}_2$ be a sequence of random variables such that $\|\xi_k\|_{\mathbf{L}_2}\leq C$ for all $C\in R_+$ and $k\in N$. Construct a $\mathcal{H}$-valued random $\mathbf{K}$-variable

$$\xi(\omega)=\sum_{k=1}^{\infty}\lambda_k\xi_k(\omega)\varphi_k.$$

Completion of the linear span of the set $\{\lambda_k\xi_k\varphi_k\}$ in the norm

$$\|\eta\|^2_{\mathbf{H}_\mathbf{K}\mathbf{L}_2}=(\sum_{k=1}^{\infty}\lambda_k^2\mathbf{D}\xi_k)^{1/2}$$

is called the *space of ($\mathcal{H}$-valued) random* $\mathbf{K}$*-variables* and is denoted by $\mathbf{H}_\mathbf{K}\mathbf{L}_2$. It is easy to see that the space $\mathbf{H}_\mathbf{K}\mathbf{L}_2$ is Hilbert, and the constructed above random $\mathbf{K}$-variable $\xi=\xi(\omega)\in\mathbf{H}_\mathbf{K}\mathbf{L}_2$.

The phase space

Consider the stochastic analogue of the equation (1). Let $\mathfrak{U} = \overset{\circ}{W}{}_2^1$, $\mathfrak{F} = W_2^{-1}$ and $\{\nu_k\}$ be a sequence of eigenvalues of the Laplace operator numbered by non-increment taking into account multiplicity. We construct the spaces of random $\mathbf{K}$-values $\mathbf{U_K L}_2$, $\mathbf{F_K L}_2$ and spaces of differentiable "noise" $\mathbf{C}^l\mathbf{U_K L}_2$, $l \in \{0\} \bigcup \mathbb{N}$. Let $\mathbf{K} = \{\lambda_k\}$ be a sequence such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. Operators L, M and N are defined by formulas

$$L : \eta \to (\lambda + \Delta)\eta, \; M : \eta \to \alpha\Delta\eta, \; N : \eta \to \beta\eta^3, \; \eta \in \mathbf{U_K L}_2.$$

Then the stochastic analogue of the Hoff equation (1) is represented as an equation

$$L \overset{o}{\eta} = M\eta + N(\eta). \tag{3}$$

Definition 1.

The set $\mathbf{P_K L}_2 \subset \mathbf{U_K L}_2$ is called a *stochastic phase space* of equation (3), if
(i) probably almost every solution path $\eta = \eta(t)$ of the equation (3) lies in $\mathbf{P_K L}_2$, i.e. $\eta(t) \in \mathbf{P_K L}_2, t \in \mathbb{R}$, for almost all trajectories;
(ii) for almost all $\eta_0 \in \mathbf{P_K L}_2$ exists a solution to the problem $\eta(0) = \eta_0$ for equation (3).

Definition 2.

The set

$$\mathbf{M}_{\mathbf{K}}^{+(-)}\mathbf{L}_2 = \{\eta_0 \in \mathbf{U}_{\mathbf{K}}\mathbf{L}_2 : \|P_{l(r)}\eta_0\|_{\mathbf{U}_{\mathbf{K}}\mathbf{L}_2} \leq R_1,\ \|\eta(t,\eta_0)\|_{\mathbf{U}_{\mathbf{L}}\mathbf{L}_2} \leq R_2,\ t \in \mathbb{R}_{+(-)}\}$$

such that $\mathbf{M}_{\mathbf{K}}^{+(-)}\mathbf{L}_2$ diffeomorphic to a closed ball in $\mathbf{I}_{\mathbf{K}}^{+(-)}\mathbf{L}_2$; $\mathbf{M}_{\mathbf{K}}^{+(-)}\mathbf{L}_2$ concerns $\mathbf{I}_{\mathbf{K}}^{+(-)}\mathbf{L}_2$ at point zero; for any $\eta_0 \in \mathbf{M}_{\mathbf{L}}^{+(-)}\mathbf{L}_2$ $\|\eta(t,\eta_0)\|_{\mathbf{U}_{\mathbf{L}}\mathbf{L}_2} \to 0$ for $t \to +(-)\infty$
is called a *stable (unstable) invariant manifold of the equation* (3).

Theorem 1.

Let $\alpha\beta > 0$. Then the phase space of the equation (3) is the set

$$\mathbf{M}_{\mathbf{K}}\mathbf{L}_2 = \begin{cases} \{\chi \in \mathbf{U}_{\mathbf{K}}\mathbf{L}_2 : \sum\limits_{-\lambda=\nu_l} < (1-\alpha-\beta\chi^2)\chi, \varphi_l > \varphi_l = 0\}, \text{ if } -\lambda = \nu_l; \\ \mathbf{U}_{\mathbf{K}}\mathbf{L}_2, \text{ if } -\lambda \neq \nu_k. \end{cases}$$

Theorem 2.

Let $\alpha,\ \beta,\ \lambda \in \mathbb{R}_+$.
(i) If $\lambda \leq -\nu_1$ then the equation (3) has only a stable invariant manifold that coincides with $\mathbf{M}_{\mathbf{K}}\mathbf{L}_2$;
(ii) If $-\nu_1 < \lambda$ then there are a finite-dimensional unstable invariant the manifold $\mathbf{M}_{\mathbf{K}}^{+}\mathbf{L}_2$ and an infinite-dimensional stable invariant manifold $\mathbf{M}_{\mathbf{K}}^{-}\mathbf{L}_2$ of the equation (3) in the neighborhood of point zero.